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Parton model in QCD

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PARTON MODEL

Elastic scattering : electron — proton

——–> proton (hadron) is **NOT point-like**

Deep inelastic scattering : electron — proton

——–> proton (hadron) consists of **point-like particles-partons**

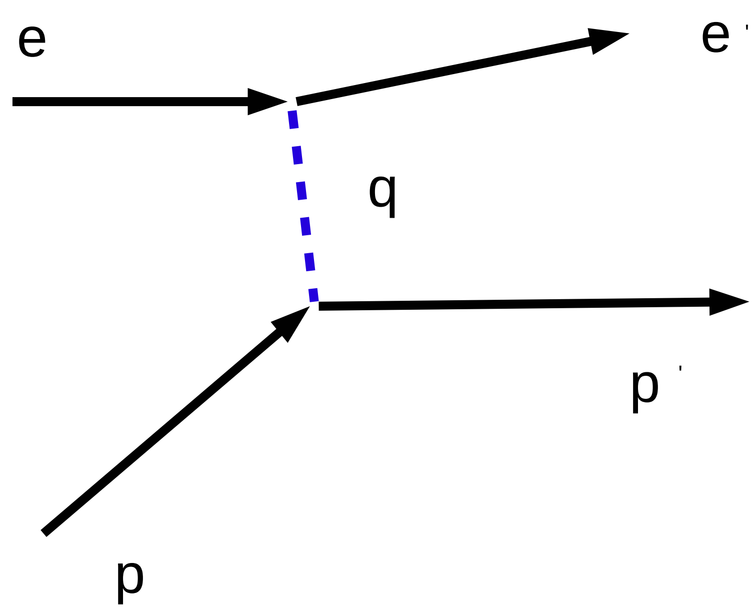
Cross section (hadron) $=$ Σ cross section (parton) $\times$ weights

Weights — probabilities in the system of infinite momentum

(Bjorken, Feynman)

WHY is a proton (hadron) NOT point-like ?

Elastic ep-scattering is described by the same Feynman diagram as $e\mu^+$-scattering:



$$M_{fi} = \bar{u}'_e \gamma^\mu u_e \frac{e^2}{q^2} \langle p' | J_\mu(q) | p \rangle .$$

(BUT the hadron current $J_\mu(q)$ can't be presented in the "QED form":

$J^\mu_h = e\overline{\psi}_h(x)\gamma^\mu\psi_h(x)$ — **CONTRADICTS** the experiment).

Relativistic + gauge invariance − >

$\langle p'|J_\mu(q)|p\rangle$ can be parametrized over 2 form factors:

$$\langle p'|J_\mu(q)|p\rangle = \bar{u}'_p \left\{\gamma^\mu F_1(q^2) - \frac{F_2(q^2)}{2M}\sigma^{\mu\nu} q_\nu\right\} u_p =$$

$$= \bar{u}'_p \left\{\gamma^\mu F_m(q^2) + 2M\left[F_e(q^2) - F_m(q^2)\right]\frac{P^\mu}{P^2}\right\} u_p,$$

where M is the proton mass and

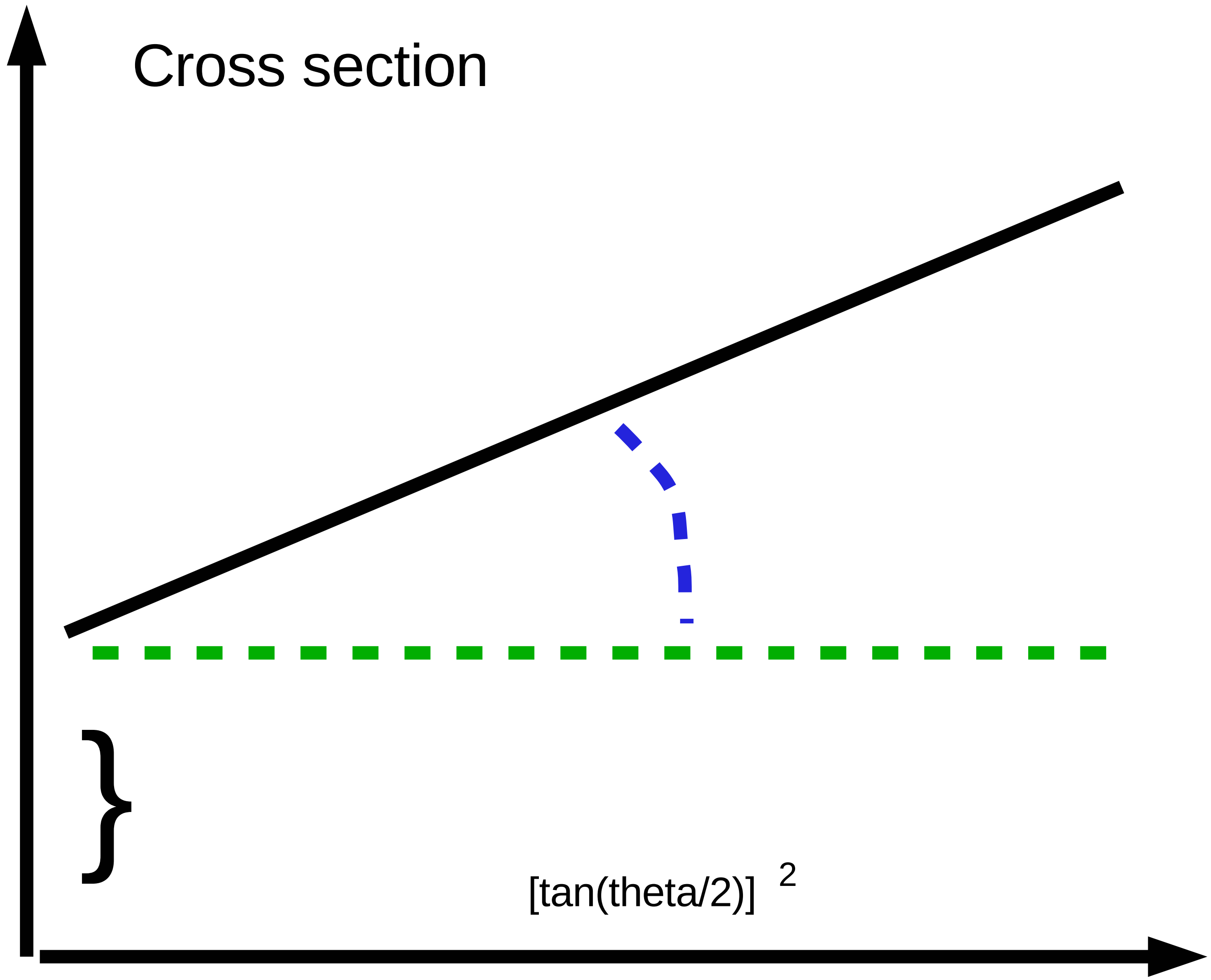
$$F_m = F_1 + F_2, \qquad F_e = F_1 + F_2\frac{q^2}{4M^2}, P = p + p'.$$

