Quantum Diffusion During Inflation and Primordial Black Holes

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Outline

- Introduction to stochastic-δN inflation
- Characteristic function formalism
- Application to primordial black holes
- Summary

Inflation

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The inflaton ϕ has classical equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,.$$

Stochastic Formalism

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Then ϕ is described by a Langevin equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi\left(N\right) \, ,$$

where $\langle \xi\left(N\right) \rangle = 0$ and $\langle \xi\left(N\right) \xi\left(N'\right) \rangle = \delta\left(N - N'\right)$, $k < aH$ and $N = \int H \mathrm{d}t$.

Inflaton evolves under Langevin equation until ϕ reaches ϕ_{end} where inflation ends.

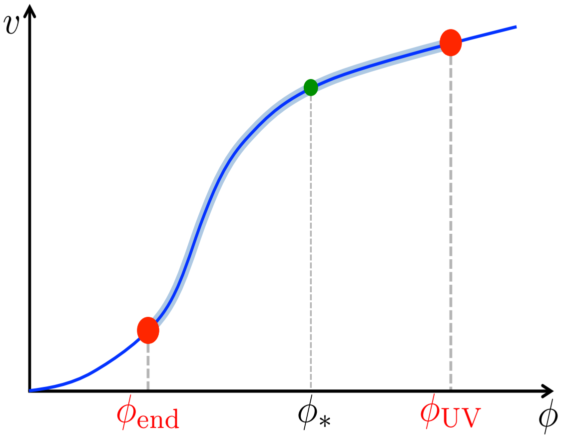

Figure 1: A reflective wall is added at ϕ_{uv} to prevent the field from exploring arbitrarily large values.

Separate Universe (Wands et al, 2000)

The primordial curvature perturbation ζ is

$$\zeta(t, \mathbf{x}) = N(t, \mathbf{x}) - N_0(t) \equiv \delta N \,,$$

where N is the local number of e-folds of inflation, and N_0 is the amount of expansion in an unperturbed universe.

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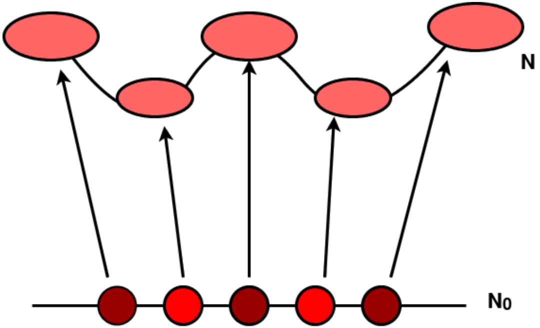

Figure 2: N_0 is at a zero curvature surface, final slice is constant density.

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This reduces calculating curvature perturbations to calculating statistics of $\mathcal{N}$ realised under Langevin equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi\left(\mathcal{N}\right) \, .$$

Characteristic Function Formalism

We want to know about the moments of $\mathcal{N}$, and set $f_n(\phi) = \langle \mathcal{N}^n(\phi) \rangle$. Characteristic function $\chi_{\mathcal{N}}(t, \phi)$ is

$$\begin{aligned} \chi_{\mathcal{N}}(t, \phi) &= \left\langle e^{it\mathcal{N}(\phi)} \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} f_n(\phi) \,. \end{aligned}$$

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$\chi_{\mathcal{N}}$ is related to the PDF $P(\mathcal{N},\phi)$ by

$$P\left(\delta\mathcal{N},\phi\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it[\delta\mathcal{N}+\langle\mathcal{N}\rangle(\phi)]}\chi_{\mathcal{N}}\left(t,\phi\right)\mathrm{d}t\,.$$

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Characteristic function satisfies the PDE

$$\left[\frac{\partial^2}{\partial\phi^2} - \frac{v'}{v^2}\frac{\partial}{\partial\phi} + \frac{it}{vM_{\rm Pl}^2}\right]\chi_{\mathcal{N}}(t,\phi) = 0\,.$$

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Inverse Fourier transforming this gives the full PDF!

