The ABC of Feynman integrals

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Talk given on July 2nd, 2018 at

African Conference on Fundamental Physics and Applications (ACP)

Namibian University of Science and Technology (NUST), Windhoek, Namibia

Cluster of Excellence PRiSMA Precision Physics, Fundamental Interactions and Structure of Matter

G|FK

erc European Research Council Established by the European Commission

Academic career

- 2005-2008: PhD thesis
LAPTH, Annecy (France)

- 2008-2011: Postdoc
Humboldt university, Berlin (Germany)

- 2011-2015: Long-term member
IAS, Princeton (USA)

- 2015-2018: Full professor
JGU Mainz (Germany)

- since 2018: Director
MPP Physik, Munich (Germany)

(some) funding opportunities Germany

- PhD program IMPRS
  International Max Planck Research School
  on Elementary Particle Physics

  Max-Planck-Institut
  für Physik

  www.mpp.mpg.de/en/

- French-German PhD exchange

  www.cdfa-physique.fr

- German Academic Exchange Service (DAAD)

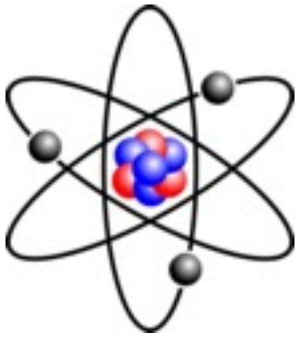
  www.daad.de/en/

Research area - Elementary particle physics

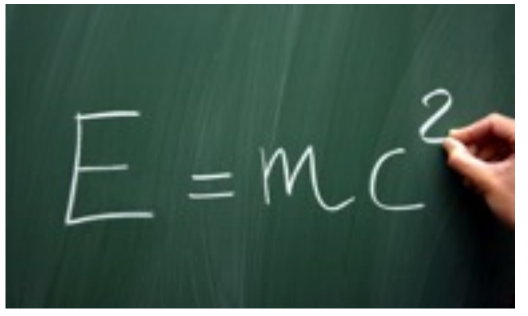
What are the fundamental constituents of nature?

What physical laws describe them?

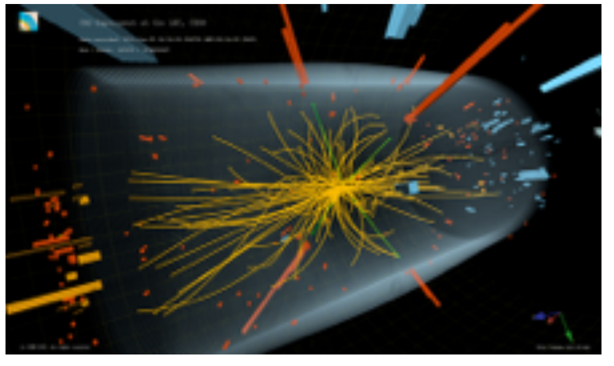
- quantum mechanics (early 20th century) describes atoms



- at higher energies, particles can be created and destroyed; described by quantum field theory

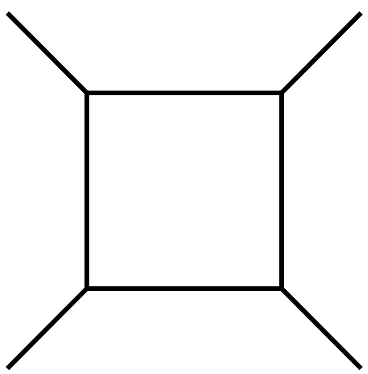


- study via scattering processes



Motivation

- scattering amplitudes needed for collider physics
- important challenge:
  Feynman loop integrals



This talk:

- novel ideas for evaluating them
- systematic approach via differential equations