Then the differential cross section in the proton rest frame

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left[\frac{F_e^2 + bF_m^2}{1+b} + 2bF_m^2 \tan^2\frac{\theta}{2}\right],$$

where $b = \frac{-q^2}{4M^2}$, θ is the scattering angle of electron.

Cross section (normalized) as a function of $\tan^2 \frac{\theta}{2}$



Slope + intersection −− > form factors

Proton: at $q^2 \to 0$

$$F_e^p(0) = 1, \qquad F_m^p(0) = 2.79.$$

Neutron:

$$F_e^n(0) = 0, \qquad F_m^n(0) = -1.91.$$

From SLAC experiment (1964):

$$F_e^p(q^2) = \frac{F_m^p(q^2)}{2.79} = \frac{F_m^n(q^2)}{|1.91|} = \left(\frac{1}{1 + |q^2|/m_g^2}\right)^2,$$

with m_g^2 =0.71 GeV^2 and $F_e^n(q^2) = 0$.

In the Breit frame

(where $\vec{P} = \vec{p} + \vec{p}' = 0 \quad (q^0 = 0,\ \vec{q} = 2\vec{p}' = -2\vec{p})$):

$$e\rho(\vec{r}) = e\frac{1}{(2\pi)^3}\int F(-\vec{q}^2)e^{i\vec{q}\vec{r}}d^3q$$

is the spatial distribution of charge density.

In the case:

$F = \text{const} \quad --> \quad \rho(\vec{r}) = \delta(\vec{r}) \quad -> \quad$ point-like

Thus from SLAC experiment:

1. Proton and neutron are not point-like
(the form factors are dependent of q^2)

2. The Fourier of form factors are well approximated by:

$$\rho(r) = \rho(0)\exp\left(-r/a\right)$$

with

$$a = 0.23 \text{ fm}$$

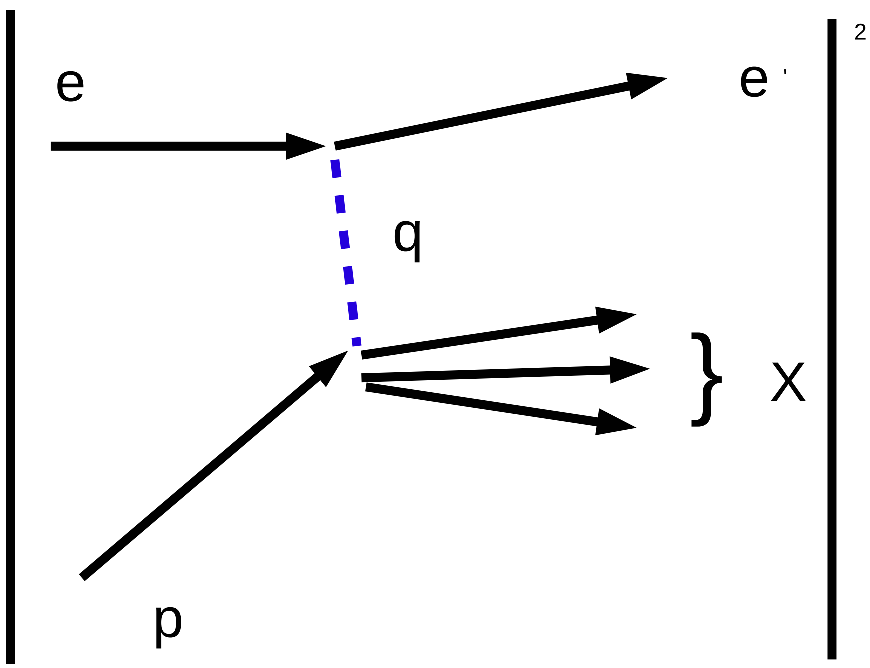
$$\langle r_E^2\rangle_p = \langle r_M^2\rangle_p = \langle r_M^2\rangle_n \approx 0.7 \text{ fm}^2$$