Simple Example

As an example, we take the potential

$$v(\phi) = v_0 \left(\frac{\phi}{M_{\rm Pl}} \right)^2 .$$

The computational program is then

- solve our ODE for $\chi_{\mathcal{N}}(t, \phi)$

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In this case, we can solve everything analytically!

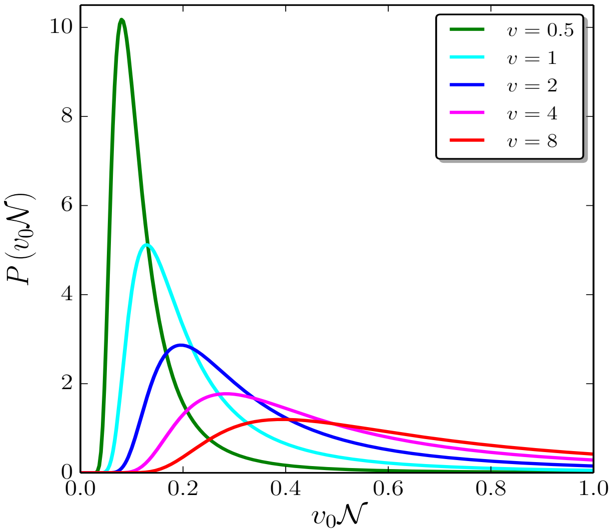

Figure 3: Plot of the PDF of $\mathcal{N}$ against $\mathcal{N}$, for the potential $v(\phi) = v_0 \left(\frac{\phi}{M_{\rm Pl}}\right)^2$.

If $\zeta > \zeta_{\rm c}$, collapse to form PBHs

The number of PBHs produced is then calculated from the probability distribution $P(\delta\mathcal{N}, \phi)$ of these large perturbations using

$$\beta\left[M\left(\phi\right)\right] = 2\int_{\zeta_{\rm c}}^{\infty} P\left(\delta\mathcal{N}, \phi\right) {\rm d}\delta\mathcal{N}\,.$$

Application to Primordial Black Holes (PBHs)

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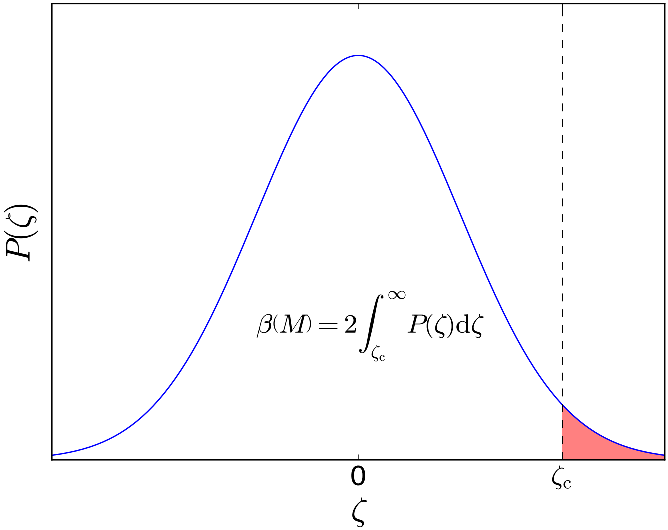
$$\beta\left[M\left(\phi\right)\right] = 2\int_{\zeta_c}^{\infty} P\left(\delta\mathcal{N}, \phi\right) \mathrm{d}\delta\mathcal{N}\,.$$

This gives the mass fraction of the universe contained in PBHs

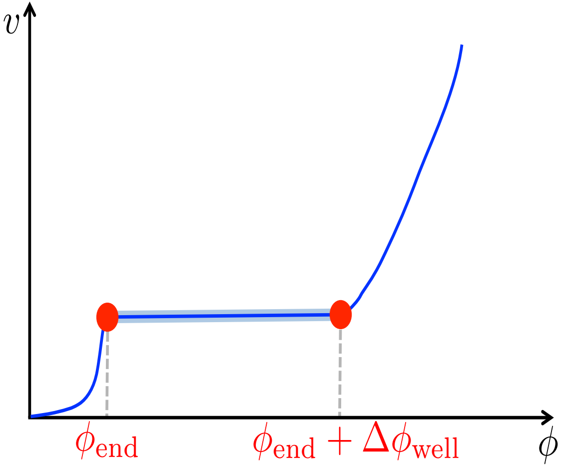
Gaussian Example

It is typically assumed ζ has a Gaussian distribution.



