ENTANGLEMENT AND THE INFRARED

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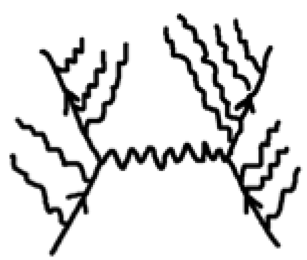

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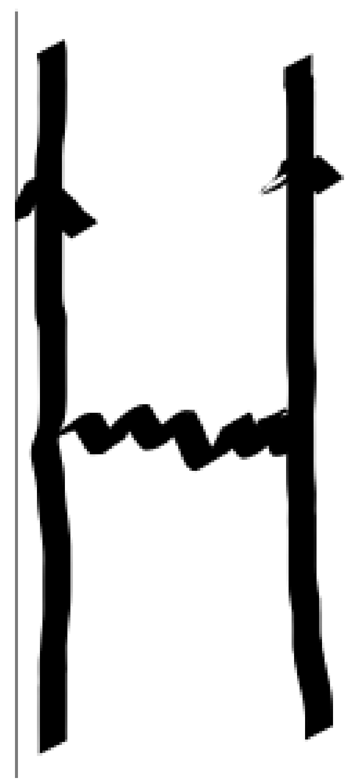
Prologue:

- The infrared sectors of quantum electrodynamics and perturbative quantum gravity have recently been of interest to possible resolutions of the black hole information paradox.
- $\exists$ New large gauge transformations, new conserved charges and super-selection rules.
- All of this brings up fundamental issues in quantum electrodynamics and perturbative quantum gravity, even without black holes.
- We will take a simple information theoretic look at the infrared in QED. Perturbative quantum gravity is similar (and perhaps even more interesting) but not as well-defined a quantum field theory since it is not renormalizable.