as the proton size

Why does a proton (hadron) consist of point-like particles-partons ?

For deep inelastic ep-scattering at

$$\sqrt{-q^2} \gg M_p \simeq 1 \text{ GeV}$$



the measured probability of inclusive process is propotional to :

$$|M|^2 = \left(\bar{u}'_e \gamma_\nu u_e\right)^* \left(\bar{u}_e \gamma_\mu u_e\right) \left(\frac{e^2}{q^2}\right)^2 \sum_X \langle p|J_\nu(-q)|X\rangle \cdot \langle X|J_\mu(q)|p\rangle$$

$$\cdot 2\pi\delta\left((p+q)^2 - M_X^2\right),$$

where

$$W_{\mu\nu} = \sum_X \langle p|J_\nu(-q)|X\rangle\langle X|J_\mu(q)|p\rangle 2\pi\delta\left((p+q)^2 - M_X^2\right)$$

is the subject of study.

Again, Relativistic + gauge invariance − >

$$\frac{M}{4\pi} W_{\mu\nu} = W_2(q^2, x)\left(p_\mu - q_\mu \frac{(p\cdot q)}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right)$$

$$W_1(q^2, x) M^2 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right),$$

where $x = -\frac{q^2}{2(pq)} = \frac{|q^2|}{2M\nu}$, ν is the energy loss in the rest frame of the proton with mass M.

The inclusive cross section of this process:

$$\frac{d\sigma}{d\nu d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point.elastic}} \left[W_2(x,q^2) + 2W_1(x,q^2)\tan^2\frac{\theta}{2}\right].$$

These structure functions $\nu W_2(q^2,x)$ and $W_1(q^2,x)$ at $-q^2 \geq 1$ GeV^2 can be related with the nucleon form factors F_e^2, F_m^2 and $\delta(x-1)$ since for elastic scattering

$p' - p = q$, $p' = p + q$, $p'^2 = p^2 + 2pq + q^2$, $q^2 = -2pq \Longrightarrow -\frac{q^2}{2pq} = x = 1.$

From first experiment:

$\nu W_2(q^2,x)$ and $W_1(q^2,x)$ at $-q^2 \geq 1$ GeV^2

were independent of scale q^2 (SCALING)

————> elastic scattering on point-like partons

Parton model = impulse approximation = constituents as free

$------>$ VALID at

$$\tau \sim \frac{1}{\Delta E} \sim \frac{|\vec{p}|}{k_\perp^2 \max}$$

— the relaxation time is considerably larger than the collision time

$$\tau_{\text{collision}} \sim \frac{1}{|q|}.$$

Thus

$$|q| \gg \frac{k_\perp^2 \max}{|\vec{p}|}$$

and in the "frame" $|\vec{p}| \to \infty$ the assumption, that $k_\perp \mathrm{max}$ is restricted, is quite ENOUGH :

Toy model: proton consists of 2 massless partons with
$x\vec{P} + \vec{k}_\perp$ is the momentum of one parton
$(1-x)\vec{P} - \vec{k}_\perp$ is the momentum of other parton

Then:

$$\boldsymbol{\Delta E = \sqrt{\left(x\vec{P} + \vec{k}_\perp\right)^2} + \sqrt{\left((1-x)\vec{P} - \vec{k}_\perp\right)^2} - P \approx}$$

$$\boldsymbol{\approx xP\left(1 + \frac{k_\perp^2}{2x^2p^2}\right) + (1-x)P\left[1 + \frac{k_\perp^2}{2(1-x)^2P^2}\right] - P =}$$

$$\boldsymbol{\frac{k_\perp^2}{2p}\left[\frac{1}{x} + \frac{1}{1-x}\right] \simeq \frac{k_\perp^2}{2P} \sim \frac{k_\perp^2}{P}}$$

Quantum mechanics teaches us: the "life time" of such configuration:

$$\boldsymbol{\tau = \frac{1}{\Delta E} \approx \frac{P}{k_\perp^2} \gg \frac{1}{|q|} = t_{\rm collision}.}$$

————————

More realistic model (*Drell, Yan, Phys. Rev. Lett. 24, 181 (1970)*):

proton consists of infinite number of massless noninteracting partons:

$|n\rangle = a_1^+\left(\vec{k}_1\right)\ldots a_n^+\left(\vec{k}_n\right)|0\rangle$ — n-parton Fock state

$\langle n|p\rangle = f_p(k_1\ldots k_n)$ is the probability amplitude to find the n-parton configuration with the given momenta in the proton.

