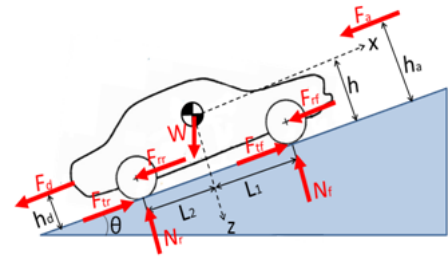


LONGITUDINAL beam DYNAMICS RECAP



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CERN, BE-OP



Beam Instrumentation
Tuusula, Finland, 2-15/6/2018

CAS Beam Instrumentation, Tuusula, June 2018

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Summary of the lecture:

- Introduction
- Linac: Phase stability
- Synchrotron:
 - Synchronous Phase
 - Dispersion Effects in Synchrotron
 - Stability and Longitudinal Phase Space Motion
 - Equations of motion
- Injection Matching
- Hamiltonian of Longitudinal Motion
- Appendices: some derivations and details

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Particle types and acceleration

The accelerating system will depend upon the **evolution** of the **particle velocity**:

- **electrons** reach a **constant velocity** (~speed of light) at relatively low energy
- **heavy particles** reach a constant velocity only at very high energy
 - we need different types of resonators, optimized for different velocities
 - the **revolution frequency** will **vary**, so the **RF frequency** will be **changing**

Particle rest mass:

electron 0.511 MeV

proton 938 MeV

²³⁹U ~220000 MeV

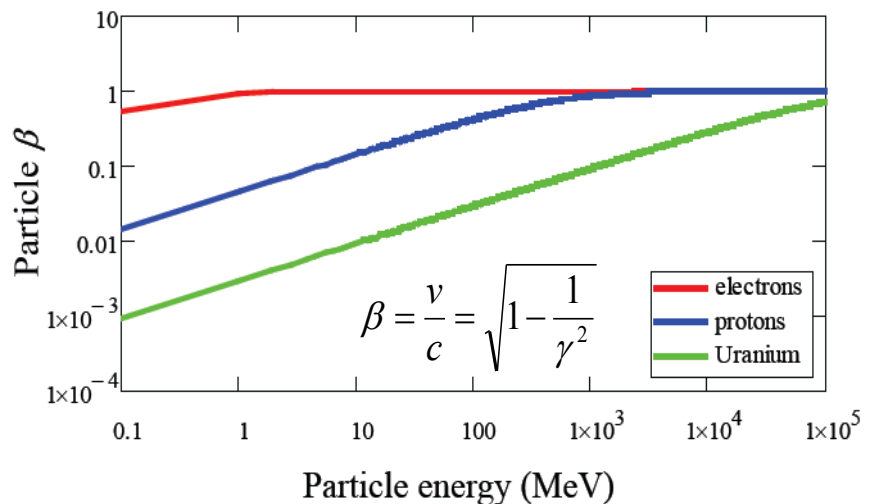
Total Energy: $E = \gamma m_0 c^2$

Relativistic gamma factor:

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c$$



Acceleration + Energy Gain

May the force
be with you!



To accelerate, we need a **force in the direction of motion!**

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{d\vec{p}}{dt} = e \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

2nd term always perpendicular to motion => **no acceleration**

Hence, it is necessary to have an **electric field E** (preferably) **along the direction of the initial momentum (z)**, which changes the momentum p of the particle.

$$\frac{dp}{dt} = eE_z$$

In relativistic dynamics, total **energy E** and **momentum p** are **linked by**

$$E^2 = E_0^2 + p^2 c^2 \Rightarrow dE = v dp \quad (2E dE = 2c^2 p dp \Leftrightarrow dE = c^2 mv / E dp = v dp)$$

The rate of **energy gain per unit length** of acceleration (along z) is then:

$$\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic **energy gained** from the field along the z path is:

$$dW = dE = q E_z dz \rightarrow W = q \int E_z dz = qV \quad \begin{array}{l} - V \text{ is a potential} \\ - q \text{ the charge} \end{array}$$

Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{d\vec{p}}{dt} = e \left(\vec{E} + \vec{v} \times \vec{B} \right)$

2nd term always perpendicular to motion \Rightarrow no acceleration

Relativistic Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \quad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c$$

$$E^2 = E_0^2 + p^2 c^2 \quad \longrightarrow \quad dE = v dp$$

$$\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = e E_z$$

$$dE = dW = e E_z dz \quad \rightarrow \quad W = e \int E_z dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin \omega_{RF} t = \hat{E}_z \sin \phi(t)$$

$$\int \hat{E}_z dz = \hat{V}$$

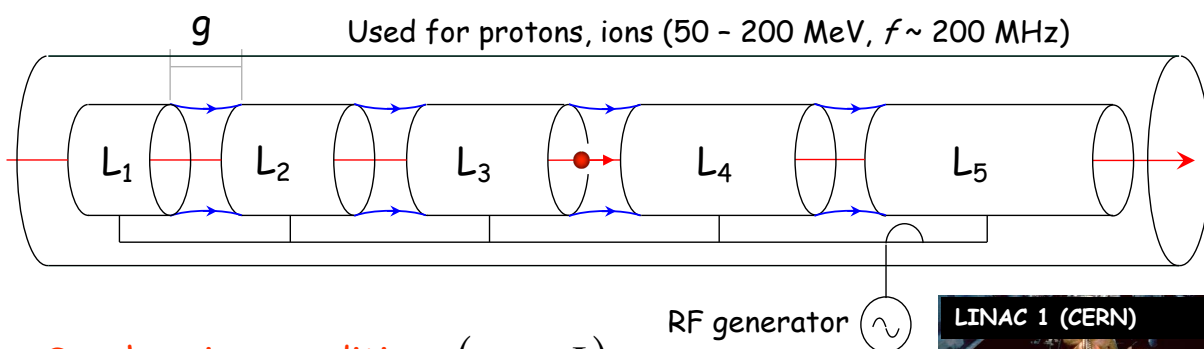
$$W = e \hat{V} \sin \phi$$

(neglecting transit time factor)

The field will change during the passage of the particle through the cavity \Rightarrow effective energy gain is lower

Radio Frequency (RF) acceleration: Alvarez Structure

Electrostatic acceleration limited by insulation possibilities \Rightarrow use RF fields



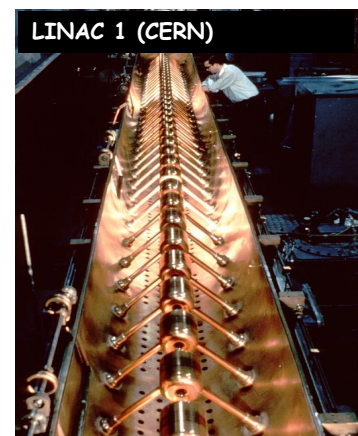
Synchronism condition ($g \ll L$)



$$L = v_s T_{RF} = \beta_s \lambda_{RF}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

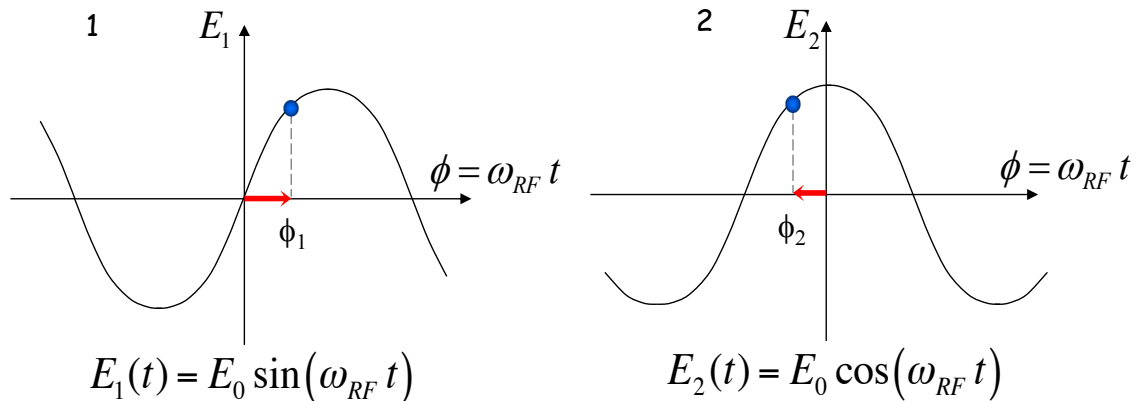
Note: - Drift tubes become longer for higher velocity
- Acceleration only for bunched beam (not continuous)



Common Phase Conventions

1. For **circular accelerators**, the origin of time is taken at the **zero crossing** of the RF voltage with positive slope
2. For **linear accelerators**, the origin of time is taken at the positive **crest** of the RF voltage

Time $t = 0$ chosen such that:

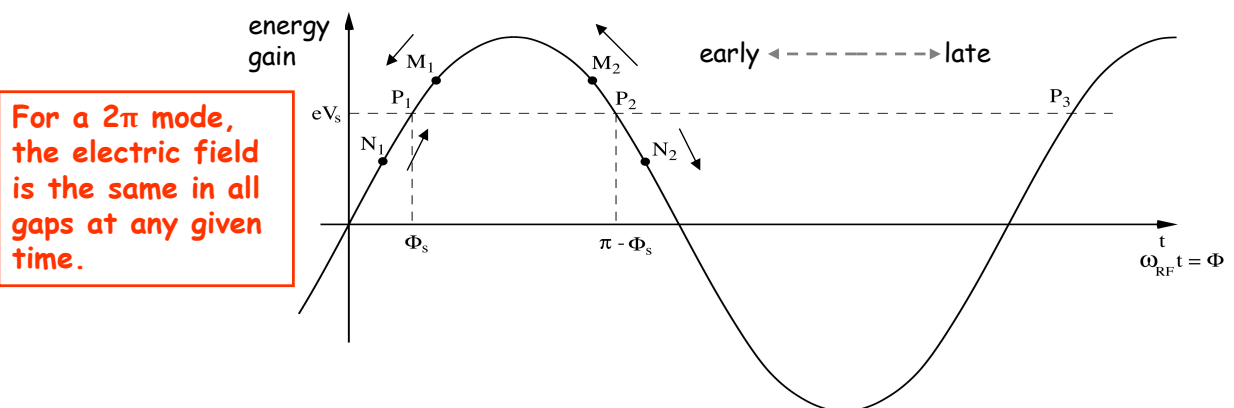


3. I will stick to **convention 1** in the following to avoid confusion

Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$eV_s = e\hat{V} \sin \Phi_s$ is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1, P_2, \dots are fixed points.