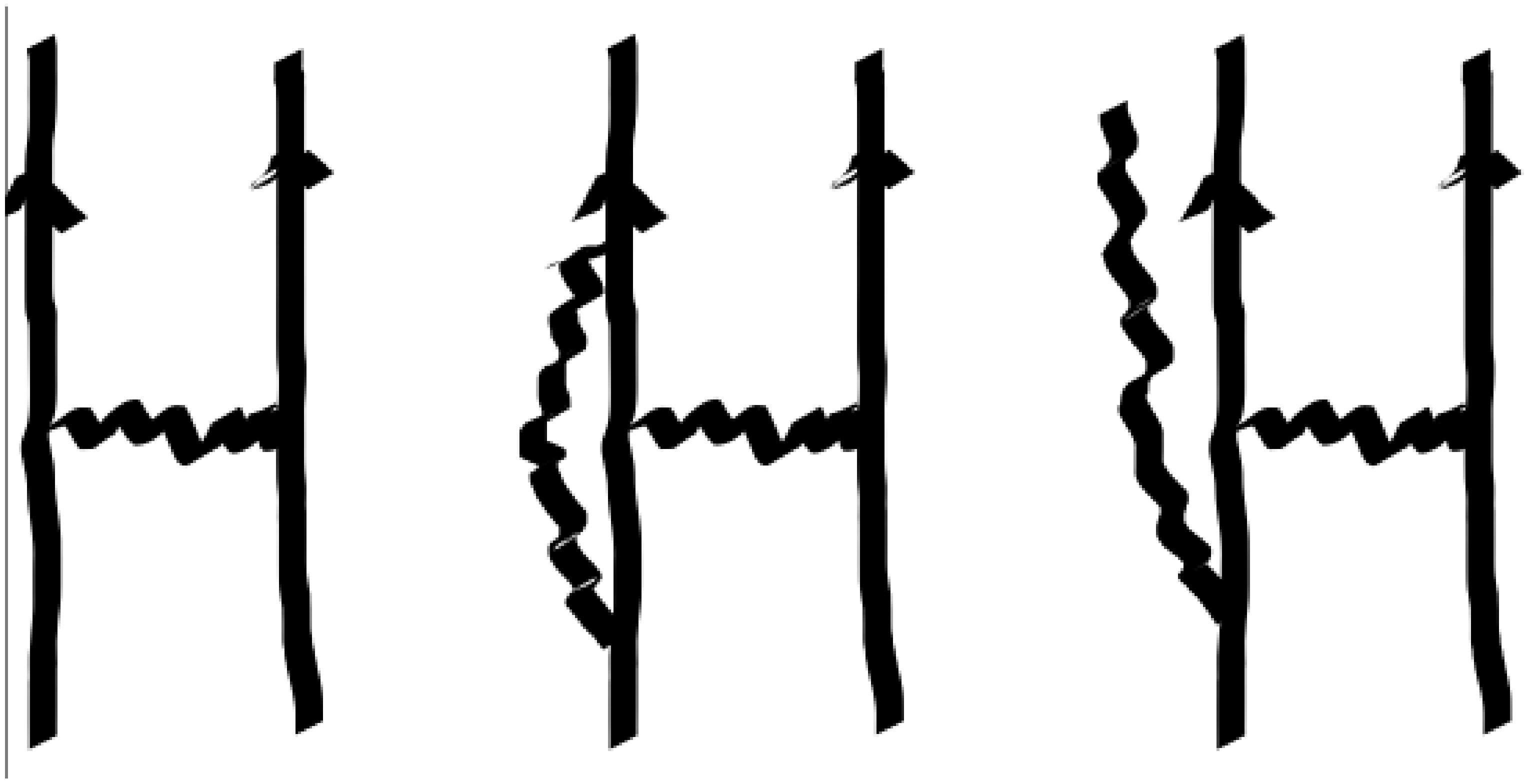
For example: Moeller Scattering

The amplitude for Moeller scattering, to 1% accuracy, is given by the tree-level Feynman diagram:

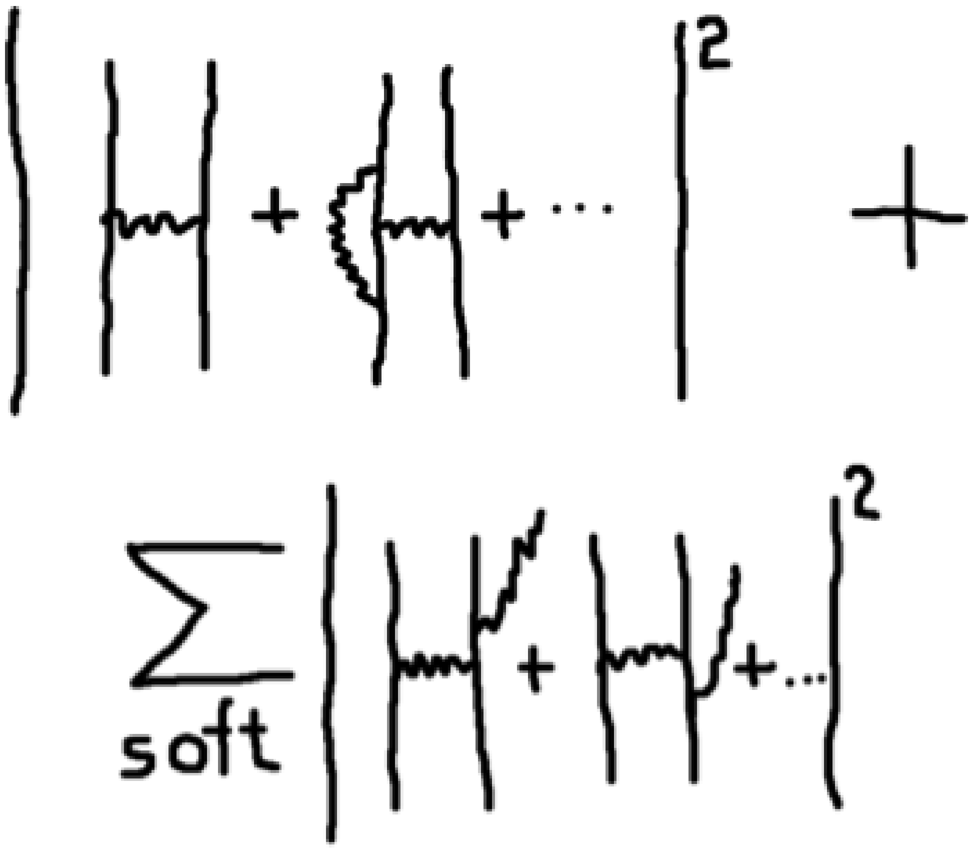


Radiative corrections to Moeller scattering:

However, to get 0.01% accuracy, there is a subtlety due to infrared divergences:



Inclusive probability

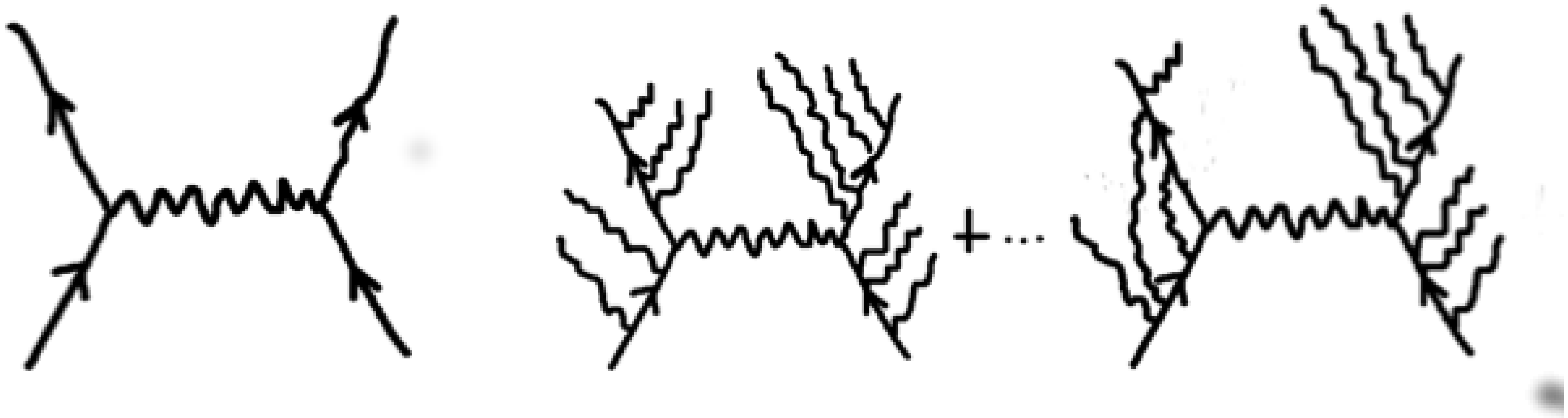


= finite

Cancellation of infrared divergences is guaranteed by unitarity of infrared cutoff S-matrix.

Infrared Catastrophe

Any scattering of charged particles is accompanied by the emission of an infinite number of soft photons



F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937)