Then the hadron matrix element squared can be written over partonic variables:

$$|M_h|^2 = \sum_{n,m,l}\langle p|n\rangle\langle n|J_\mu(-q)|m\rangle\langle m|J_\nu(q)|l\rangle\langle l|p\rangle\cdot$$

$$\cdot 2\pi\delta\left[(P_n+q)^2 - M_m^2\right].$$

The transition

$$\sum_X |X\rangle\langle X|\delta\left((p+q)^2 - M_x^2\right) = \sum_m |m\rangle\langle m|\delta\left[(p+q)^2 - M_m^2\right]$$

due to the completeness theorem of state.

Asssumption: photon interacts elastically only with one parton of type i in n-parton configuration —->$|n\rangle = |n-1\rangle|1_i\rangle$

$$|M_h|^2 = \sum_{n,m,l,i} \langle p|n-1\rangle |1_i\rangle \langle n-1| \langle 1_i| J_\mu(-q)|1_i'|m-1\rangle \cdot$$

$$\cdot 2\pi\delta\left((p_i+q)^2 - M_i^2\right) \cdot \langle m-1|1_i'|J_\mu(q)|1_i|l-1\rangle \langle l-1|\langle 1_i|p\rangle =$$

$$= \sum_{n_i} \langle p|n-1\rangle \langle 1_i|J_\mu(-q)1_i'\rangle \langle 1_i'|J_\mu(q)|1_i\rangle \cdot$$

$$\cdot \langle n-1|\langle 1_i|p\rangle 2\pi\delta\left((p_i+q)^2 - M_i^2\right) =$$

$$= \sum_{ni} \langle p|n-1\rangle |1_i\rangle \langle n-1|\langle 1_i|p\rangle \cdot \langle 1_i|J_\mu(-q)|1_i'\rangle \cdot \langle 1_i'|J_\mu(q)1_i\rangle \cdot$$

$$2\pi\delta\left((p_i+q)^2 - M_i^2\right) =$$

$$= \sum_i P(i) \cdot |M_i|^2 2\pi\delta\left[(p_i+q)^2 - M_i^2\right]$$

The matrix element squared for the i-parton with spin $1/2$ and ξ as the fraction of the longitudinal proton momentum can be calculated:

$$|M_i|^2 = \frac{e_i^2}{2} Sp\left[(\xi p)\gamma_\mu(\xi p + q)\gamma_\nu\right] 2\pi\delta\left((\xi p + q)^2 - M_\xi^2\right) \approx$$

$$\approx e_i^2 2\left[(\xi p)_\mu(\xi p + q)_\nu + (\xi p)_\nu(\xi p + q)_\mu - g_{\mu\nu}(\xi p)(\xi p + q)\right] \cdot$$

$$\cdot 2\pi\delta\left[2\xi(pq) + q^2\right] \approx e_i^2 2\left[2\xi^2 p_\mu p_\nu - g_{\mu\nu}\xi(pq)\right]\frac{2\pi}{2(pq)}\delta(\xi - x) =$$

$$= 4\pi e_i^2\left[2\xi^2 p_\mu p_\nu - \xi(pq)g_{\mu\nu}\right] \cdot \frac{\delta(\xi - x)}{2(pq)}.$$

In accordance with the previous parametrization:

$$|M_h|^2 = \frac{4\pi}{M} W_2(p_\mu p_\nu) - 4\pi M W_1 g_{\mu\nu} =$$

$$\sum_i \int \frac{f^i(\xi)d\xi}{\xi} \cdot 4\pi e_i^2\left[2\xi^2 p_\mu p_\nu - g_{\mu\nu}\xi(pq)\right]\frac{\delta(\xi - x)}{2(pq)}.$$

Finally (after comparison):

$$\frac{pq}{M}W_2 = \nu W_2 = x\sum e_i^2 f^i(x),$$

$$2MW_1 = \sum e_i^2 f^i(x) = \frac{(pq)}{xM}W_2$$

The inclusive cross section in this parton (Bjorken-Feynman) model:

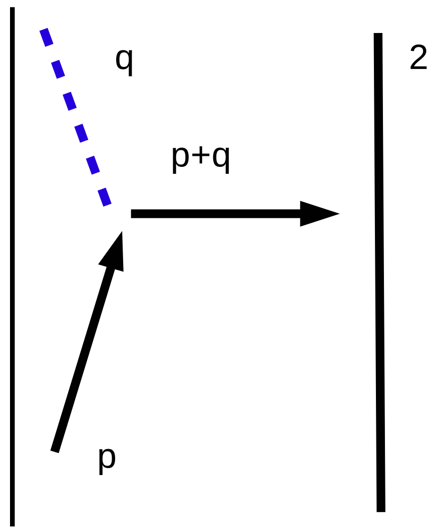
$$\frac{d^2\sigma}{dx dQ^2} = \sum_a \left(\frac{d\sigma}{dQ^2}\right)_{\text{elastic}} f_h^a(x), \quad Q^2 = -q^2.$$

(Partons with other spins give results contradicted experiment.)

