Transverse Dynamics I

JAI Accelerator Physics Course
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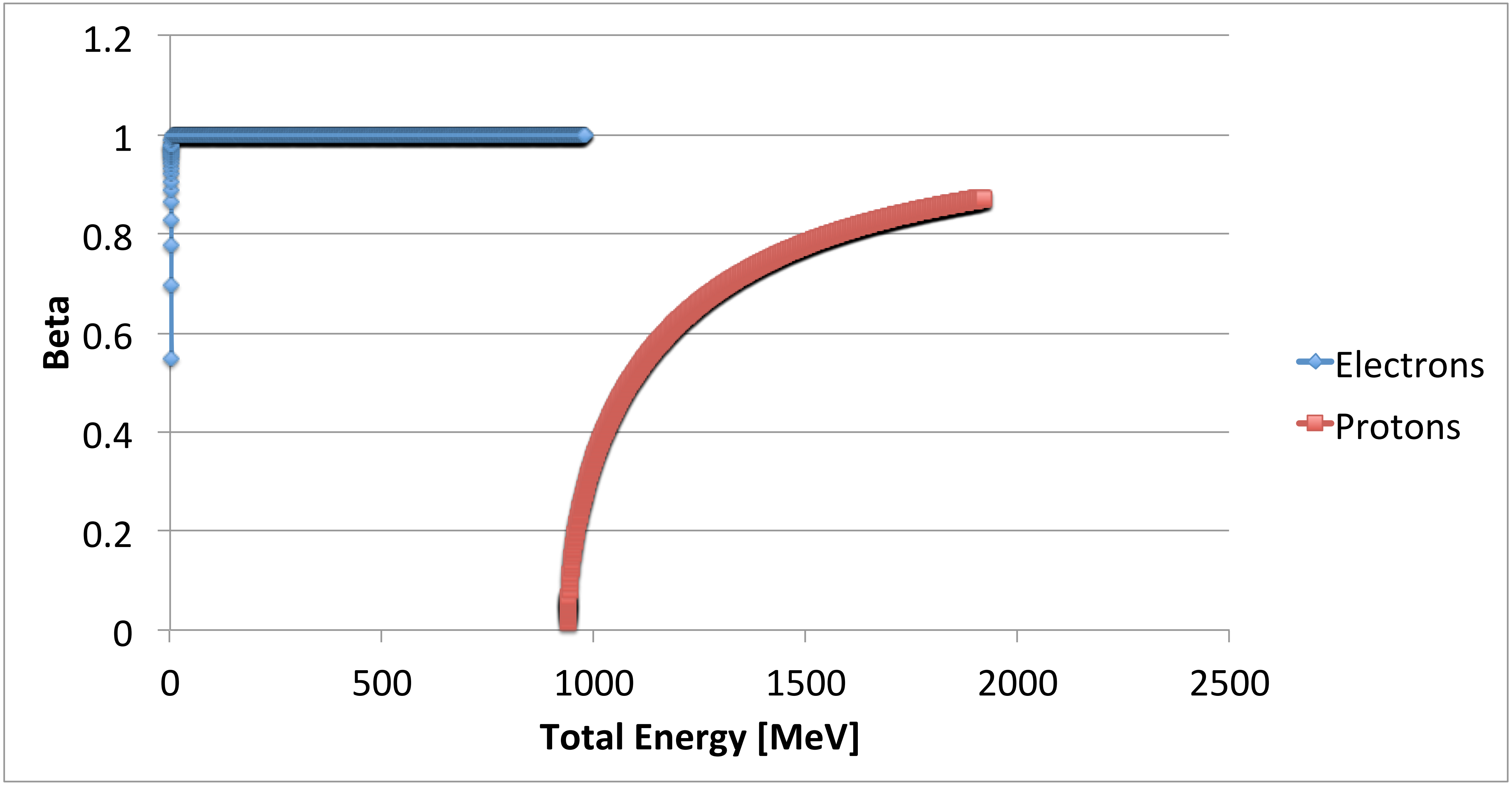
Acknowledgements

These lectures have been produced with the advice and some content from Ted Wilson, whose book is the main text for this course.