Stochastic Limit

Inflationary models that produce $\zeta > \zeta_c$ can be approximated by a flat potential at the end of inflation, so $v \simeq v_0$.



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The PDF in this limit is given by

$$P\left(\mathcal{N},\phi\right) = -\frac{\pi}{2\mu^2}\vartheta_2'\left(\frac{\pi}{2}x, e^{-\frac{\pi^2}{\mu^2}\mathcal{N}}\right) ,$$

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where

$$\mu^2 = \frac{\Delta\phi_{\mathsf{well}}^2}{v_0 M_{\mathrm{Pl}}^2}, \qquad x = \frac{\phi - \phi_{\mathsf{end}}}{\Delta\phi_{\mathsf{well}}},$$

and ϑ_2 is the second elliptic theta function.

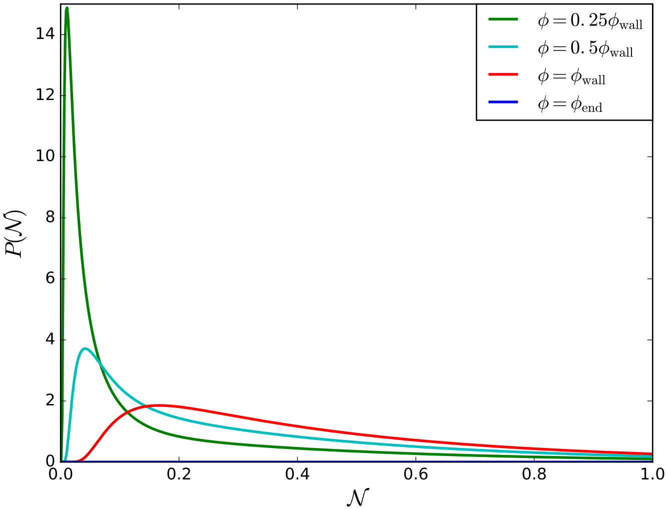

Figure 4: The PDF we obtain for a flat potential.

Mass fraction

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The expression we find depends on ϕ, μ and $\zeta_{\rm c}$.

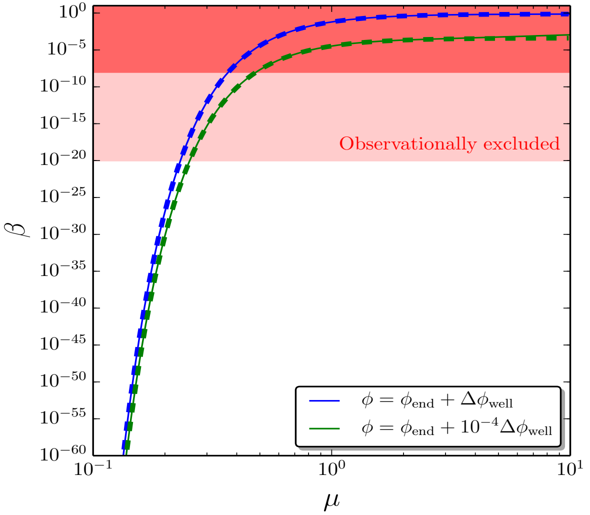

Figure 5: The mass fraction β is plotted as a function of μ, with $\zeta_c = 1$.

Summary

- The stochastic-$\delta\mathcal{N}$ formalism is needed to analyse curvature perturbations and PBH formation.
- We developed a characteristic function formalism to calculate the PDF of large fluctuations.
  - Taking a classical limit $v \ll 1$ allows us to recover the classical results, and provide a first NG correction.
- In the stochastic limit we derived a new constraint on μ (height of well vs width of well).

Future Work

- Recent PBH models (Garcia-Bellido et al, 2017) exhibit slow-roll violation, i.e. a phase of ultra slow-roll. Formalism needs to be checked here.
- Study higher-order corrections to the tail of the Gaussian, even in the classical case
- Extend the formalism to include multi-field inflation.