If an **energy increase** is transferred into a **velocity increase** \Rightarrow

M_1 & N_1 will move towards P_1 \Rightarrow **stable**

M_2 & N_2 will go away from P_2 \Rightarrow **unstable**

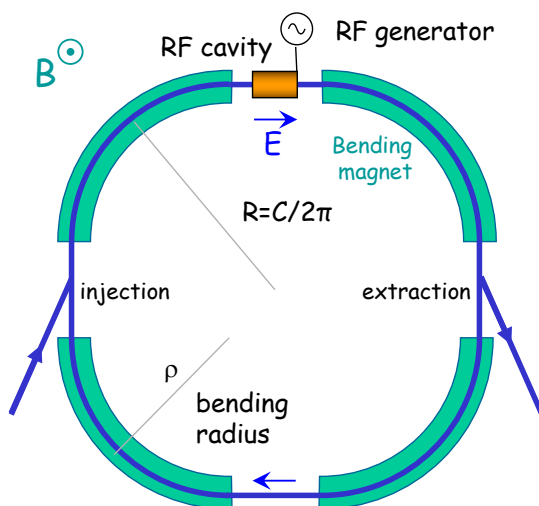
(Highly relativistic particles have no significant velocity change)

Circular accelerators

Cyclotron (not covered here)

Synchrotron

Circular accelerators: The Synchrotron



1. **Constant orbit** during acceleration
2. To keep particles on the closed orbit, **B should increase** with time
3. ω and ω_{RF} **increase** with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition

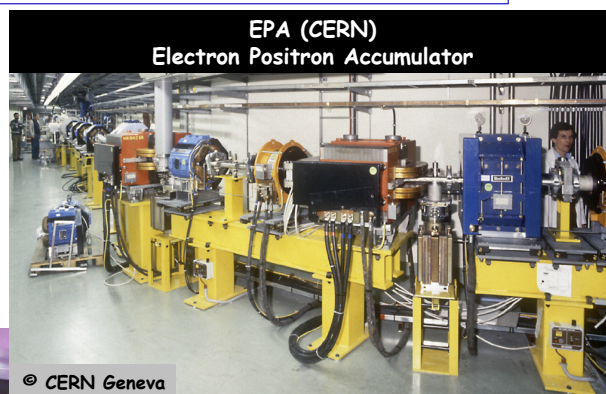
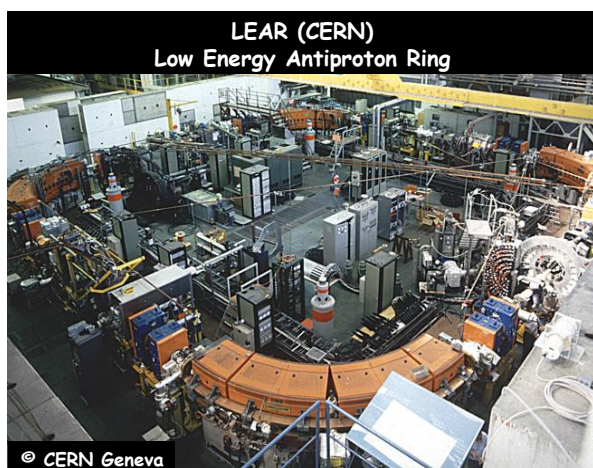


$$T_s = h T_{RF}$$

$$\frac{2\pi R}{v_s} = h T_{RF}$$

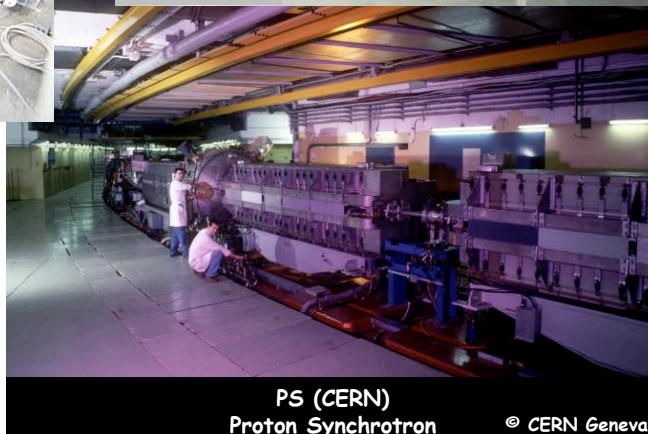
h integer,
harmonic number:
number of RF cycles
per revolution

Circular accelerators: The Synchrotron



Examples of different
proton and electron
synchrotrons at CERN

+ LHC (of course!)

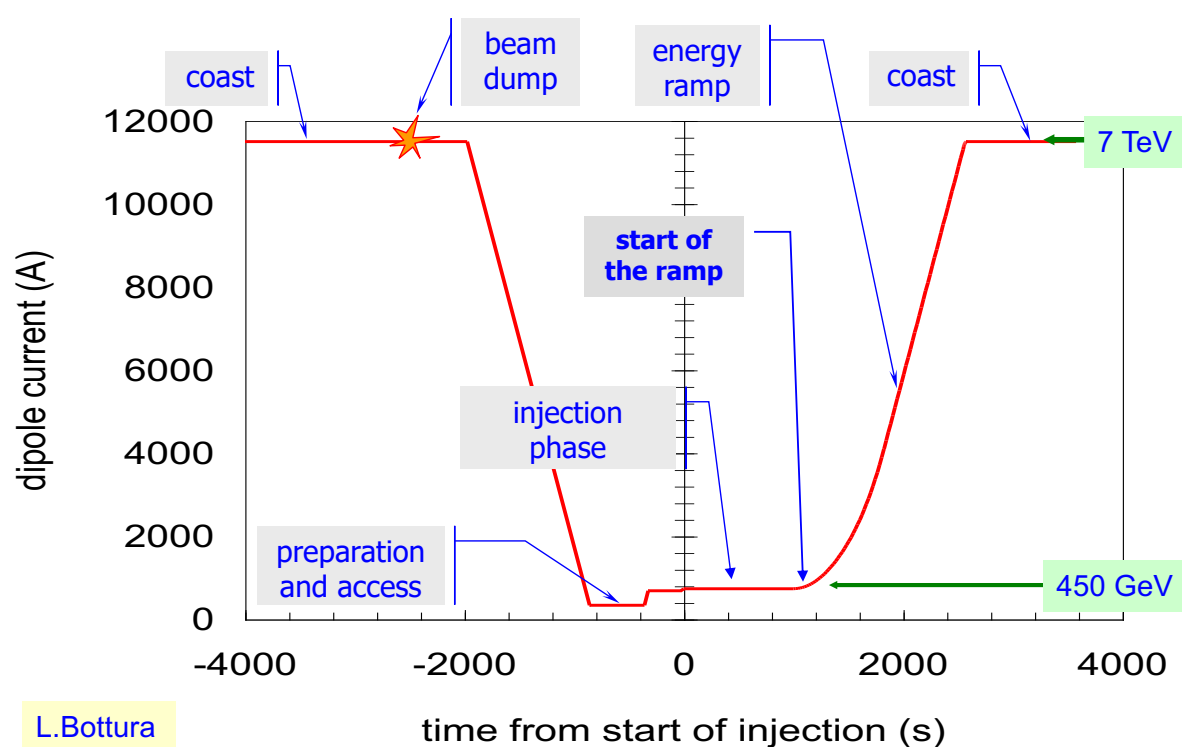


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The Synchrotron - LHC Operation Cycle

The magnetic **field** (dipole current) is **increased during** the **acceleration**.



L.Bottura

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The Synchrotron - Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \Rightarrow \frac{dp}{dt} = e\rho \dot{B} \Rightarrow (\Delta p)_{turn} = e\rho \dot{B} T_r = \frac{2\pi e\rho R \dot{B}}{v}$$

Since: $E^2 = E_0^2 + p^2 c^2 \Rightarrow \Delta E = v \Delta p$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R \dot{B} = e\hat{V} \sin \phi_s$$

Stable phase ϕ_s changes during energy ramping!

$$\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \rightarrow \phi_s = \arcsin \left(2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \right)$$

- The number of **stable synchronous particles** is equal to the **harmonic number h**. They are equally spaced along the circumference.
- Each synchronous particle satisfies the relation $p=eB\rho$. They have the nominal energy and follow the nominal trajectory.

The Synchrotron - Frequency change

During the energy ramping, **the RF frequency increases** to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

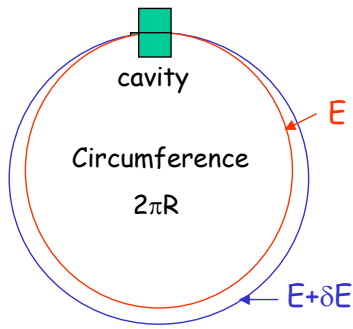
Hence: $\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{ec^2}{E_s(t)} \frac{\rho}{R_s} B(t)$ (using $p(t) = eB(t)\rho$, $E = mc^2$)

Since $E^2 = (m_0 c^2)^2 + p^2 c^2$ the **RF frequency** must **follow** the variation of the **B field** with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0 c^2 / e c \rho)^2 + B(t)^2} \right\}^{1/2}$$

This asymptotically tends towards $f_r \rightarrow \frac{c}{2\pi R_s}$ when B becomes large compared to $m_0 c^2 / (e c \rho)$
which corresponds to $v \rightarrow c$

Dispersion Effects in a Synchrotron



p =particle momentum

R =synchrotron physical radius

f_r =revolution frequency

A particle slightly shifted in momentum will have a

- dispersion orbit and a **different orbit length**
- a **different velocity**.