Contents

- Reminder: relativity
- Magnetic rigidity
- Transverse dynamics in a cyclotron
- AVF cyclotrons
- Synchrotrons - weak focusing
- Magnet types and multipoles
- Synchrotrons - strong focusing

Reminder: relativity



Can keep gaining in energy, but the velocity no longer increases…

Reminder: relativity

	β	cp	T	E	γ
$\beta =$	β	$\frac{cp/E}{\sqrt{(E_0/cp)^2+1}}$	$\sqrt{1-(1+T/E_0)^{-2}}$	$\sqrt{1-(E_0/E)^2} = \frac{cp}{E}$	$\sqrt{1-\gamma^{-2}}$
$cp =$	$E_0/\sqrt{\beta^{-2}-1}$ $= E\beta$	cp	$[T(2E_0+T)]^{1/2}$ $= T((\gamma+1)/(\gamma-1))^{1/2}$	$\sqrt{E^2-E_0^2} = E\beta$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$cp/\beta\gamma$ $= E(1-\beta^2)^{1/2}$	$cp(\gamma^2-1)^{-1/2}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
$T =$	$\left[\frac{1}{\sqrt{1-\beta^2}}-1\right]E_0$	$\sqrt{E_0^2+c^2p^2}-E_0$ $= cp\left((\gamma-1)/(\gamma+1)\right)^{1/2}$	T	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$(1-\beta^2)^{-1/2}$	$\frac{cp}{E_0\beta} = \left[1-\left(\frac{cp}{E_0}\right)^2\right]^{1/2}$	$1+T/E_0$	E/E_0	γ

Adapted from pp.7 Chao & Tigner, Handbook of Accelerator Physics & Engineering

Question

To calculate the bending magnetic field needed in a particular accelerator, do we care about the beam energy, velocity or momentum??

A. Kinetic Energy
B. Velocity
C. Momentum

Magnetic Rigidity

- A very useful quantity in accelerator physics, gives a measure of how hard it is to bend particles of a certain momentum

Lorentz force

$$F = q\vec{E} + q\vec{v} \times \vec{B}$$

In presence of perpendicular B field

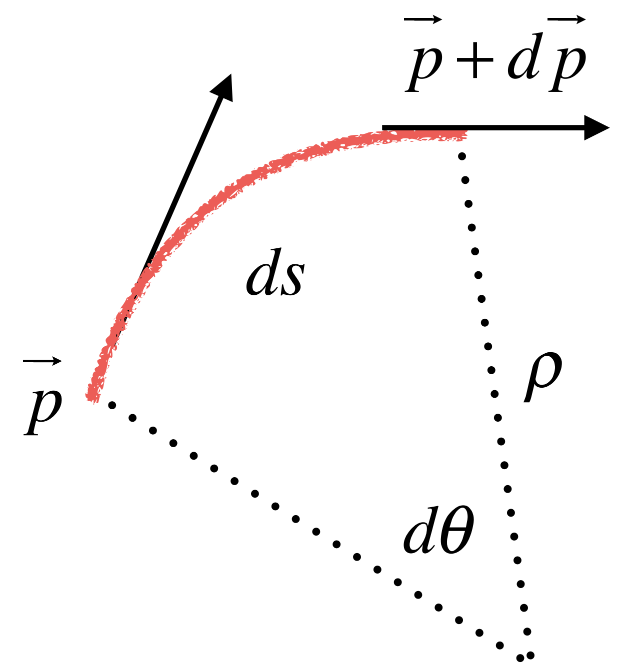
$$F = q\frac{ds}{dt}B \quad [1]$$

Using [1] and [3] we get:

$$B\rho = \frac{pc}{qc} \quad [4a]$$

In useful units:

$$B\rho[T.m] = 3.3356 \cdot pc[GeV] \quad [4b]$$



$$\frac{1}{\rho} = \frac{d\theta}{ds} \quad [2]$$

$$\frac{d\vec{p}}{dt} = |p|\frac{d\theta}{dt}$$

$$\frac{d\vec{p}}{dt} = |p|\frac{d\theta}{ds}\frac{ds}{dt} \quad \text{sub in [2]}$$