D. R. Yennie, S. C. Frautschi, H. Suura, Ann. Phys. 13, 379 (1961)

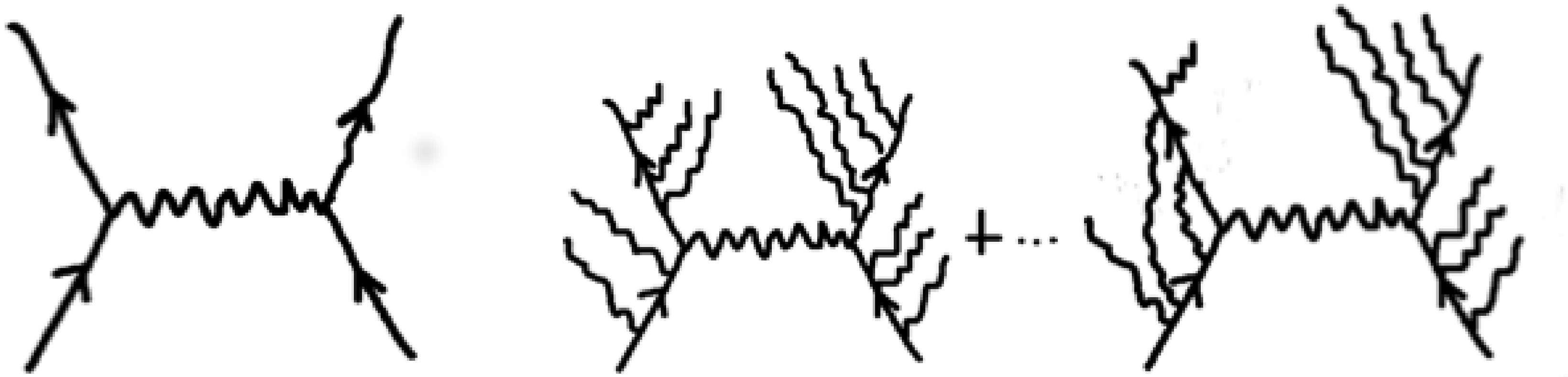
soft photon theorems

S. Weinberg, Phys. Rev. 140, B516 (1965)

soft graviton theorem

Information loss?

Soft photons which escape detection have polarizations and directions of propagation.



How much information do they carry away with them?

G.Grignani,GWS, Phys. Lett. B 772 (2017) 699.

D.Carney,L.Chaurette,D.Neuenfeld, GWS,
Phys.Rev.Lett.119(2017)no.18,180502
Phys.Rev. D97 (2018) no.2, 025007
arXiv:1803.02370

Information loss due to entanglement:

Composite system of two qubits: $|\ >_1 \otimes |\ >_2$

If subsystem $|\ >_2$ **becomes inaccessible, how much information about** $|\ >_1$ **do we lose?**

Unentangled state: $|\psi>= \left[\alpha|\uparrow>_1 + \sqrt{1-|\alpha|^2}|\downarrow>_1\right] \otimes |\uparrow>_2$

Entangled state:

$$|\psi>= \left[\alpha|\uparrow>_1 \otimes |\uparrow>_2 + \sqrt{1-|\alpha|^2}|\downarrow>_1 \otimes |\downarrow>_2\right]$$

Reduced density matrix: $\rho = \mathrm{Tr}_2\, |\psi><\psi|$

Unentangled state: $\rightarrow$

$$\rho = \left[\alpha|\uparrow>_1 + \sqrt{1-|\alpha|^2}|\downarrow>_1\right] \left[{}_1<\uparrow|\alpha^* + {}_1<\uparrow|\sqrt{1-|\alpha|^2}\right]$$

Entangled state: $\rightarrow$

$$\rho = |\alpha|^2|\uparrow>_1<\uparrow| + (1-|\alpha|^2)|\downarrow>_1<\downarrow| = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & 1-|\alpha|^2 \end{bmatrix}$$

So what?

Unentangled state: $\rightarrow$

$$\rho = \left[\alpha|\uparrow>_1 + \sqrt{1-|\alpha|^2}|\downarrow>_1\right]\left[{}_1<\uparrow|\alpha^* + {}_1<\uparrow|\sqrt{1-|\alpha|^2}|\right]$$

$$\rho = \begin{bmatrix} |\alpha|^2 & \alpha\sqrt{1-|\alpha|^2} \\ \alpha^*\sqrt{1-|\alpha|^2} & 1-|\alpha|^2 \end{bmatrix}$$