As a result of both effects the revolution frequency changes with a "**slip factor η** ":

$$\eta = \frac{d f_r / f_r}{d p / p} \Rightarrow$$

Note: you also find η defined with a minus sign!

Effect from orbit defined by **Momentum compaction factor**: $\alpha_c = \frac{dL/L}{dp/p}$

Property of the beam optics:
(derivation in Appendix)

$$\alpha_c = \frac{1}{L} \int_c \frac{D_x(s)}{\rho(s)} ds_0$$

Dispersion Effects - Revolution Frequency

The **two effects** of the **orbit length** and the particle **velocity** change the revolution frequency as:

$$f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} \underset{\substack{\text{definition of momentum} \\ \text{compaction factor}}}{=} \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c \right) \frac{dp}{p}$$

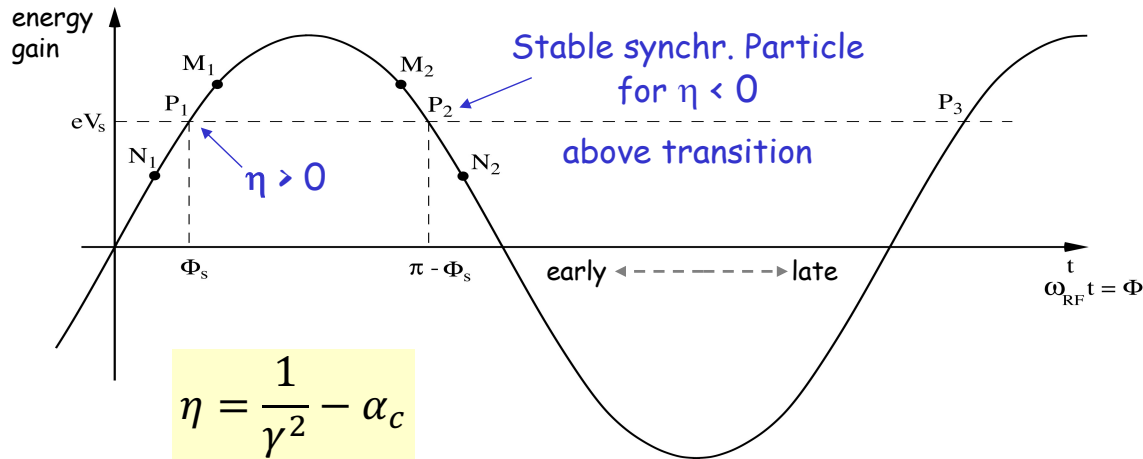
$$p = mv = \beta \gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d(1-\beta^2)^{-1/2}}{(1-\beta^2)^{-1/2}} = \underbrace{(1-\beta^2)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$$

Slip factor: $\eta = \frac{1}{\gamma^2} - \alpha_c$ or $\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}$ with $\gamma_t = \frac{1}{\sqrt{\alpha_c}}$

At **transition energy**, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

- From the definition of η it is clear that an **increase in momentum** gives
- **below transition** ($\eta > 0$) a **higher revolution frequency** (increase in velocity dominates) while
 - **above transition** ($\eta < 0$) a **lower revolution frequency** ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.



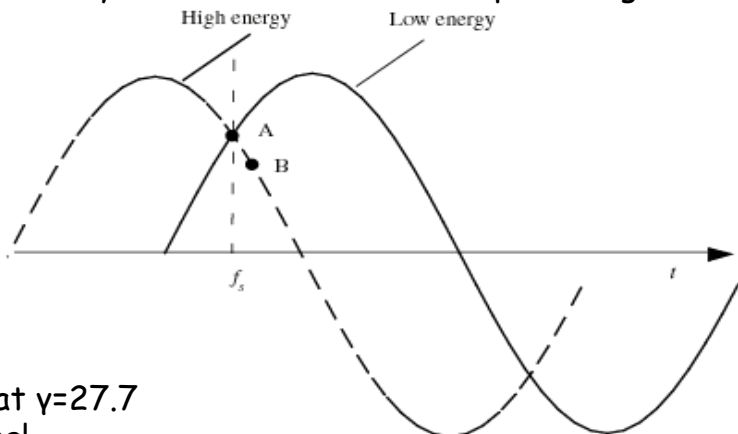
Crossing Transition

At **transition**, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a '**phase jump**'.

$$\alpha_c \sim \frac{1}{Q_x^2}$$

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$



In the PS: γ_t is at ~ 6 GeV

In the SPS: $\gamma_t = 22.8$, injection at $\gamma = 27.7$

=> no transition crossing!

In the LHC: γ_t is at ~ 55 GeV, also far below injection energy

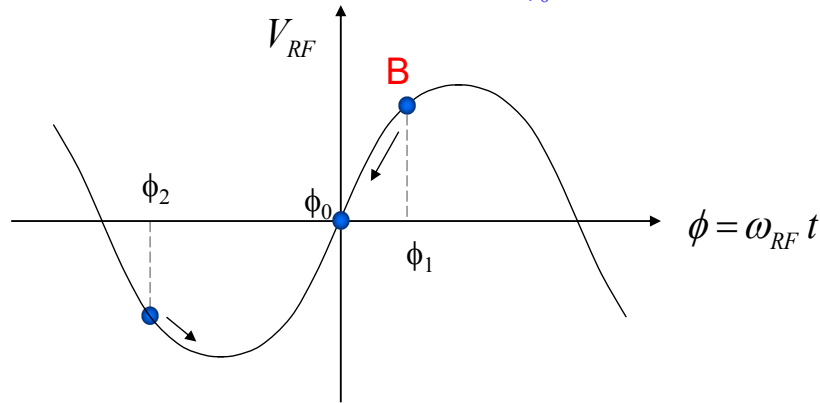
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): $B = \text{const.}$, below transition $\gamma < \gamma_t$

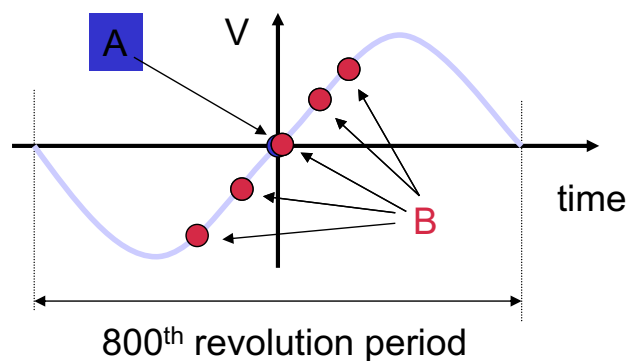
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

- Φ_1
- The particle **B** is accelerated
 - Below transition, an energy increase means an increase in revolution frequency
 - The particle arrives earlier - tends toward ϕ_0



- ϕ_2
- The particle is decelerated
 - decrease in energy - decrease in revolution frequency
 - The particle arrives later - tends toward ϕ_0

Synchrotron oscillations

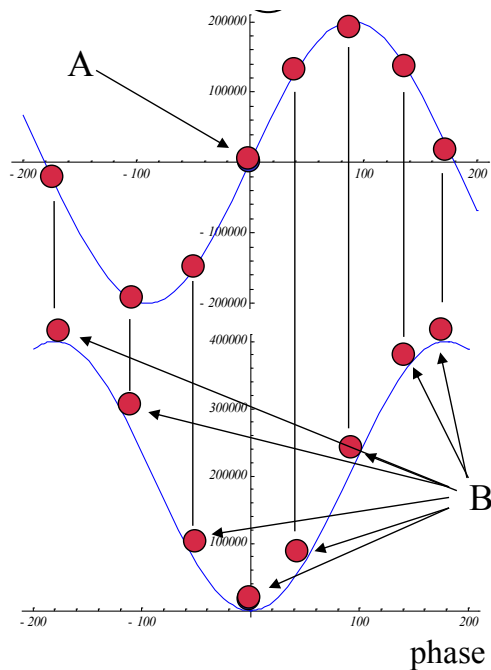


Particle **B** is performing **Synchrotron Oscillations** around synchronous particle **A**.

The amplitude depends on the initial phase and energy.

The **oscillation frequency** is much **slower than** in the **transverse** plane. It takes a large number of revolutions for one complete oscillation. Restoring electric force smaller than magnetic force.

The Potential Well

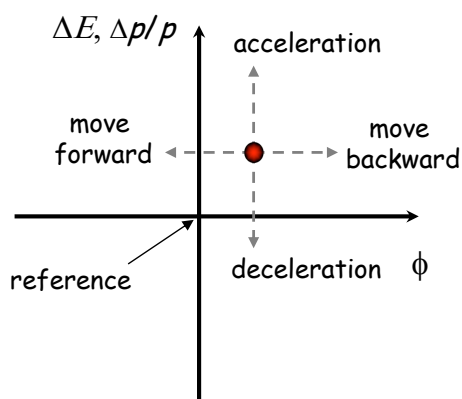


Cavity voltage

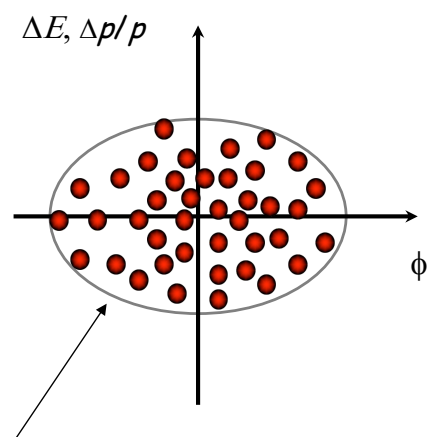
Potential well

Longitudinal phase space

The **energy - phase oscillations** can be drawn in **phase space**:



The particle trajectory in the phase space ($\Delta p/p, \phi$) describes its longitudinal motion.