$$\frac{d\vec{p}}{dt} = \frac{|p|}{\rho}\frac{ds}{dt} \quad [3]$$

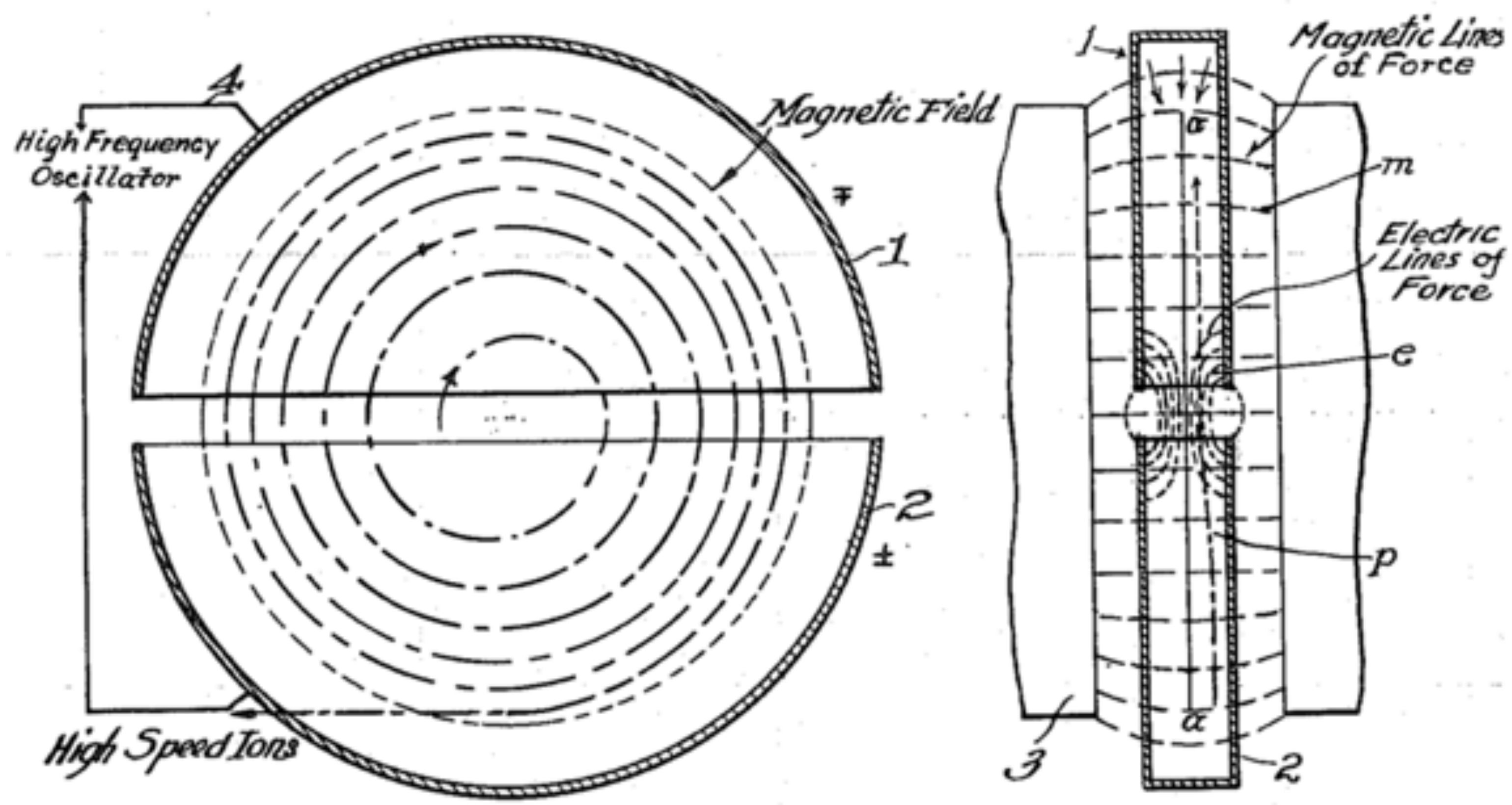
Cyclotrons - transverse

In a constant field, a charged particle executes a circular orbit, with radius ρ and frequency ω