Entangled state: $\rightarrow$

$$\rho = |\alpha|^2|\uparrow>_1<\uparrow| + (1-|\alpha|^2)|\downarrow>_1<\downarrow| = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & 1-|\alpha|^2 \end{bmatrix}$$

These density matrices differ in their off-diagonal terms

Probability of finding system 1 in state $\frac{1}{\sqrt{2}}(|\uparrow>_1 + |\downarrow>_1)$

Unentangled state: $\rightarrow P = \frac{1}{2}\left|\alpha + \sqrt{1-|\alpha|^2}\right|^2$

Entangled state: $\rightarrow P = \frac{1}{2}|\alpha|^2 + \frac{1}{2}(1-|\alpha|^2) = \frac{1}{2}$

x **no interference**

Quantifying Entanglement:

Unentangled state: $\rightarrow$

$$\rho = \left[\alpha|\uparrow>_1 + \sqrt{1-|\alpha|^2}|\downarrow>_1\right]\left[{}_1<\uparrow|\alpha^* + {}_1<\uparrow|\sqrt{1-|\alpha|^2}|\right]$$

$$\rho = \begin{bmatrix} |\alpha|^2 & \alpha\sqrt{1-|\alpha|^2} \\ \alpha^*\sqrt{1-|\alpha|^2} & 1-|\alpha|^2 \end{bmatrix}$$

Entangled state: $\rightarrow$

$$\rho = |\alpha|^2|\uparrow>_1<\uparrow| + (1-|\alpha|^2)|\downarrow>_1<\downarrow| = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & 1-|\alpha|^2 \end{bmatrix}$$

Entanglement entropy: $S = -\mathrm{Tr}\ \rho \ln \rho$

Unentangled state $S = 0$

Entangled state $S = -|\alpha|^2 \ln|\alpha|^2 - (1-|\alpha|^2)\ln(1-|\alpha|^2) \neq 0$

S-Matrix and out-going density matrix:

Scattering: in-states evolve to a superposition of in-states, with coefficients the S-matrix elements

$$|\alpha> \;\rightarrow\; \sum_{\beta,\gamma} S^{\dagger}_{\alpha,\beta\gamma}\,|\beta\gamma>$$

where γ are soft photons.

$$|\alpha><\alpha| \;\rightarrow\; \sum_{\beta\gamma} S^{\dagger}_{\alpha,\beta\gamma}\,|\beta\gamma> \sum_{\tilde{\beta}\tilde{\gamma}} <\tilde{\beta},\tilde{\gamma}|\;S_{\tilde{\beta}\tilde{\gamma},\alpha}$$

The S-matrix is infrared divergent.

Infrared divergences cancel from inclusive transition probabilities, i.e. from the **diagonal** elements of the reduced density matrix

$$\rho = \sum_{\hat{\gamma}} <\hat{\gamma}| \left[\sum_{\beta\gamma} S^{\dagger}_{\alpha,\beta\gamma}\,|\beta\gamma> \sum_{\tilde{\beta}\tilde{\gamma}} <\tilde{\beta},\tilde{\gamma}|\;S_{\tilde{\beta}\tilde{\gamma},\alpha}\right] |\hat{\gamma}>$$

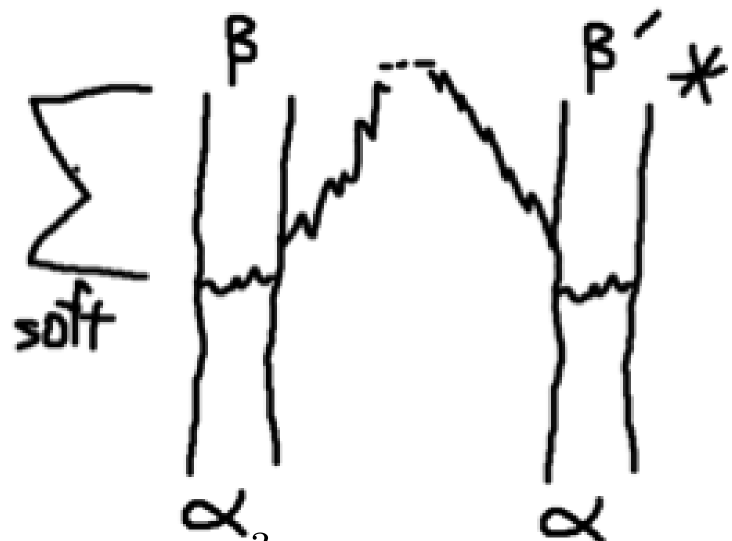
What about off-diagonal matrix elements of ρ?