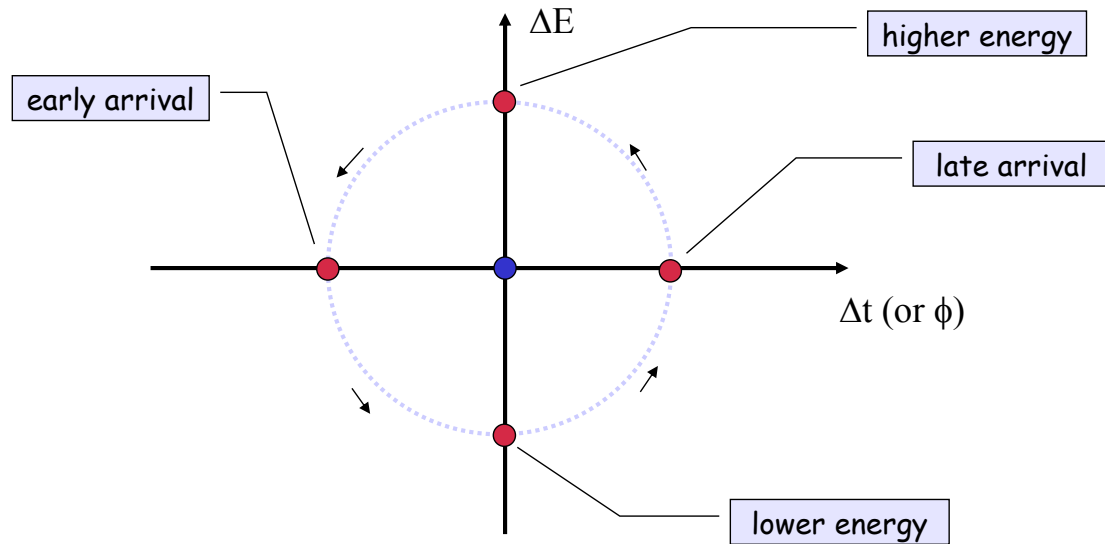


Emittance: phase space area including all the particles

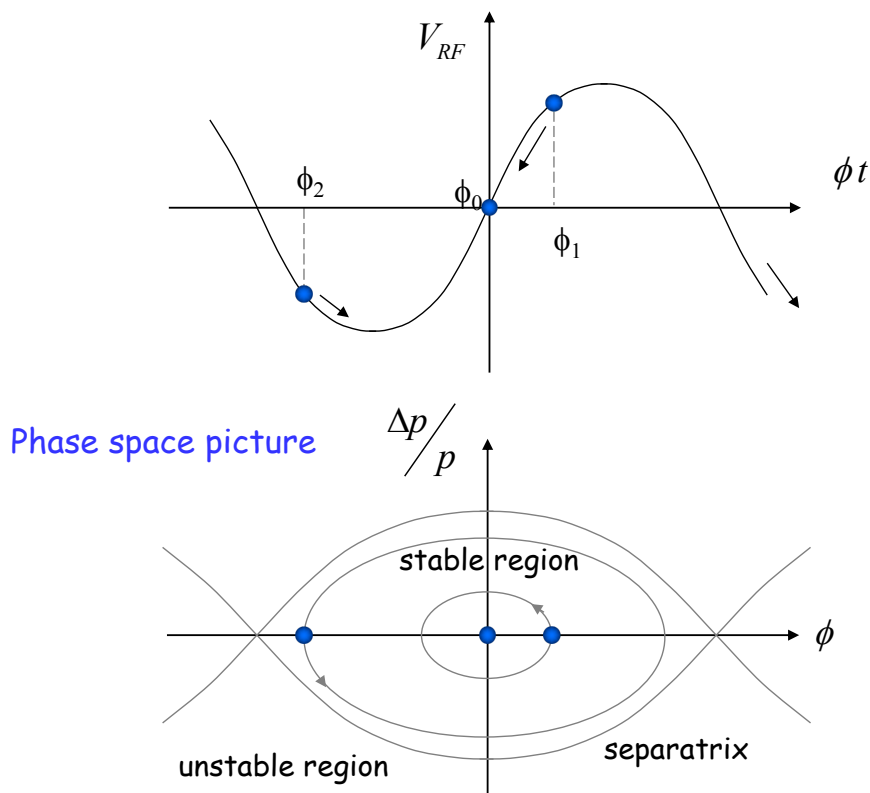
NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Longitudinal Phase Space Motion

Particle **B** performs a synchrotron oscillation around synchronous particle **A**.
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations - No acceleration

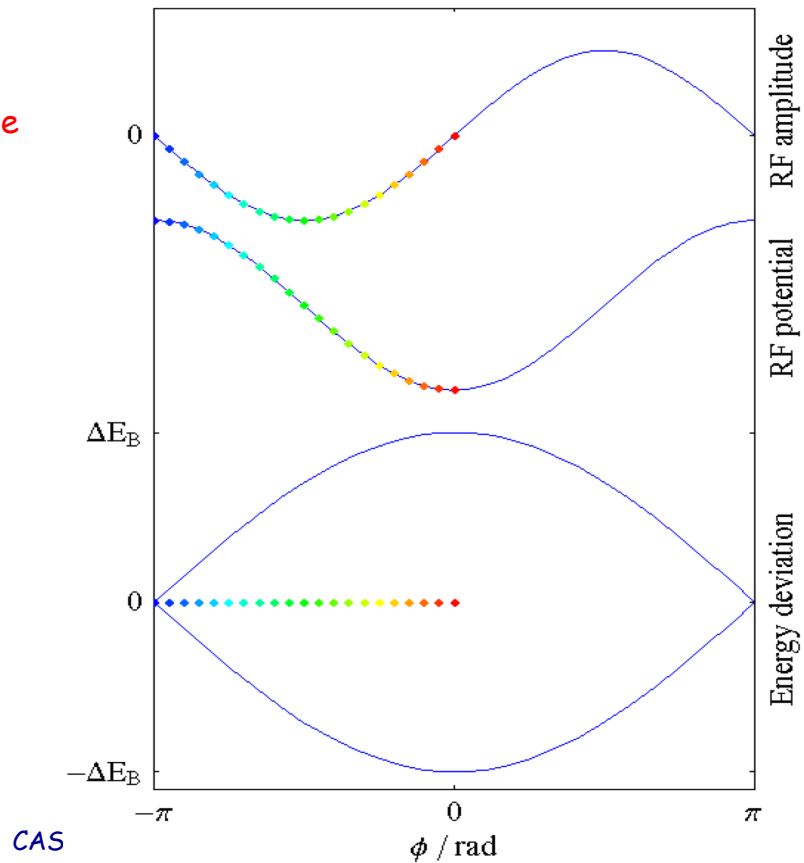


Synchrotron motion in phase space

The restoring force is **non-linear**.

⇒ speed of motion depends on position in phase-space

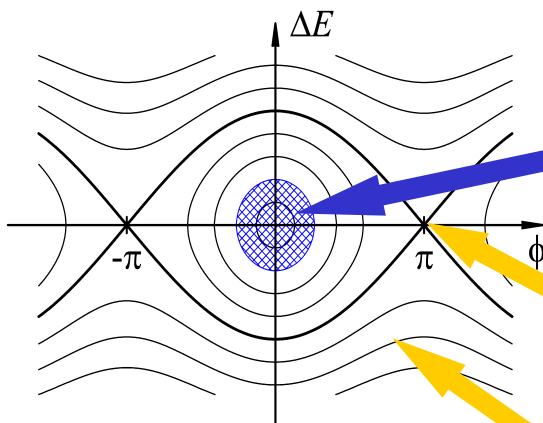
(here shown for a stationary bucket)



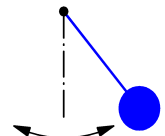
Synchrotron motion in phase space

ΔE - ϕ phase space of a **stationary bucket**
(when there is **no acceleration**)

Dynamics of a particle
Non-linear, conservative oscillator → e.g. pendulum



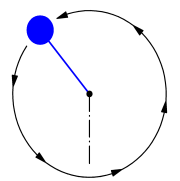
Particle inside the separatrix:



Particle at the unstable fix-point



Particle outside the separatrix:

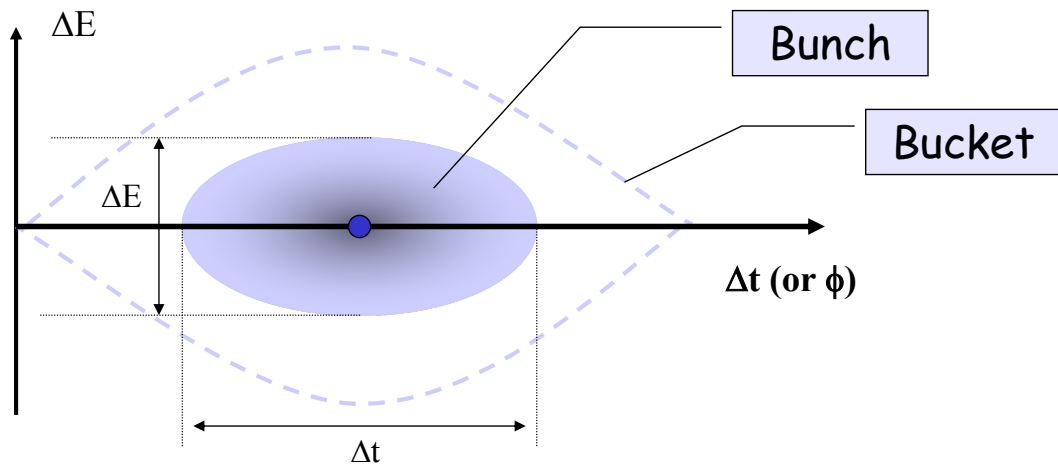


Bucket area: area enclosed by the separatrix

The area covered by particles is the longitudinal emittance.

(Stationary) Bunch & Bucket

The **bunches** of the beam **fill** usually **a part of** the **bucket** area.