PARTON MODEL IN QCD

Let us consider electron scattering on a quark with a charge e_i in perturbative QCD

1-order:



" Hadron tensor"

$$W_{\mu\nu} = \frac{e_i^2}{2} Sp\left[\hat{p}\gamma_\mu(p+q)\gamma_\nu\right] 2\pi\delta\left((p+q)^2\right).$$

At $2(pq) \gg p^2$ $(x = -\frac{q^2}{2pq})$

$$\delta\left((p+q)^2\right) \approx \delta(1-x)\frac{1}{2(pq)}.$$

Further calculating Sp and comparing with a general form of $W_{\mu\nu}$, we obtain

$$2MW_1 = e_i^2\delta(1-x) = \frac{pq}{xM}W_2(x,q^2),$$

This is, in fact, included in common parton model.

———————————————————————————

LLA : leading logarithm approximation (Gribov-Lipatov approximation)

———- >

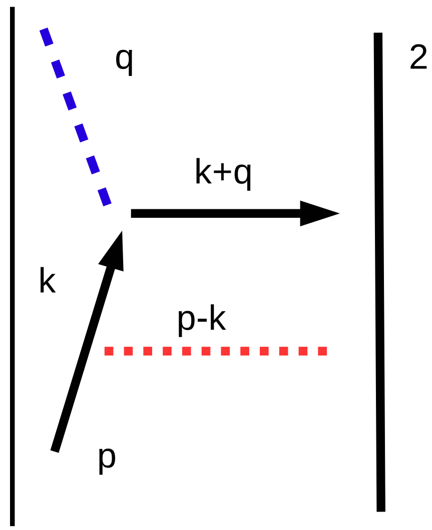
to take into account diagrams only with **enhancement:**

$$\sim \left(\alpha \ln |q^2|\right)^n .$$

———- > parton picture (+ due to *physical gauge*)

Dokshitzer, Dyakonov, Troyan, Phys. Rep. 58, N5, 269 (1980), ...

2-order: (quark radiates gluon before hard scattering)



" Hadron tensor"

$$W_{\mu\nu} = e_i^2 g^2 \frac{N^2-1}{2N} \int \frac{d^4k}{(2\pi)^4} 2\pi\delta\left((k+q)^2\right) 2\pi\delta\left((p-k)^2\right) \cdot$$

$$\cdot \frac{1}{k^4} \cdot \frac{1}{2} Sp\left[\hat{p}\gamma_\rho \hat{k}\gamma_\mu (k+q)\gamma_\nu \hat{k}\gamma_\sigma\right] d_{\sigma\rho}(p-k).$$

$\frac{N^2-1}{2N} = \frac{1}{N} t^a_{ij} t^a_{ji} = C_2$ **is the colour factor (averaging +summing)**

Sudakov technique:

$$\boldsymbol{k = \alpha q' + \beta p' + k_\perp,}$$

$$q' = q + xp, \qquad q'^2 = p'^2 = 0,$$

$$p' = p + \frac{\mu^2}{2(pq)} q', \qquad p^2 = -\mu^2 < 0, \qquad 2(pq) = -\frac{q^2}{x},$$

$$\boldsymbol{(k_\perp p') = (k_\perp q') = 0}$$

$$\boldsymbol{k^2 = \alpha\beta 2(p'q') - \bar{k}_\perp^2 = -\beta\mu^2 - \frac{k_\perp^2}{1-\beta}}$$

$$\boldsymbol{\int d^4k \Longrightarrow \int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} d\beta \pi (pq) \int_0^{\infty} dk_\perp^2,}$$

α- and β-integrations are taken with δ-functions, then:

$$W_{\mu\nu} = \frac{e_i^2 g^2}{16\pi}\frac{1}{(pq)}\int\limits_{\mu^2 x}^{2(pq)}\frac{d|k^2|}{|k^2|}\frac{1}{x}C_2\frac{1+x^2}{1-x+\frac{|k^2|}{2(pq)}\frac{\beta}{A}}\cdot Sp\left[x\hat{p}\gamma_\mu(x\hat{p}+q)\gamma_\nu\right].$$

In the process the gluon propagator is chosen to be (*planar gauge, physical gauge*)

$$d_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{(kn)}, n = Aq' + Bp'$$

With a logarithmic accuracy:

$$\int\limits_{\mu^2 x}^{2(pq)(1-x)}\frac{d|k^2|}{|k^2|} = \ln\frac{|q^2|(1-x)}{\mu^2 x^2} + 0(1) = \ln\frac{|q^2|}{\mu^2} + 0(1),$$

(*x and $(1-x)$ are not parametrically small, otherwise* $\ln\frac{1}{x}$, $\ln\frac{1}{1-x}$ *should be collected*)