Entanglement entropy from Moeller Scattering:

$$S = -\mathrm{Tr}\rho \ln \rho = -\sum_i \rho_i \ln \rho_i$$

Density matrix = pure state + trace...

$\rho = \left[\quad S^\dagger|\alpha >< \alpha|S \quad \right]_{\beta\beta'} +$



Eigenvalues 0 and 1 perturbed by $\left(\frac{e^2}{4\pi}\right)^3 \ln \frac{\Lambda}{m_{\rm ph}}$

The density matrix eigenvalues, ρ_i, are logarithmically infrared divergent at order e^6.

Soft photon theorem applied to the density matrix:

The S-matrix is infrared divergent $\rightarrow$ define it with a fundamental IR cutoff $m_{\text{ph.}}$, $S^{m_{\text{ph.}}}_{\alpha,\tilde{\beta}\gamma}$

Final state: $\sum_{\beta\gamma\tilde{\beta}\tilde{\gamma}} S^{m_{\text{ph.}}\dagger}_{\beta\gamma,\alpha} |\beta\gamma >< \tilde{\beta}\tilde{\gamma}| S^{m_{\text{ph.}}}_{\alpha,\tilde{\beta}\tilde{\gamma}}$

Trace soft photons $m_{\text{ph.}} \leq \omega \leq \Lambda_2 =$**"detector resolution"**

Soft photon theorem (valid when $m_{\text{ph.}} << \Lambda_2 << \alpha, \beta, \tilde{\beta}$):

$$\rho^{\text{out}}_{\beta\ \tilde{\beta}} = \sum_{\gamma} \Theta(E_T - \sum E_i) \prod_i \Theta(\Lambda_2 - |k_i|) S^{m_{\text{ph.}}\dagger}_{\beta\gamma,\alpha} S^{m_{\text{ph.}}}_{\alpha,\tilde{\beta}\gamma}$$

$$= S^{m_{\text{ph.}}\dagger}_{\beta,\alpha} S^{m_{\text{ph.}}}_{\alpha,\tilde{\beta}} \left(\frac{\Lambda_2}{m_{\text{ph}}}\right)^{\tilde{A}_{\alpha\beta,\alpha\tilde{\beta}}} F(E_T)\ ,\ \ F(\infty) = 1$$

where

$$A_{X,Y} = -\sum_{n\in X, m'\in Y} \frac{e_n e_{n'} \eta_n \eta'_n}{8\pi\beta_{nn'}} \ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right]$$

$$\beta_{nn'} = \text{relative relativistic velocity}$$

Soft Photon Theorem II:

Change IR cutoff on internal loops from $m_{\mathrm{ph.}}$ to Λ_1:

$$S^{m_{\mathrm{ph.}}}_{\alpha,\tilde{\beta}} = S^{\Lambda_1}_{\alpha,\tilde{\beta}} \left(\frac{m_{\mathrm{ph}}}{\Lambda_1} \right)^{\frac{1}{2} A_{\alpha\tilde{\beta},\alpha\tilde{\beta}}}$$

where

$$A_{X,Y} = - \sum_{n\in X, m'\in Y} \frac{e_n e_{n'} \eta_n \eta'_n}{8\pi \beta_{nn'}} \ln \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right]$$

$$\beta_{nn'} = \text{relative relativistic velocity}$$

Soft photon theorem applied to the density matrix:

$m_{\rm ph}$ **photon mass as fundamental infrared cutoff**

$\Lambda_1 =$ **Feynman diagram cutoff;** $\Lambda_2 =$ **detector resolution**

E_T**=total energy of soft photons**

$\alpha\beta\tilde{\beta} >> \Lambda_1, \Lambda_2, E_T >> m_{\rm ph}$

Summary – soft photon theorem implies:

$$\rho_{\beta\tilde{\beta}} = S^{\dagger\Lambda_1}_{\beta,\alpha} S^{\Lambda_1}_{\tilde{\beta},\alpha} \left(\frac{m_{\rm ph}}{\Lambda_1}\right)^{\frac{1}{2}A_{\alpha\beta,\alpha\beta}} \left(\frac{m_{\rm ph}}{\Lambda_1}\right)^{\frac{1}{2}A_{\alpha\tilde{\beta},\alpha\tilde{\beta}}} \left(\frac{\Lambda_2}{m_{\rm ph}}\right)^{\tilde{A}_{\alpha\beta,\alpha\tilde{\beta}}} F(E_T)$$

$$\sim m_{\rm ph}^{\Delta A}\ ,\ \Delta A = \tfrac{1}{2}A_{\alpha\beta,\alpha\beta} + \tfrac{1}{2}A_{\alpha\tilde{\beta},\alpha\tilde{\beta}} - A_{\alpha\beta,\alpha\tilde{\beta}} \geq 0$$

$$A_{X,Y} = -\sum_{n\in X, m'\in Y} \frac{e_n e_{n'} \eta_n \eta'_n}{8\pi\beta_{nn'}} \ln\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right]$$

$\beta_{nn'}$ =relative relativistic velocity

- A generic density matrix element is proportional $\sim m_{\mathrm{ph}}^{\Delta A}$, where $\Delta A \geq 0$ and depends on incoming and outgoing four-momenta.
- $\Delta A = 0$ for diagonal elements of the density matrix (transition probabilities)
- Generically, $\Delta A > 0$ for off-diagonal elements
- The inequality is saturated, $\Delta A = 0$, and density matrix element nonzero only when the set of outgoing currents match:

$$\beta = \left\{ \frac{e_1 p_1^\mu}{2\omega(p_1)}, \ldots, \frac{e_n p_n^\mu}{2\omega(p_n)} \right\}$$

equals

$$\tilde{\beta} = \left\{ \frac{\tilde{e}_1 \tilde{p}_1^\mu}{2\omega(\tilde{p}_1)}, \ldots, \frac{\tilde{e}_{\tilde{n}} \tilde{p}_{\tilde{n}}^\mu}{2\omega(\tilde{p}_{\tilde{n}})} \right\}$$

- **decoherence** momentum eigenstates are pointer basis

Example: Compton scattering

$\rho_{k',q';\tilde{k}',\tilde{q}'} = m_{\rm ph}^{\frac{e^2}{4\pi^2}\left[\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}-1\right]}$, β =relative electron velocity

Exponent ≥ 0. Exponent $= 0$ only when $\beta = 0$.

As $m_{\rm ph} \to 0$, $\rho_{k',q';\tilde{k}',\tilde{q}'} = 0$ **unless** $k'_\mu = \tilde{k}'_\mu$.

Implication: *Diagonal elements* of the density matrix are the transition probabilities for QED processes.