$$\omega_0 = qB_z / m$$

$$\rho = mv / qB_z$$

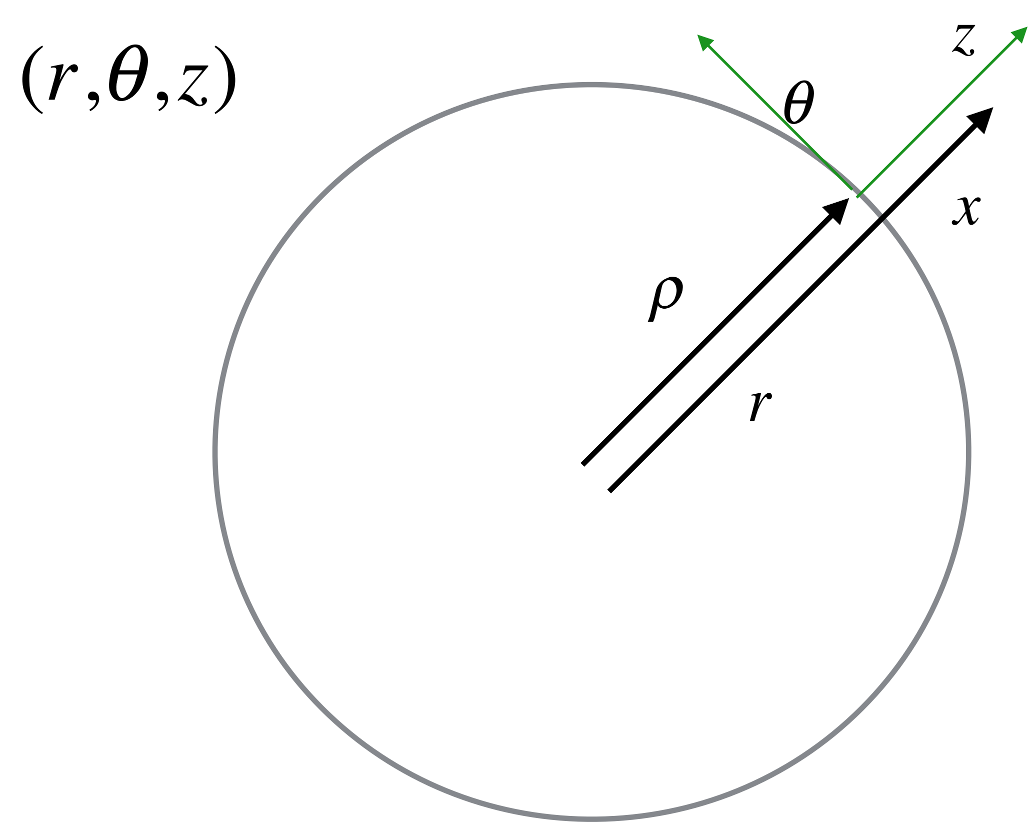
$$\omega_0 = v_\theta / \rho$$



The Cyclotron, from E. Lawrence's 1934 patent

Weak focusing in cyclotrons

Steenbeck 1935, Kerst and Serber 1941



Closed orbit in median plane

x is a small orbit deviation

$$r = \rho + x = \rho(1 + x/\rho) \quad (5)$$

Expand B field around orbit:

$$B_z = B_{z,0} + \frac{\partial B_z}{\partial x} x$$

Define field index:

$$n = -\frac{\rho}{B_{z,0}} \frac{\partial B_z}{\partial x} \quad (6)$$

Therefore:

$$B_z = B_{z,0}\left(1 - \frac{nx}{\rho}\right) \quad (7)$$

$$k = -\frac{1}{B_{z,0}\rho} \frac{\partial B_z}{\partial x}$$

nb. field index k can also be defined.

Looking at the horizontal restoring force

$$F_x = \frac{mv_\theta^2}{\rho} - qv_\theta B_z \qquad (8)$$

(centrifugal force - magnetic force)

And combining (5) and (7) we end up with (assuming $x \ll \rho$):

$$F_x = -\frac{mv_\theta^2}{\rho}\frac{x}{\rho}(1-n)$$

We can then get to this equation of motion:

$$\ddot{x} + \frac{v_\theta^2}{\rho^2}(1-n)x = 0 \qquad (9)$$

or $$\ddot{x} + \omega^2 x = 0$$

Harmonic oscillator with frequency $\omega = \omega_0\sqrt{1-n}$

For horizontal stability, we require $n < 1$

Because this focusing feature was discovered in the development of betatrons, we call these 'betatron oscillations'