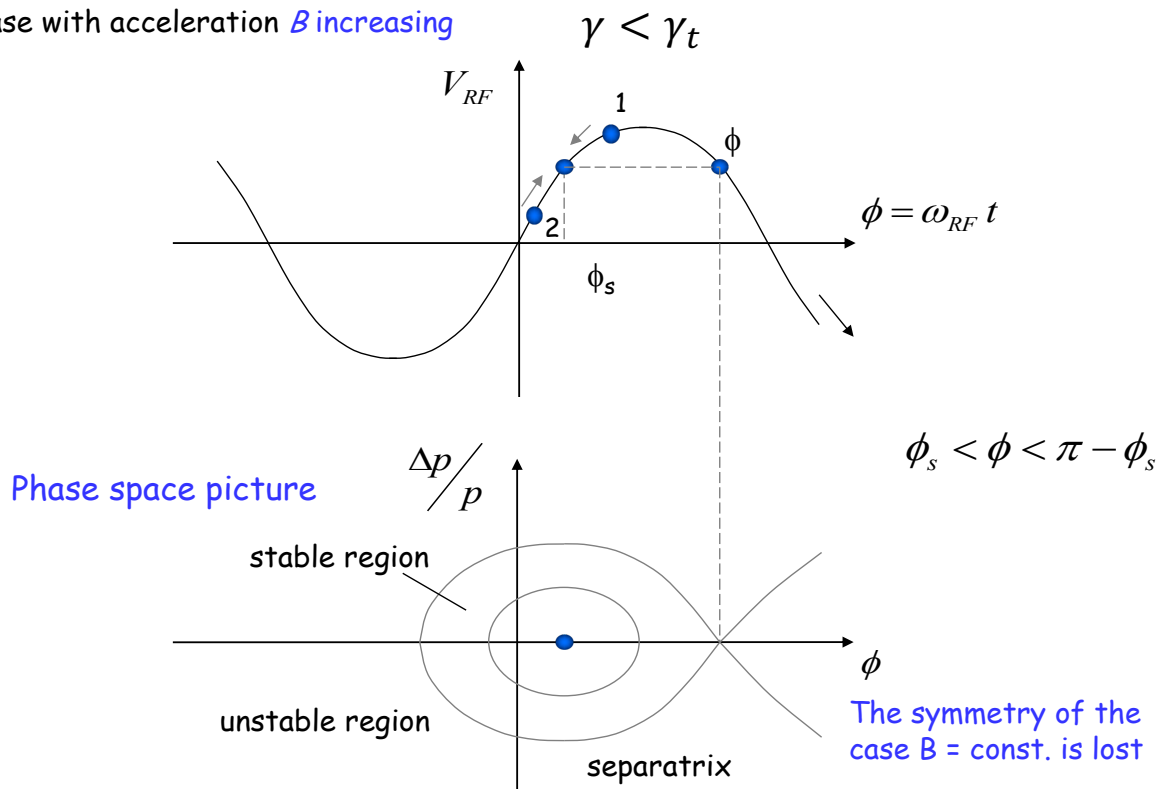
Bucket area = **longitudinal Acceptance** [eVs]

Bunch area = **longitudinal beam emittance** = $4\pi \sigma_E \sigma_t$ [eVs]

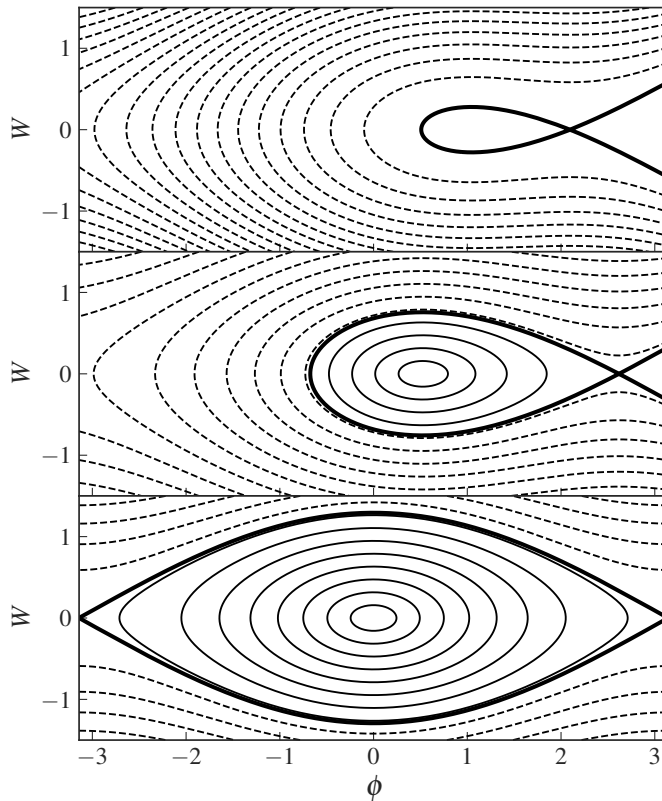
Attention: Different definitions are used!

Synchrotron oscillations (with acceleration)

Case with acceleration B increasing



RF Acceptance versus Synchronous Phase



The **areas of stable motion** (closed trajectories) are called "**BUCKET**". The number of circulating buckets is equal to "**h**".

The phase extension of the **bucket is maximum** for $\phi_s = 180^\circ$ (or 0°) which means **no acceleration**.

During **acceleration**, the buckets get **smaller**, both in length and **energy acceptance** (see appendix).

\Rightarrow **Injection** preferably **without acceleration**.

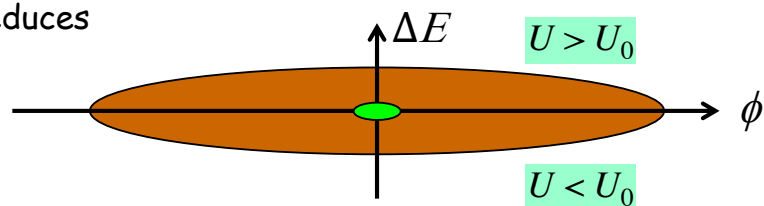
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss **energy dependant**:

$$U_0 = \frac{4}{3} \pi \frac{r_e}{(m_0 c^2)^3} \frac{E^4}{\rho}$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

\Rightarrow The **synchrotron motion** is **damped** toward an **equilibrium bunch length** and **energy spread**.

$$\sigma_\tau = \frac{\alpha}{\Omega_s} \left(\frac{\sigma_\epsilon}{E} \right)$$

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the **energy** gained by the particle and the **RF phase** experienced by the same particle.

Since there is a **well defined synchronous particle** which has always the same **phase** ϕ_s , and the nominal **energy** E_s , it is sufficient to follow other particles with respect to that particle.

So let's introduce the following reduced variables:

revolution frequency :	$\Delta f_r = f_r - f_{rs}$
particle RF phase :	$\Delta\phi = \phi - \phi_s$
particle momentum :	$\Delta p = p - p_s$
particle energy :	$\Delta E = E - E_s$
azimuth angle :	$\Delta\theta = \theta - \theta_s$

Equations of Longitudinal Motion

In these reduced variables, the **equations of motion** are (see Appendix):

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h \eta \omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h \eta \omega_{rs}} \dot{\phi} \quad 2\pi \frac{d}{dt} \left(\frac{\Delta E}{\omega_{rs}} \right) = e \hat{V} (\sin\phi - \sin\phi_s)$$

↓

deriving and combining

↓

$$\frac{d}{dt} \left[\frac{R_s p_s}{h \eta \omega_{rs}} \frac{d\phi}{dt} \right] + \frac{e \hat{V}}{2\pi} (\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will simplify in the following...

Small Amplitude Oscillations

Let's assume constant parameters R_s , p_s , ω_s and η :

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s} (\sin\phi - \sin\phi_s) = 0 \quad \text{with} \quad \Omega_s^2 = \frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now **small** phase **deviations** from the reference particle:

$$\sin\phi - \sin\phi_s = \sin(\phi_s + \Delta\phi) - \sin\phi_s \cong \cos\phi_s \Delta\phi \quad (\text{for small } \Delta\phi)$$

and the corresponding linearized motion reduces to a **harmonic oscillation**:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0 \quad \text{where } \Omega_s \text{ is the } \textbf{synchrotron angular frequency}.$$

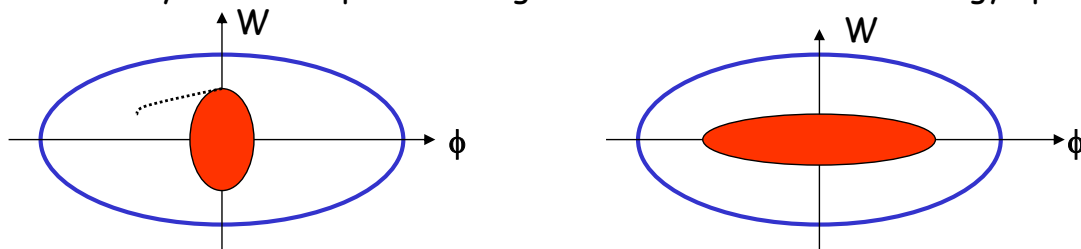
The **synchrotron tune** ν_s is the number of synchrotron oscillations per revolution:

$$\nu_s = \Omega_s / \omega_r$$

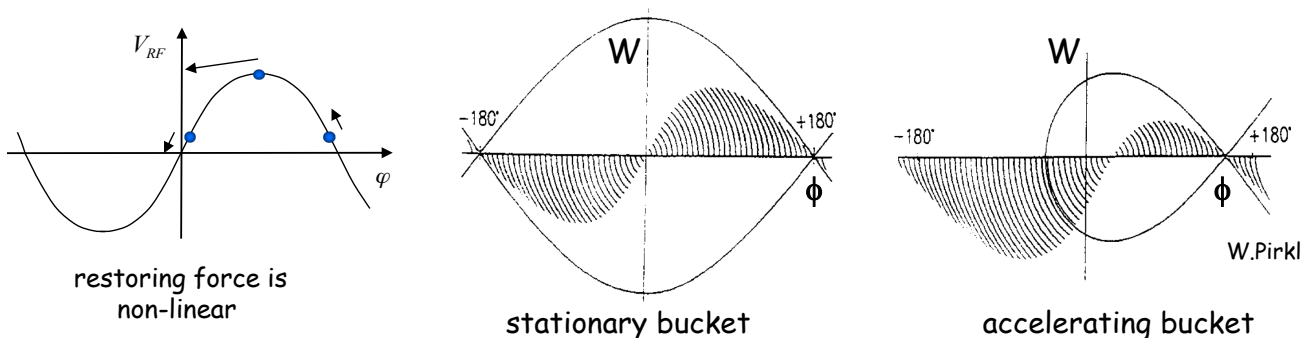
See Appendix for large amplitude treatment and further details.