$\rho_{k',q';k',q'}$ **= Probability of** $|k, q >\to |k'q' >$

Off-diagonal elements vanish $\rho_{k',q';\tilde{k}',\tilde{q}'} = 0, \; k \neq \tilde{k}'$

Probability $|k, q >\to \frac{1}{\sqrt{2}}|k'_1, q'_1 > + \frac{1}{\sqrt{2}}|k'_2, q'_2 >$

equals

$\frac{1}{2}\cdot$**Probability** $|k, q >\to |k'_1, q'_1 >$

$+$

$\frac{1}{2}\cdot$**Probability** $|k, q >\to |k'_2, q'_2 >$

Limitations: What if the photon has a mass?

$$\rho_{k',q';\tilde{k}',\tilde{q}'} = (m_{\rm ph})^{\frac{e^2}{4\pi^2}\left[\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}-1\right]}$$

"experimental" bound: $m_{\rm ph} \sim 10^{-32} m_{\rm el}$

$$\sim e^{-0.1\beta^2} \; \beta << 1 \; , \;\; \sim \left(\frac{1-\beta}{2}\right)^{0.1} \; \beta \sim 1$$

Extreme off-diagonal elements are emphasized

Finite time, etc: $m_{\rm ph.} \to \frac{\hbar}{c^2 \cdot {\rm time}}$

Infrared safe "dressed states"

For each charged particle, add a coherent state of soft photons:

$$|p> \rightarrow |p>_D \equiv W(p)|p>$$

$$W(p) = \exp\left\{\sum_{\ell}\int_0^{\Lambda}\frac{d^3k}{2\sqrt{\vec{k}^2+m_{\rm ph}^2}}\left[\frac{p\cdot\epsilon_\ell(k)}{p\cdot k}a_\ell^\dagger(k)-\frac{p\cdot\epsilon_\ell^*(k)}{p\cdot k}a_\ell(k)\right]\right\}$$

$$m_{\rm ph} << \Lambda << p \quad k\cdot\epsilon_\ell(k) = 0$$

$\tilde{S}_{\alpha\beta} \equiv_D < \alpha|S|\beta>_D$ is infrared finite. Out-state can be a pure state

$$|\alpha>_D < \alpha| \;\rightarrow\; \tilde{\rho} = \sum_{\beta}\tilde{S}^\dagger_{\alpha,\beta}\;|\beta>_D\sum_{\tilde{\beta}}\;{}_D<\tilde{\beta}|\;\tilde{S}_{\tilde{\beta},\alpha}$$

$$\mathrm{Tr}_{\rm soft\ photons}\tilde{\rho} = \left(\frac{m_{\rm ph}}{\Lambda}\right)^{\Delta A}$$

Conclusions:

- The solution of the infrared problem in quantum electrodynamics (and in perturbative quantum gravity) leads to a fundamental decoherence of final states.
- There are other "infrared safe" approaches. **V.Chung, Phys.Rev.140, B1110 (1965); T.W.B.Kibble, J.Math.Phys.9, 315 (1968); P.P.Kulish, L.D.Faddeev, Theor.Math.Phys.4, 745 (1970); J.Ware, R.Saotome, R.Akhoury, JHEP10, 159 (2013), 1308.6285.** Same decoherence when in-coming state is "infrared safe" coherent state.
- Proper description of incoming wavepackets requires infrared safe incoming states. Decoherence remains.
- Could such a decoherence be observable?

Black hole information paradox

In a theory of quantum gravity, the collision of two high-energy particles (i.e. gravitons) could produce a black hole which would the evaporate by emitting Hawking radiation.
Pure quantum state of two incoming particles evolves to thermal state of Hawking radiation.

$$|\psi >= \sum_E |E, \tilde{E} > \quad , \quad \rho = \sum_E e^{-\beta_H E} \, |E >< E|$$

Strominger's idea: (**A.Strominger, arXiv:1706.07143**): soft gravitons purify the Hawking radiation

$$|\psi >= \sum_E |E, \mathrm{soft} > \ , \ \rho = \mathrm{Tr}_{\mathrm{soft}} |\psi >< \psi| = \sum_E e^{-\beta_H E} |E >< E|$$

But $|\psi >= \sum_E |E, \mathrm{soft}, \tilde{E} >$. Monogamy of entanglement.