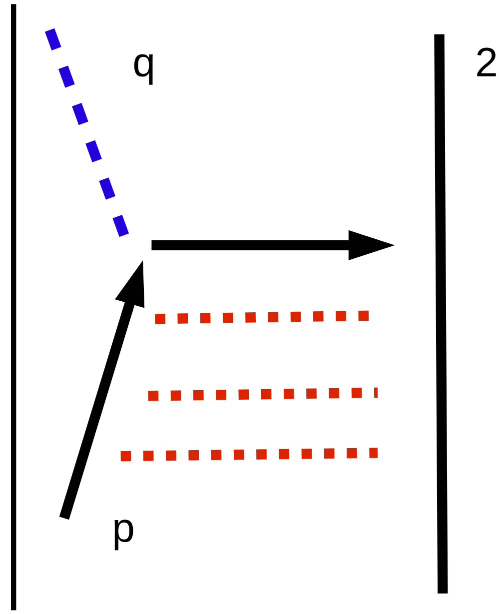
Thus, the one-rung contribution to the structure functions:

$$\frac{(pq)}{xM}W_2(x,q^2) = 2MW_1(x,q^2) = e_i^2\frac{\alpha}{4\pi}\ln\frac{|q^2|}{\mu^2}\Phi_F^F(x),$$

$$\Phi_F^F(x) = 2C_2\frac{1+x}{1-x} - DGLAP\ \ kernel,\ \ \alpha = g^2/4\pi.$$

Further one can show that

1) Contribution of quark ladder with n-rung



is proportional to

$$\sim \frac{1}{n!}\left(\frac{\alpha}{4\pi}\ln\frac{|q^2|}{\mu^2}\right)^n .$$

2) Non-ladder diagrams (without probability interpretation)

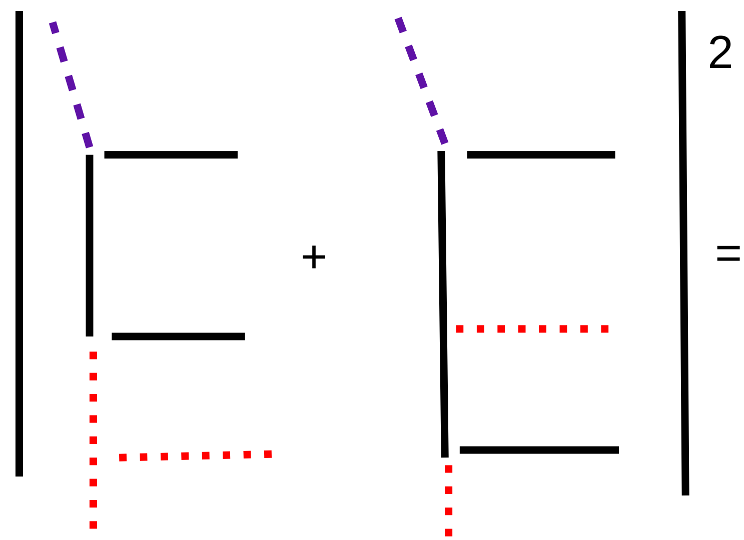
are NOT strenghtened by large $\ln \frac{|q^2|}{\mu^2}$

due to *planar gauge, physical gauge*

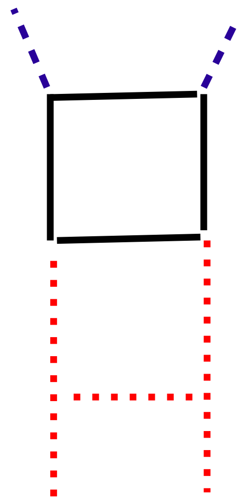
In Feynman gauge, for instance,

$$\boldsymbol{d_{\mu\nu}(k) = g_{\mu\nu}}$$

the interference contributions are strenghtened by large $\ln \frac{|q^2|}{\mu^2}$ **ALSO.**



Imaginary part of



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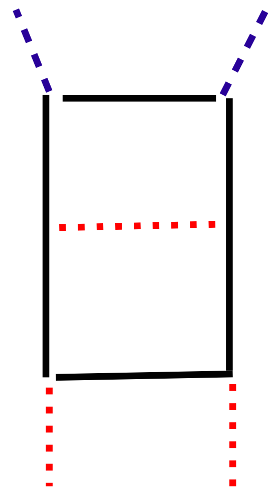

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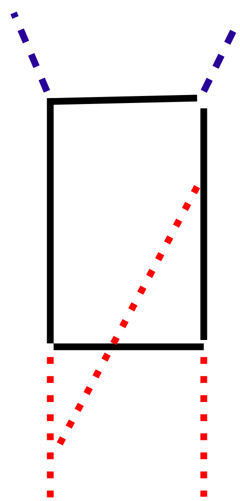

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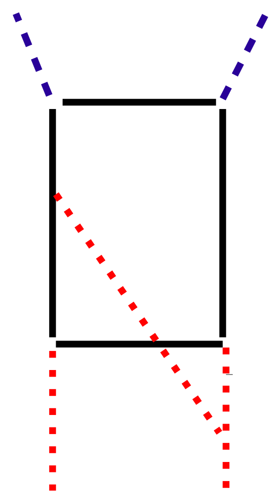