For vertical stability (see later), we require $n > 0$

Alternative (equivalent) formulation…

Alternatively (cf. Ted Wilson), let's start with the equation of motion in cylindrical co-ordinates (from Lorentz force) in theta…

$$\frac{d(m\dot{\rho})}{dt} + m\rho\dot{\theta} = q[\dot{z}B_{\theta} - \rho\dot{\theta}B_z] \quad (10)$$

if particles have same velocity $\rho\dot{\theta} = v_0 = \dot{z}$

$$\frac{d}{dt}\left(m\frac{d\rho}{dt}\right) + \frac{m{v_0}^2}{\rho} + qv_0 B_z = 0$$

Substituting for small variations and changing from t to s:

$$\frac{d}{dt} = v_0\frac{d}{ds} \qquad \Delta B_z = B_z - B_0 \qquad x = \rho - \rho_0$$

$$\frac{d(m\dot{r})}{dt} - mr\dot{\theta}^2 = q\left[r\dot{\theta}B_z - \dot{z}B_{\theta}\right]$$

$$\frac{d(mr\dot{\theta})}{dt} + m\dot{r}\dot{\theta} = q\left[\dot{z}B_r - \dot{r}B_z\right]$$

$$\frac{d(m\dot{z})^2}{dt} = q\left[rB_{\theta} - r\dot{\theta}B_r\right]$$

Cylindrical co-ordinates

We get:

$$\frac{1}{mv_0}\frac{d}{ds}\left(p_0\frac{dx}{ds}\right) + \frac{x}{{\rho_0}^2} + \frac{1}{\rho_0}\frac{\Delta B_z}{B_0} = 0 \quad (11)$$

Alternative (equivalent) formulation…

Taylor expand field about the orbit…

$$B_z = B_0 + \frac{\partial B_z}{\partial x} x$$

Define field index as before

$$k = -\frac{1}{B_0 \rho} \frac{\partial B_z}{\partial x}$$

This gives horizontal focusing:

$$\boxed{\frac{1}{p_0} \frac{d}{ds} \left(p_0 \frac{dx}{ds} \right) + \left(\frac{1}{\rho^2} - k \right) x = 0}$$

Harmonic motion with oscillations per turn:

$$Q_x = \sqrt{\frac{1}{\rho^2} - k}, \qquad Q_z = \sqrt{k}$$

Weak focusing in cyclotrons

- In reality, have a slightly decreasing field with radius

$$n = \rho^2 k \qquad\qquad 0 \le n \approx -\frac{\partial B_z}{\partial x} \le 1$$

- With relativity… for isochronicity we know we need:

$$B(r) = \gamma(r) B_0 \qquad \text{because} \qquad \omega_{rev} = \frac{qB(r)}{\gamma(r) m_0}$$

ie. need an increasing field (n<0)

which is not compatible with a decreasing field, n>0

AVF cyclotron

Thomas, 1938

Increase vertical focusing by introducing hills & valleys

This introduces a variation in B_θ

We define the flutter factor

$$F = \frac{\langle B^2 \rangle - \langle B \rangle^2}{\langle B \rangle^2} \approx \frac{(B_{Hill} - B_{Valley})^2}{8\langle B \rangle^2}$$

The betatron frequency turns out to be:

$$\nu_z^2 = n + \frac{N^2}{N^2 - 1} F + \ldots > 0$$

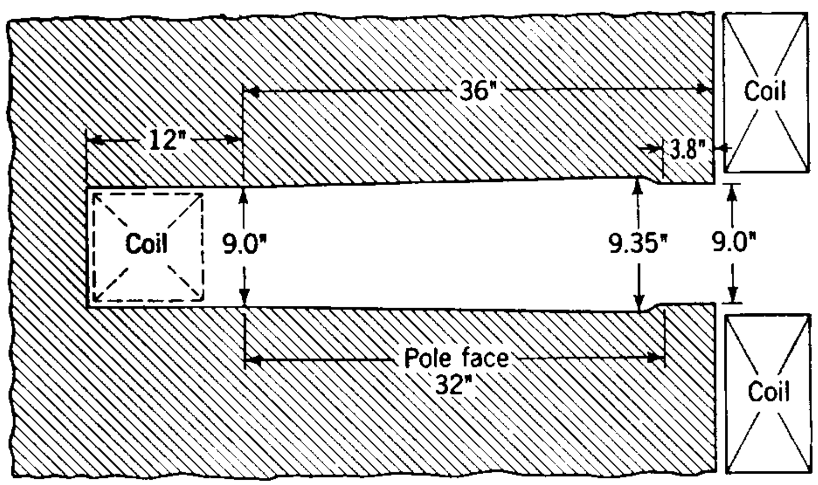
Focusing limit:

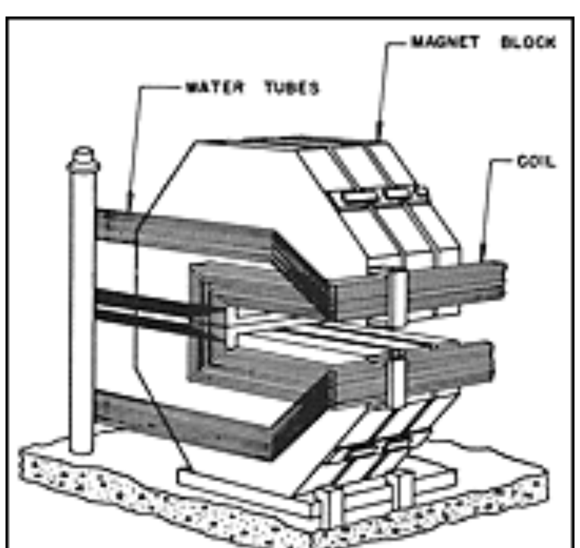
$$\frac{N^2}{N^2 - 1} F > -n = \gamma^2 - 1$$

Note: for high energies we want a large flutter factor, so B_valley = 0 -> separated sector cyclotron

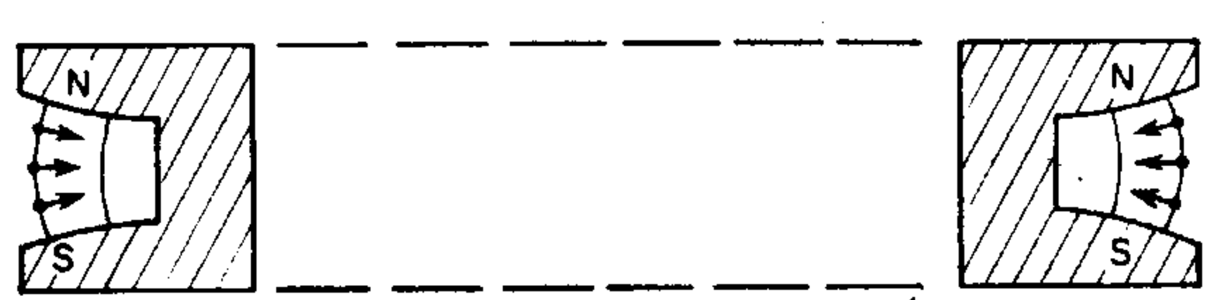
Synchrotrons

"Particles should be constrained to move in a circle of constant radius thus enabling the use of an annular ring of magnetic field … which would be varied in such a way that the radius of curvature remains constant as the particles gain energy through successive accelerations" - Marcus Oliphant, 1943