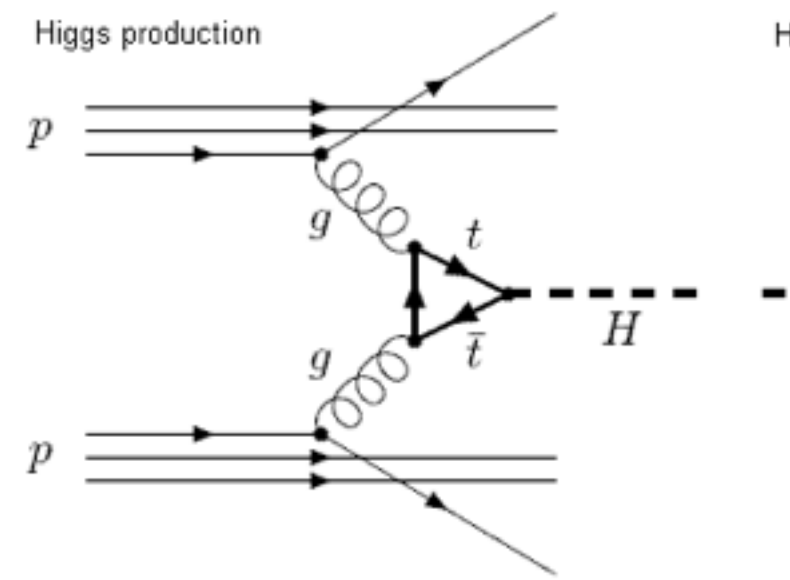
Example Feynman integral

- Integral appearing in Higgs production

$$\int \frac{d^4k}{i\pi^2} \frac{1}{(m_t^2 - k^2)(m_t^2 - (k+p_1)^2)(m_t^2 - (k-p_2)^2)} =$$

$$= -\frac{1}{2s} \log^2 \left(\frac{\sqrt{1 - 4m_t^2/s} - 1}{\sqrt{1 - 4m_t^2/s} + 1} \right)$$

$$s = (p_1 + p_2)^2$$



- One loop: only logarithm and dilogarithm needed

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_2(z) = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1 - t_2}$$

- Questions:
  - what functions will appear at higher loops?
  - how to compute them in an efficient way?

recent new developments

- special functions:
  - better understanding of multiple polylogarithms (class of iterated integrals)
  - ‘symbols’ describing main properties
- predict properties of answer from rational loop integrand
  - ‘leading singularities’
- canonical differential equations
  - simple basis of functions
  - compute integrals systematically

toy example differential equations

- basis functions:

$$\vec{f} = \begin{pmatrix} \epsilon^2 \, \mathrm{Li}_2(1-x) \\ \epsilon^1 \, \log(x) \\ \epsilon^0 \end{pmatrix} . \qquad \partial_x \vec{f} = \begin{pmatrix} \epsilon^2 \, \log(x) \\ \epsilon^1 \\ 0 \end{pmatrix} .$$

- differential equations

$$\partial_x \vec{f} = \epsilon \left[\frac{1}{x} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{1-x} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \vec{f} .$$

- all singularities manifest $x \in \{0\,,1\,,\infty\}$

- iterative solution defines $\vec{f}(x,\epsilon) = \sum_{k\geq 0} \epsilon^k f^{(k)}(x)\,.$
special functions:

- ϵ keeps track of number of integrations

Feynman integrals from canonical differential equations

[JMH, PRL 110 (2013) 25]

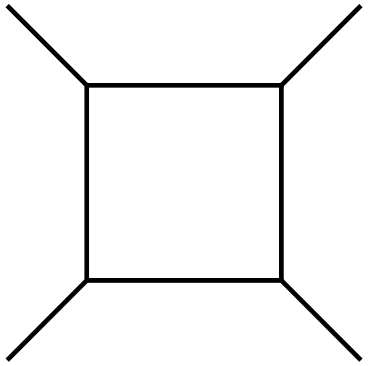
- differential equations for basis integrals $\vec{f}$
- choose canonical basis (guided by integrand properties)

Example: one dimensionless variable x ; $D = 4 - 2\epsilon$

$$\partial_x \vec{f}(x;\epsilon) = \epsilon \sum_k \frac{A_k}{x - x_k} \vec{f}(x;\epsilon)$$

- elegant description: Feynman integrals specified by:
  (1) set of 'letters' (set of singularities x_k)
  (2) set of constant matrices A_k
- defines class of special functions