Injection: Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



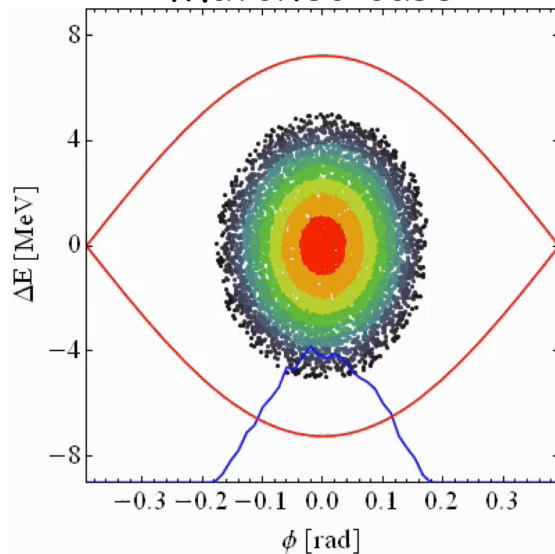
For **larger amplitudes**, the angular phase space motion is slower
(1/8 period shown below) \Rightarrow can lead to **filamentation** and **emittance growth**



Effect of a Mismatch (2)

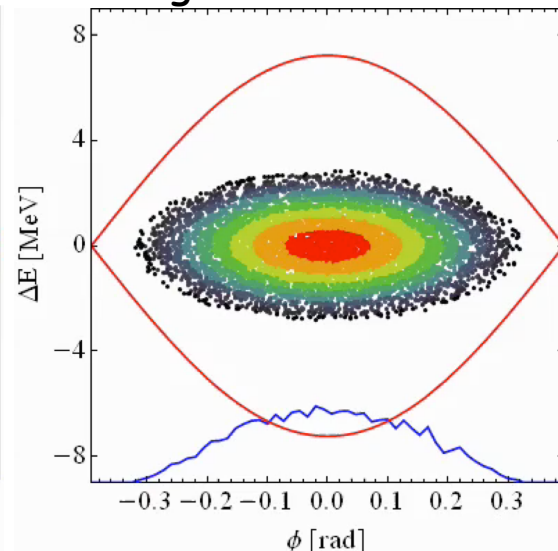
- Long. emittance is only preserved for **correct RF voltage**

Matched case



→ Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

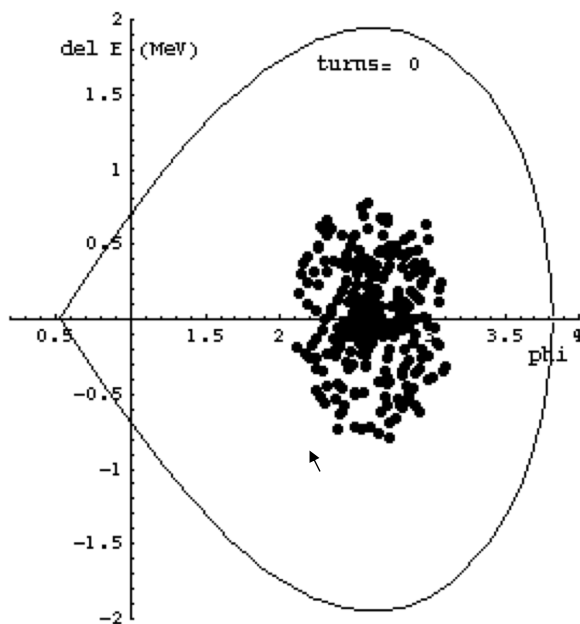


→ Dilution of bunch results in increase of long. emittance

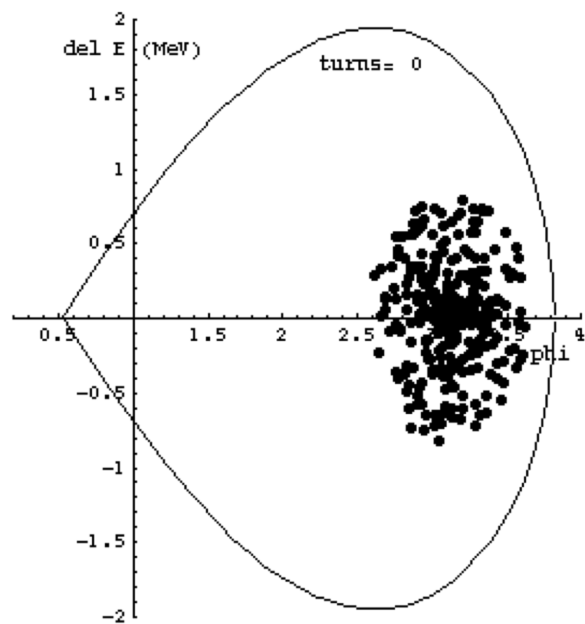
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



matched beam

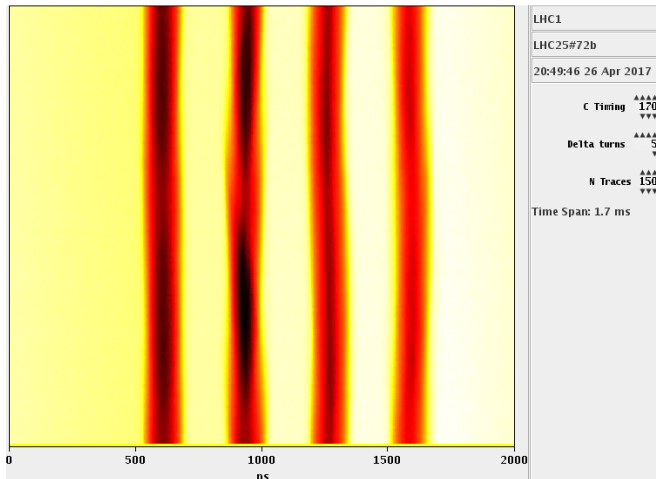


mismatched beam - **phase error**

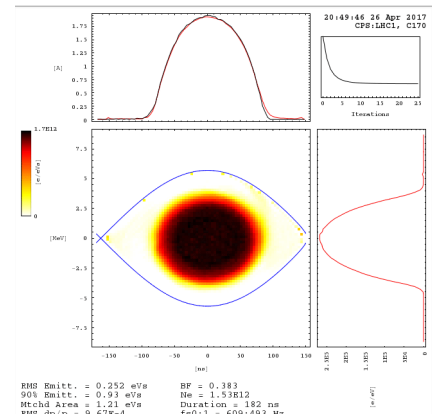
Phase Space Tomography

We can reconstruct the phase space distribution of the beam.

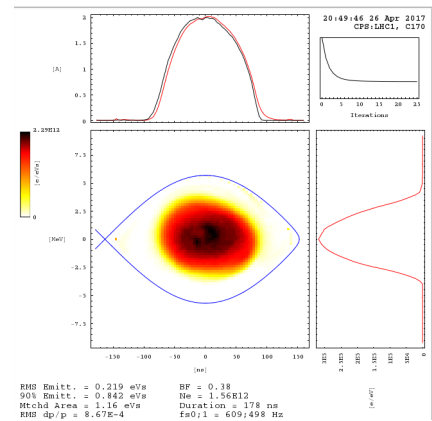
- Longitudinal bunch profiles over a number of turns
- Parameters determining Ω_s



1st
bunch



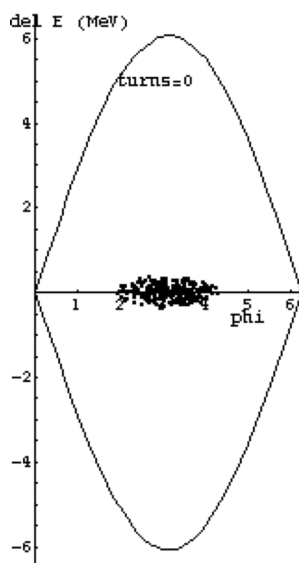
2nd
bunch



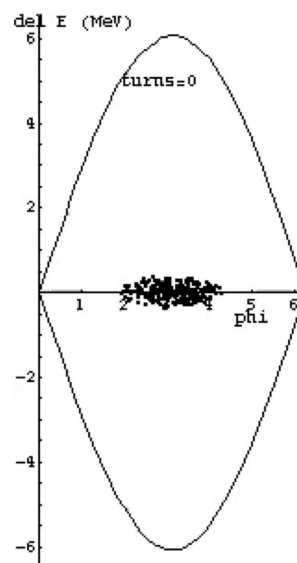
Bunch Rotation

Phase space motion can be used to make short bunches.

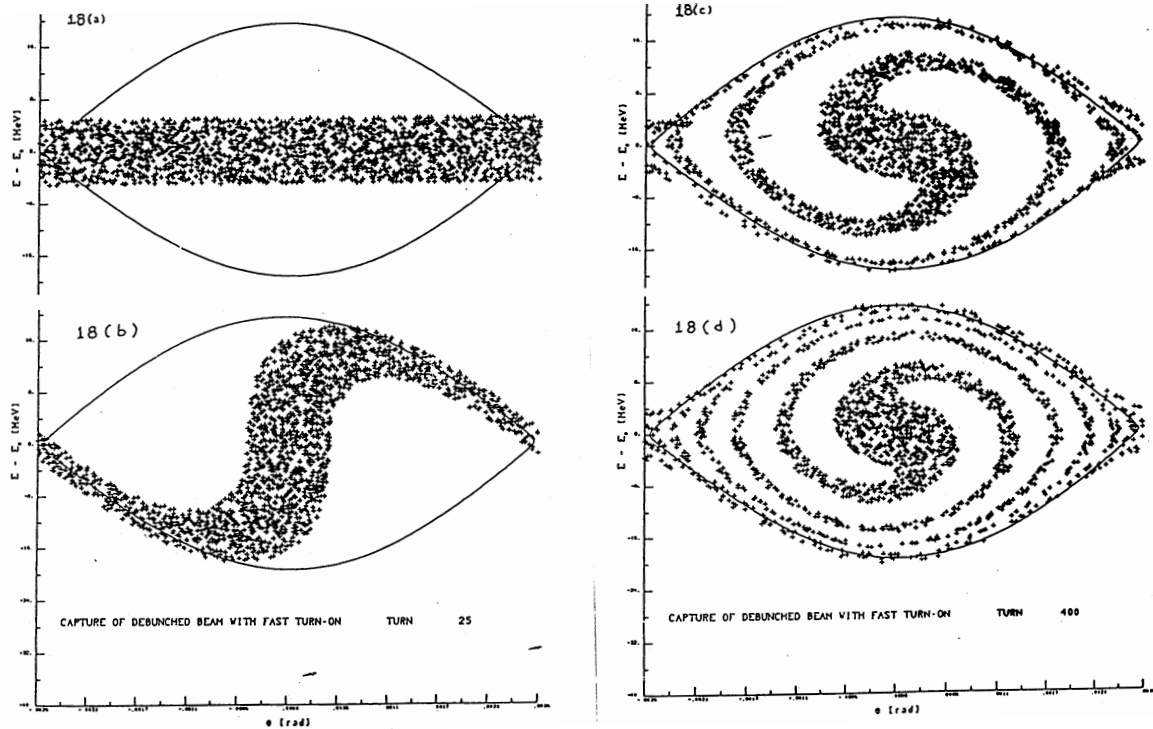
Start with a long bunch and extract or recapture when it's short.