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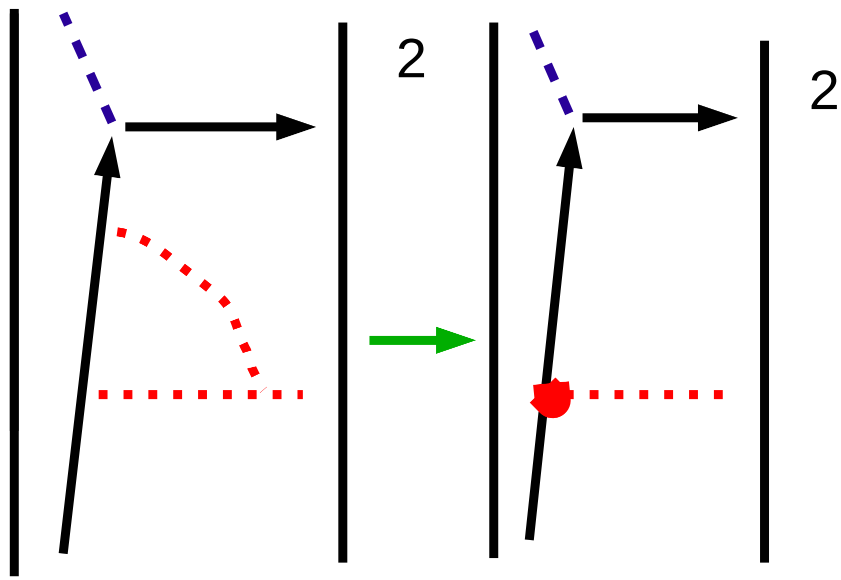
3) The corrections to the propogators and vertex functions in ladder kinematics:

Dokshitzer, Dyakonov, Troyan, Phys. Rep. 58, N5, 269 (1980), ...



$$\boldsymbol{\alpha} \longrightarrow \boldsymbol{\alpha_s(k^2)}.$$

Thus, one can introduce quantities which incorporate common features of all partonometry processes but do not depend on their specific form. These foundamental quantities are called by *parton distributions.*

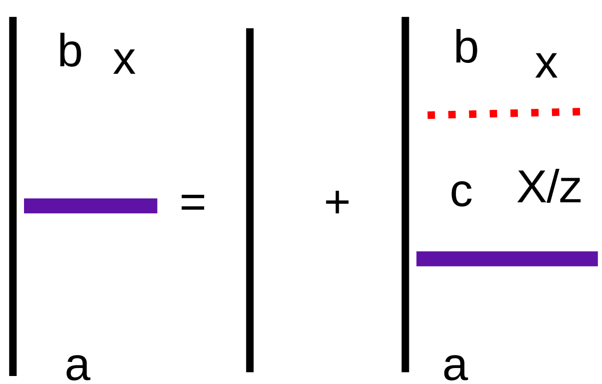
At scattering on parton of type a the structure functions are:

$$\frac{(pq)}{Mx}W_{2a}(x,q^2) = 2MW_{1a}(x,q^2) = \sum_{F=q,\bar{q}} e_F^2 D_a^F(x,\mu^2,q^2).$$

$$a = q, \bar{q}, g.$$

(in improved parton model there is contributions from g !)

For the ladder-type diagrams one can write an integral equations for parton distributions, treating 2-particle irreducible block as Bethe-Salpeter kernels



—–> DGLAP equations:

$$\frac{dD_a^b(x,t)}{dt} = \sum_c \int_x^1 \frac{dz}{z} P_c^b(z) D_a^c\left(\frac{x}{z}, t\right),$$

with

$$t = \int_{\mu^2}^{|q^2|} \frac{\alpha_s(k^2)}{4\pi} \frac{dk^2}{k^2}$$

$D_a^b(x,0) = \delta_a^b \delta(1-x)$ (at $q^2 = \mu^2$) are initial conditions.

The kernels $P_c^b(z)$ include the δ-function virtual term at $a = b$ unlike kernels $\Phi_c^b(z)$ calculated above (2-order).

Now comes the day of reckoning: in fact, leptons are scattered on hadrons and we must relate the observed structure functions of hadrons to the structure functions of q and g above.

In the ladder kinematics the large logarithms $\ln \frac{|q^2|}{\mu^2}$ is made by the broad range of integration with respect to virtualnesses:

$$|q^2| \gg \ldots |k_i^2| \gg \ldots \mu^2.$$

Thus the time of ladder formation

$$\tau_{\text{ladder}} \sim \frac{1}{|q^2|}$$

is close to the interactions time in common parton model, i.e.

$$\tau_{\text{ladder}} \ll \tau_{\text{relax.}} \sim \frac{1}{\mu^2}$$

is the time of formation of hadrons from q and g.