Example: one-loop four-point integral



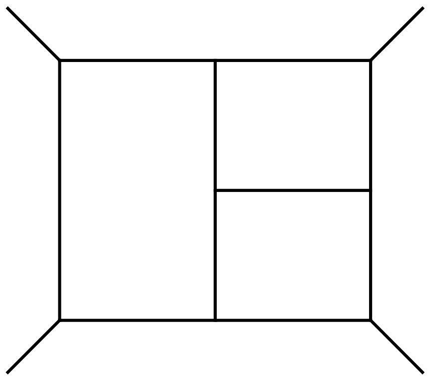
- choose basis according to [JMH, PRL 110 (2013) 25]
- differential equations $x = t/s \quad D = 4 - 2\epsilon$

$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] \vec{f}(x,\epsilon)$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

  - make singularities manifest
  - asymptotic behavior governed by matrices a, b
  - Solution: expand to any order in ϵ : $f = \sum_{k\geq 0} \epsilon^k f^{(k)}$

- Generalization to two and the loops:
same eqs., only bigger matrices!



Multi-variable case and the alphabet

- Natural generalization to multi-variable case

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

constant matrices (A_k) letters (alphabet) (α_k)

- Examples of alphabets:

4-particle on-shell $\alpha = \{x, 1+x\}$

two-variable example (from 1-loop Bhabha scattering): $\alpha = \{x\,, 1\pm x\,, y\,, 1\pm y\,, x+y\,, 1+xy\}$

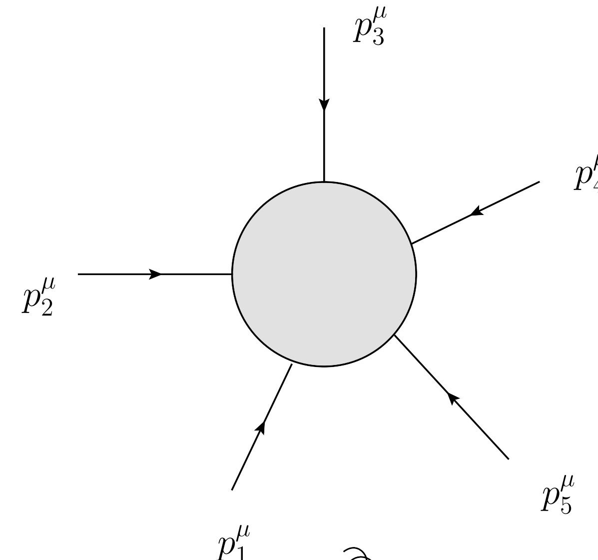
- Matrices and letters determine solution
- Immediate to solve in terms of iterated integrals

Recent application: five-particle scattering at two loops

- five independent variables

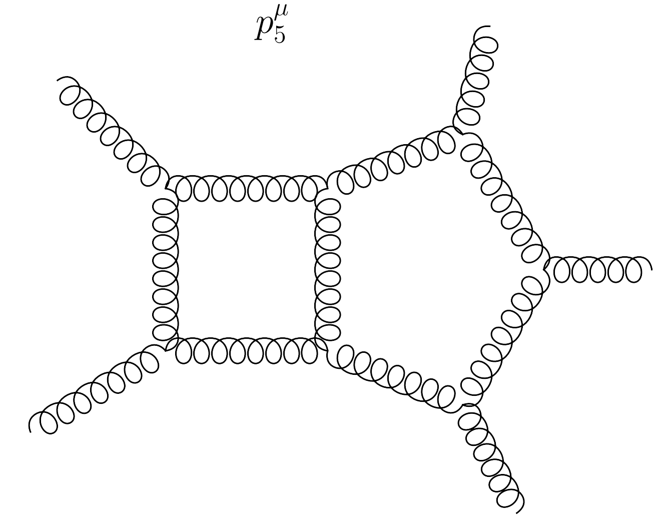
$$\vec{x} = \{s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\}$$

$$s_{ij} = (p_i + p_j)^2$$



- 61 planar master integrals

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$



- alphabet of 26 letters

- first two-loop five-particle scattering amplitude in QCD ingredient to three-jet production at NNLO

Conclusion

- new approach for computing Feynman integrals: canonical differential equations

- define novel classes of special functions

  alphabet of iterated integrals

  'symbol', identities between polylogarithms, ...

- allows to describe analytically the functions needed for multi-jet processes

  H+jets, V+jets, ...