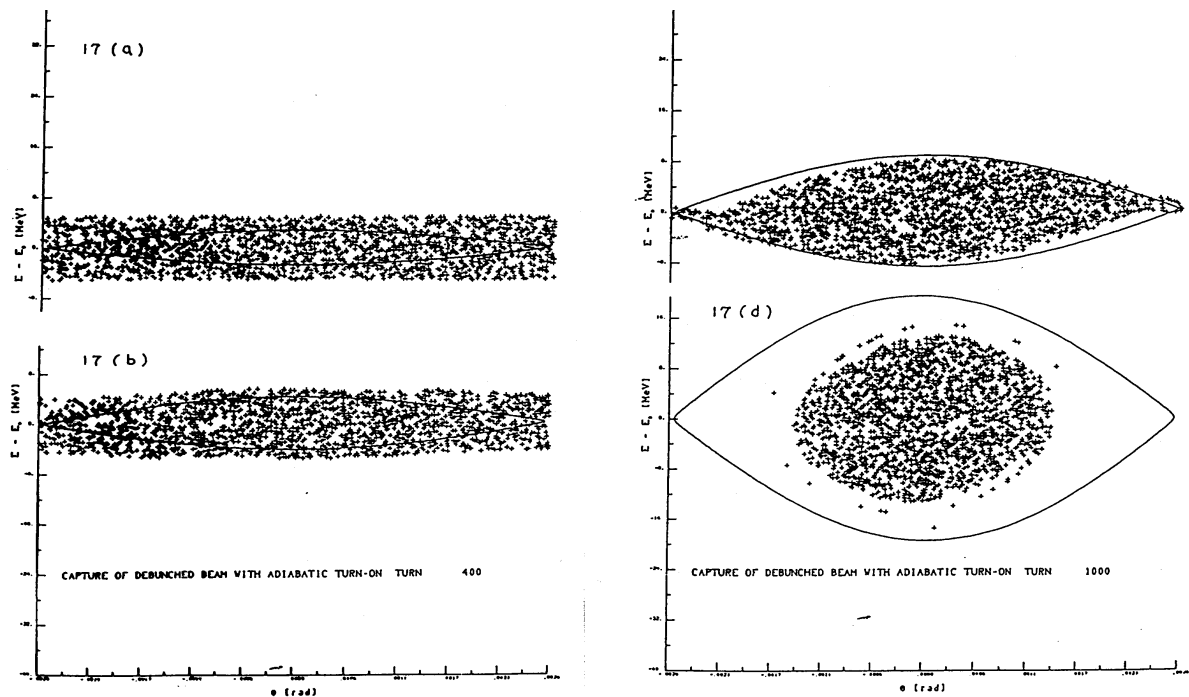
initial beam



Capture of a Debunched Beam with Fast Turn-On



Capture of a Debunched Beam with Adiabatic Turn-On

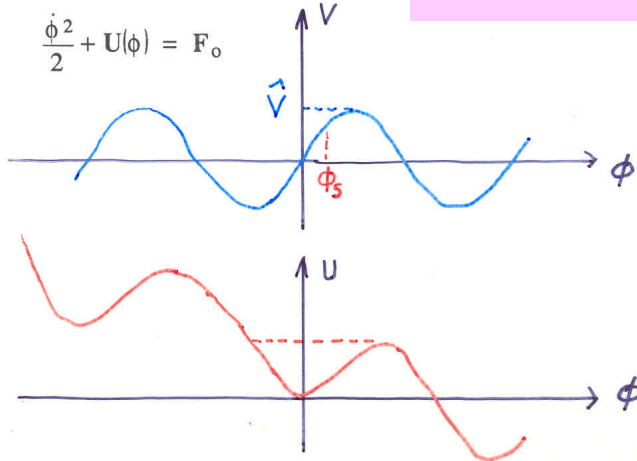


Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \quad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^\phi F(\phi) d\phi = -\frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W , leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_{rs}}$$

→

$$\frac{d\phi}{dt} = -\frac{h\eta\omega_{rs}}{pR} W$$

$$\frac{dW}{dt} = \frac{e\hat{V}}{2\pi} (\sin \phi - \sin \phi_s)$$

The two variables ϕ, W are canonical since these equations of motion can be derived from a Hamiltonian $H(\phi, W, t)$:

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W}$$

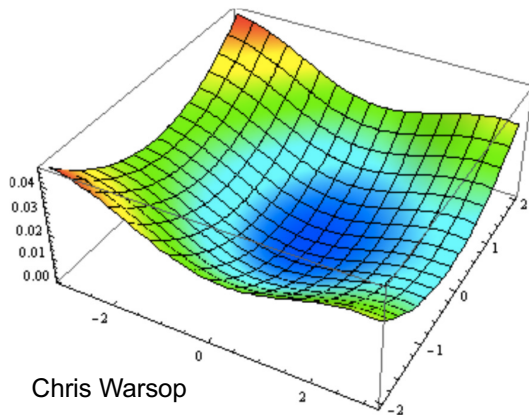
$$\frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = -\frac{1}{2} \frac{h\eta\omega_{rs}}{pR} W^2 + \frac{e\hat{V}}{2\pi} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s]$$

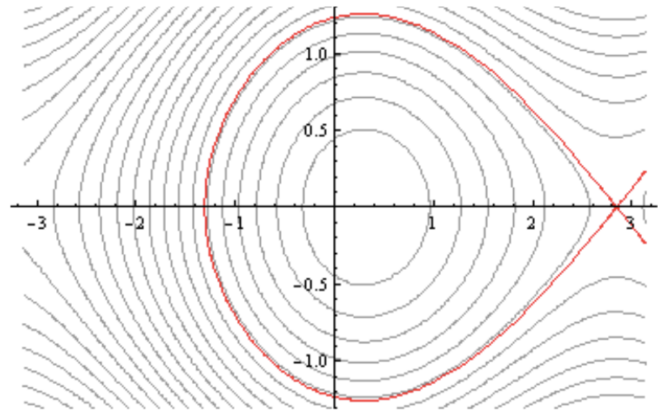
Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of $H(\phi, W)$



Contours of $H(\phi, W)$



Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, ...)

Summary

- **Synchrotron oscillations** in the longitudinal phase space (E, ϕ)
- Particles perform **oscillations around synchronous phase**
 - synchronous phase depending on acceleration
 - below or above transition
- **Bucket** is the stable region in phase space inside the **separatrix**
 - Bucket size is the **largest without acceleration**
- to **avoid filamentation** and emittance increase it is important to
 - **match the shape** of the bunch to the bucket and
 - inject with the **correct phase and energy**
- **Hamiltonian** approach can deal with fairly complicated dynamics

Bibliography

- M. Conte, W.W. Mac Kay **An Introduction to the Physics of particle Accelerators**
(World Scientific, 1991)
- P. J. Bryant and K. Johnsen **The Principles of Circular Accelerators and Storage Rings**
(Cambridge University Press, 1993)
- D. A. Edwards, M. J. Syphers **An Introduction to the Physics of High Energy Accelerators**
(J. Wiley & sons, Inc, 1993)
- H. Wiedemann **Particle Accelerator Physics**
(Springer-Verlag, Berlin, 1993)
- M. Reiser **Theory and Design of Charged Particles Beams**
(J. Wiley & sons, 1994)
- A. Chao, M. Tigner **Handbook of Accelerator Physics and Engineering**
(World Scientific 1998)
- K. Wille **The Physics of Particle Accelerators: An Introduction**
(Oxford University Press, 2000)
- E.J.N. Wilson **An introduction to Particle Accelerators**
(Oxford University Press, 2001)



And CERN Accelerator Schools (CAS) Proceedings
In particular: CERN-2014-009
Advanced Accelerator Physics - CAS

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I would like to thank everyone for the material that I have used.