Therefore the process of "ladder formation" and the process of "hadron formation" are NOT interfered and

$$D_h^a(x,t) = \sum_c \int_x^1 \frac{dz}{z} f_h^c(z) D_c^a\left(\frac{x}{z}, t\right),$$

where

$D_h^a(x,t)$ are the parton distribution functions in hadron

$D_c^a\left(\frac{x}{z}, t\right)$ is the distribution functions at parton level

$f_h^c(z)$ are unknown phenomenological functions $=$ initial conditions for the distribution functions at hadron level:

$$D_h^a(x,t)|_{t=0} = f_h^a(x).$$

and

$$\frac{dD_h^b(x,t)}{dt} = \sum_c \int_x^1 \frac{dz}{z} P_c^b(z) D_h^c\left(\frac{x}{z}, t\right),$$

Eqs. are explicitly solved by introducing the Mellin transforms

$$M_h^a(n,t) = \int_0^1 dx\ x^n\ D_h^a(x,t), \quad P_{b\to a}(n) = \int_0^1 x^n P_{b\to a}(x)dx,$$

which lead to a system of ordinary linear-differential equations at the first order:

$$dM_h^a(n,t)/dt = \sum_b M_h^b(n,t)P_{b\to a}(n).$$

The exact solution for single distributions in the moments representation can be written symbolically in a matrix form (at parton level):

$$M_a^b(n,t) = (\exp P(n)t)_a^b.$$

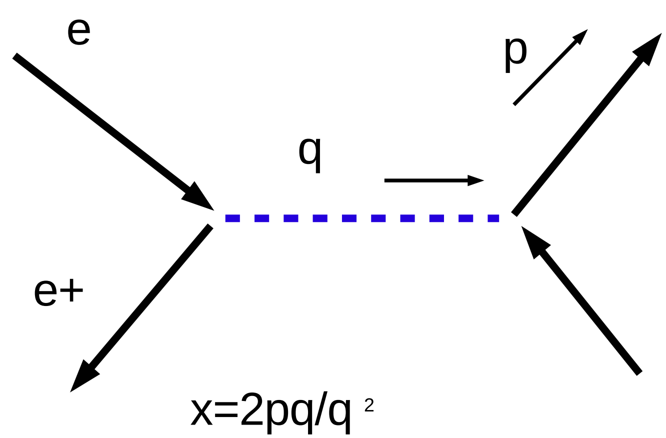
In order to obtain the distributions in x representation an inverse Mellin transformation must be performed

$$xD_h^b(x,t) = \int \frac{dn}{2\pi i} x^{-n}\ M_h^b(n,t),$$

where the integration runs along the imaginary axis to the right from all n singularities. This can be done in a general case only numerically.

$$a,b,c = q,\bar{q},g.$$

e^+e^--annihilation into q, $\bar{q}$, g



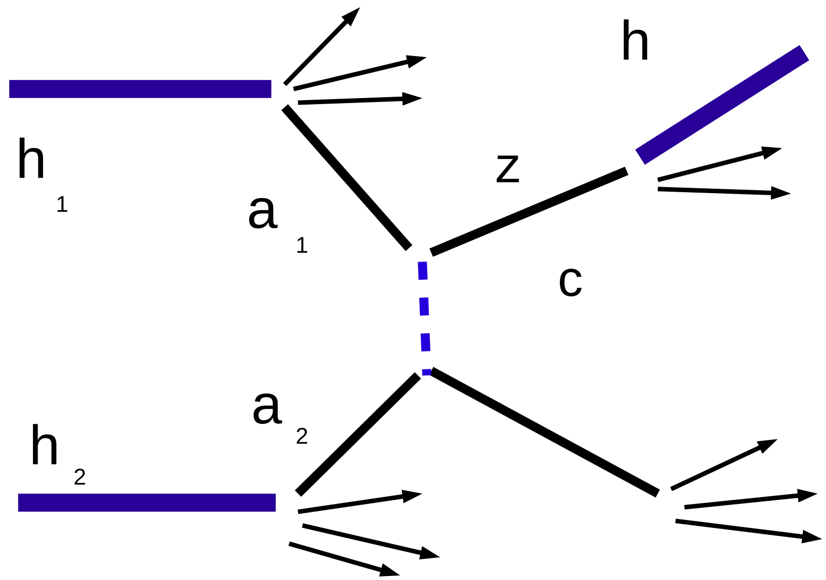
$$\frac{d\sigma(e^+e^- \rightarrow q(g) + X)}{dx} = \frac{4\pi\alpha^2}{3q^2} 3 \sum_{F=q,\bar{q}}^{2n_f} e_F^2 \overline{D}_F^{q(g)}(x, \mu^2, q^2)$$

With $\overline{D}_F^{q(g)}$ as the probability to fragment a bare parton F with the virtuality up q^2 into a dressed parton $q(g)$ having a longitudinal momentum fraction x.

Gribov-Lipatov relation at parton level:

$$D_a^b(x, \mu^2, q^2) = \overline{D}_a^b(x, \mu^2, q^2)$$

$$h_1 + h_2 \to h + X$$



$$\boldsymbol{\sigma = \sum_{a_1 a_2 c} \int dx_1 dx_2 D^{a_1}_{h_1}(x_1, q^2) D^{a_2}_{h_2}(x_2, q^2) \sigma_{a_1 a_2 \to cd} \overline{D}^h_c \left(\frac{x}{z}, q^2\right)}$$

AND ....