In particular (hope I don't forget anyone):

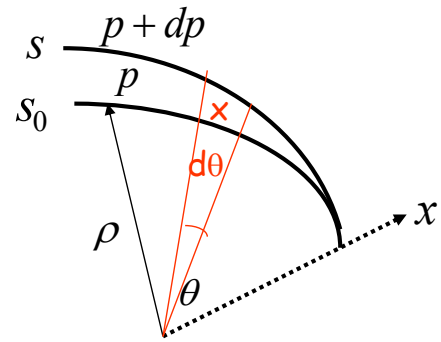
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- Roberto Corsini
- Roland Garoby
- Luca Bottura
- Chris Warsop

Appendix: Momentum Compaction Factor

$$\alpha_c = \frac{p}{L} \frac{dL}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x) d\theta$$



The elementary path difference

from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho} \frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \int_C dl = \int_C \frac{x}{\rho} ds_0 = \int_C \frac{D_x}{\rho} \frac{dp}{p} ds_0$$

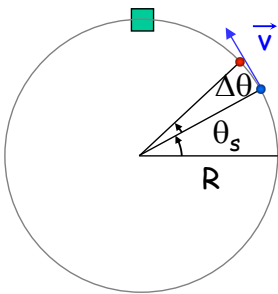
$$\alpha_c = \frac{1}{L} \int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With $p = \infty$ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle \rangle_m$ means that the average is considered over the bending magnet only

Appendix: First Energy-Phase Equation



$$f_{RF} = h f_r \Rightarrow \Delta\phi = -h \Delta\theta \quad \text{with} \quad \theta = \int \omega dt$$

particle ahead arrives earlier
 \Rightarrow smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h} \frac{d}{dt}(\Delta\phi) = -\frac{1}{h} \frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}} \left(\frac{d\omega}{dp} \right)_s$ and

$$E^2 = E_0^2 + p^2 c^2$$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h \eta \omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h \eta \omega_{rs}} \dot{\phi}$$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is: $\frac{dE}{dt} = e\hat{V}\sin\phi \frac{\omega_r}{2\pi}$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta(\dot{E}T_r) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\dot{T}_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}(T_{rs}\Delta E)$$

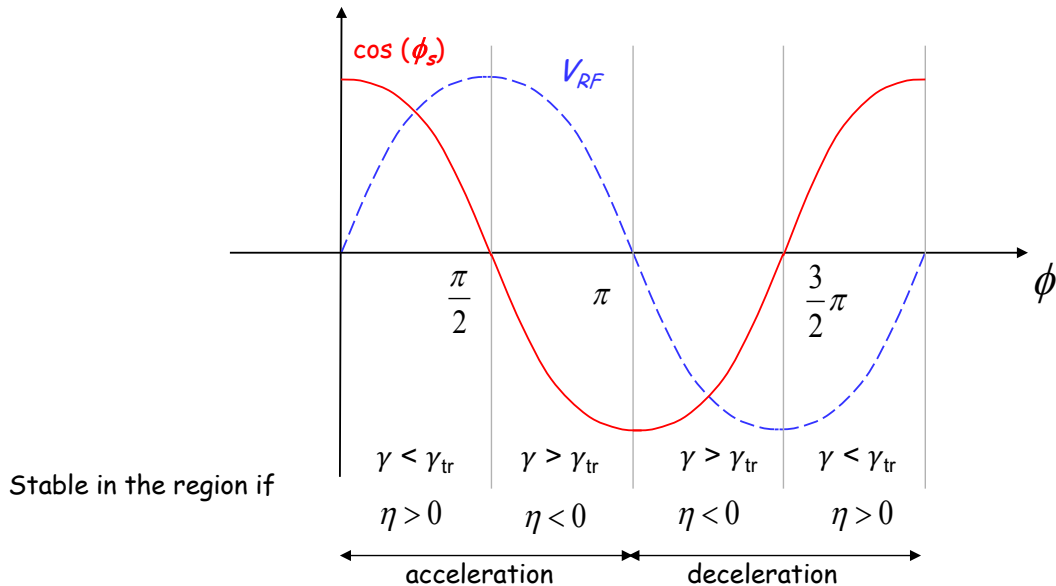
leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Appendix: Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 = \frac{e\hat{V}_{RF}\eta h\omega_s}{2\pi R_s p_s} \cos\phi_s \Rightarrow \Omega_s^2 > 0 \Leftrightarrow \eta \cos\phi_s > 0$$



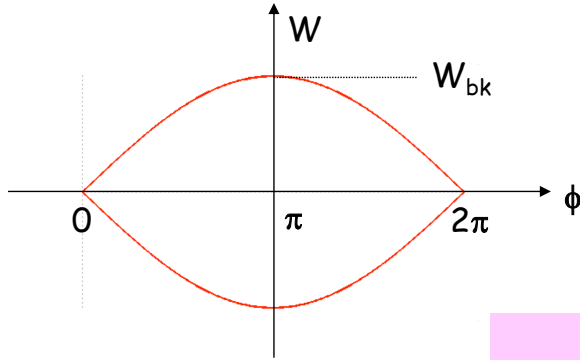
Appendix: Stationary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = \Omega_s^2$$

$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2 \frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W :



with $C=2\pi R_s$

$$W = \frac{\Delta E}{\omega_{rf}} = -\frac{p_s R_s}{h\eta\omega_{rf}} \dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W = \pm \frac{C}{\pi h c} \sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}} \sin \frac{\phi}{2} = \pm W_{bk} \sin \frac{\phi}{2}$$

Stationary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$W_{bk} = \frac{C}{\pi h c} \sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the **maximum energy acceptance**:

$$\Delta E_{\max} = \omega_{rf} W_{bk} = \beta_s \sqrt{2 \frac{-e\hat{V}_{RF} E_s}{\pi \eta h}}$$

The area of the bucket is: $A_{bk} = 2 \int_0^{2\pi} W d\phi$

Since: $\int_0^{2\pi} \sin \frac{\phi}{2} d\phi = 4$

one gets:

$$A_{bk} = 8W_{bk} = 8 \frac{C}{\pi h c} \sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$



$$W_{bk} = \frac{A_{bk}}{8}$$

Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \quad (\Omega_s \text{ as previously defined})$$

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I$$

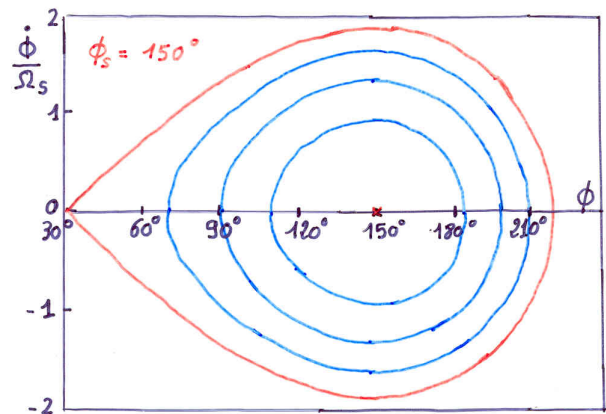
which for small amplitudes reduces to:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta\phi)^2}{2} = I' \quad (\text{the variable is } \Delta\phi, \text{ and } \phi_s \text{ is constant})$$

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi - \phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi - \phi_s$ is an extreme amplitude for a stable motion which in the phase space ($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the **separatrix**:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = -\frac{\Omega_s^2}{\cos \phi_s} (\cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos \phi_m + \phi_m \sin \phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s$$

Appendix: Energy Acceptance

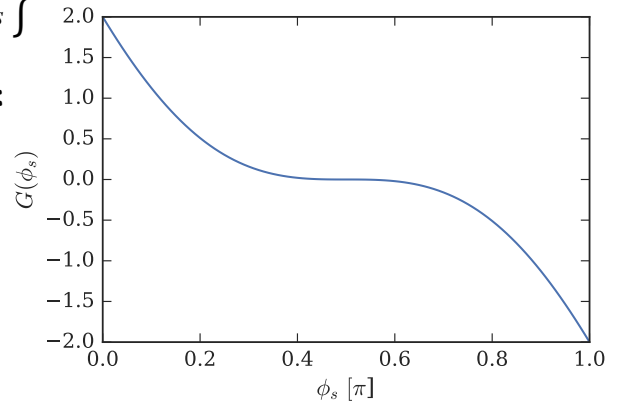
From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}_{\max}^2 = 2\Omega_s^2 \left\{ 2 + (2\phi_s - \pi) \tan \phi_s \right\}$$

That translates into an **energy acceptance**:

$$\left(\frac{\Delta E}{E_s} \right)_{\max} = \pm \beta \sqrt{\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s)}$$

$$G(\phi_s) = [2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s]$$



This "**RF acceptance**" depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

It's **largest** for $\phi_s=0$ and $\phi_s=\pi$ (**no acceleration**, depending on η).

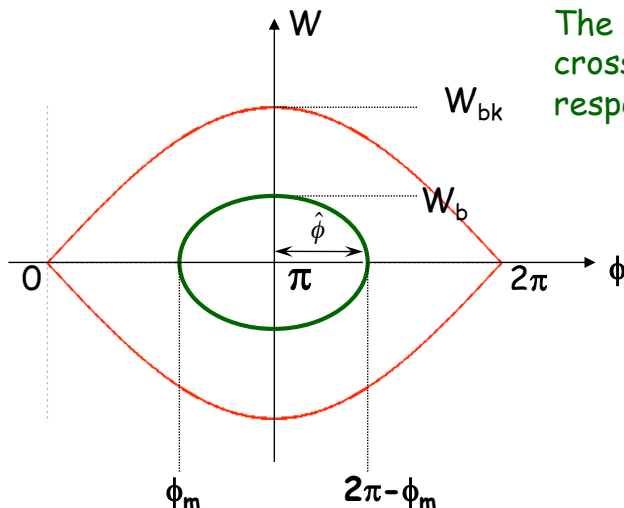
It becomes smaller during acceleration, when ϕ_s is changing

Need a **higher RF voltage** for **higher acceptance**.

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s} (\cos\phi + \phi\sin\phi_s) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = \Omega_s^2 \cos\phi_m$$

$$\dot{\phi} = \pm \Omega_s \sqrt{2(\cos\phi_m - \cos\phi)}$$

$$W = \pm W_{bk} \sqrt{\cos^2 \frac{\phi_m}{2} - \cos^2 \frac{\phi}{2}}$$

$$\cos(\phi) = 2 \cos^2 \frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos \frac{\phi_m}{2} = W_{bk} \sin \frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos \frac{\phi_m}{2}$$

$$\longrightarrow \left(\frac{\Delta E}{E_s} \right)_b = \left(\frac{\Delta E}{E_s} \right)_{RF} \cos \frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s} \right)_{RF} \sin \frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a **shorter bunch** (ϕ_m close to π , $\hat{\phi}$ small) will **require** a bigger RF acceptance, hence a **higher voltage**

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16} \sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \longrightarrow \left(\frac{16W}{A_{bk}\hat{\phi}} \right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}} \right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$