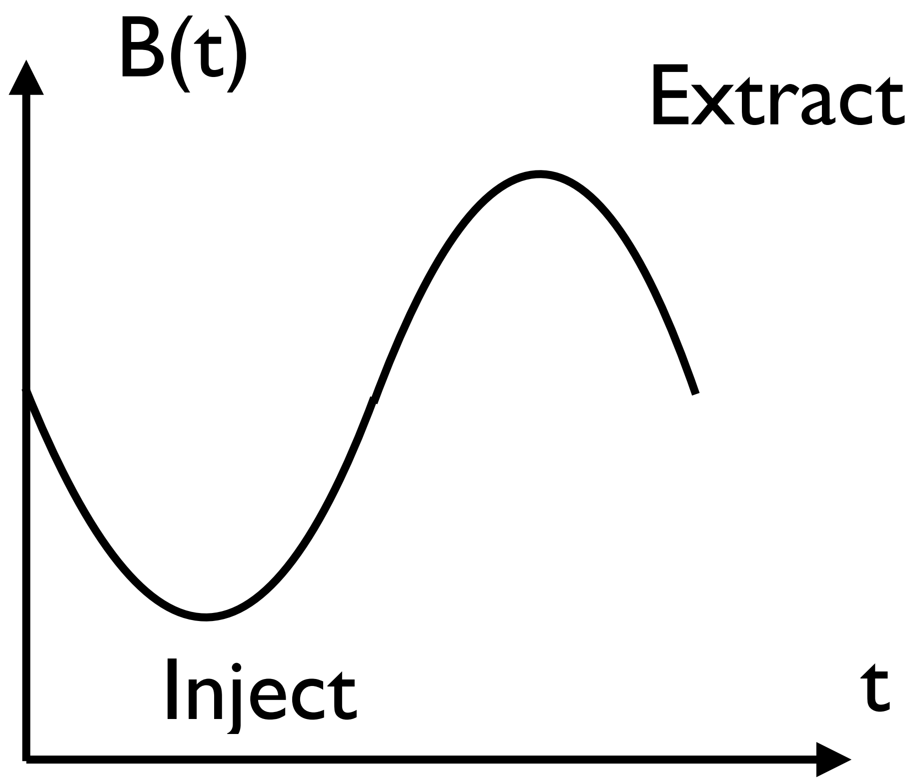




The Cosmotron, 3.3 GeV p+, BNL



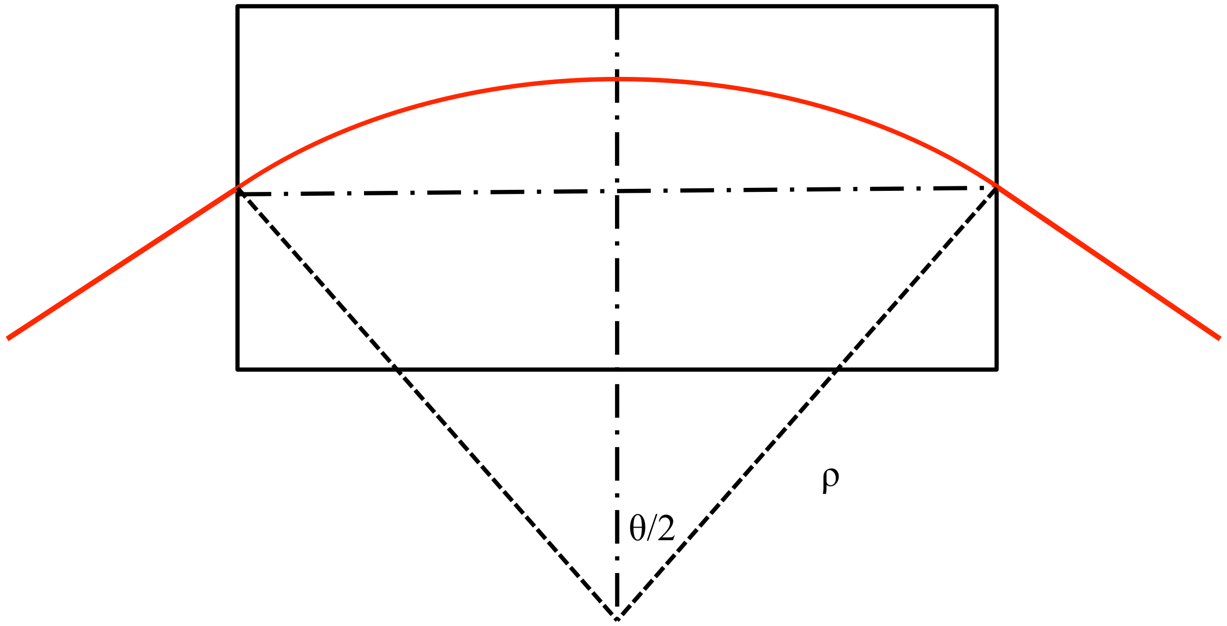
Synchrotrons



Typical synchrotron magnet cycle

Bending angle in dipole magnet

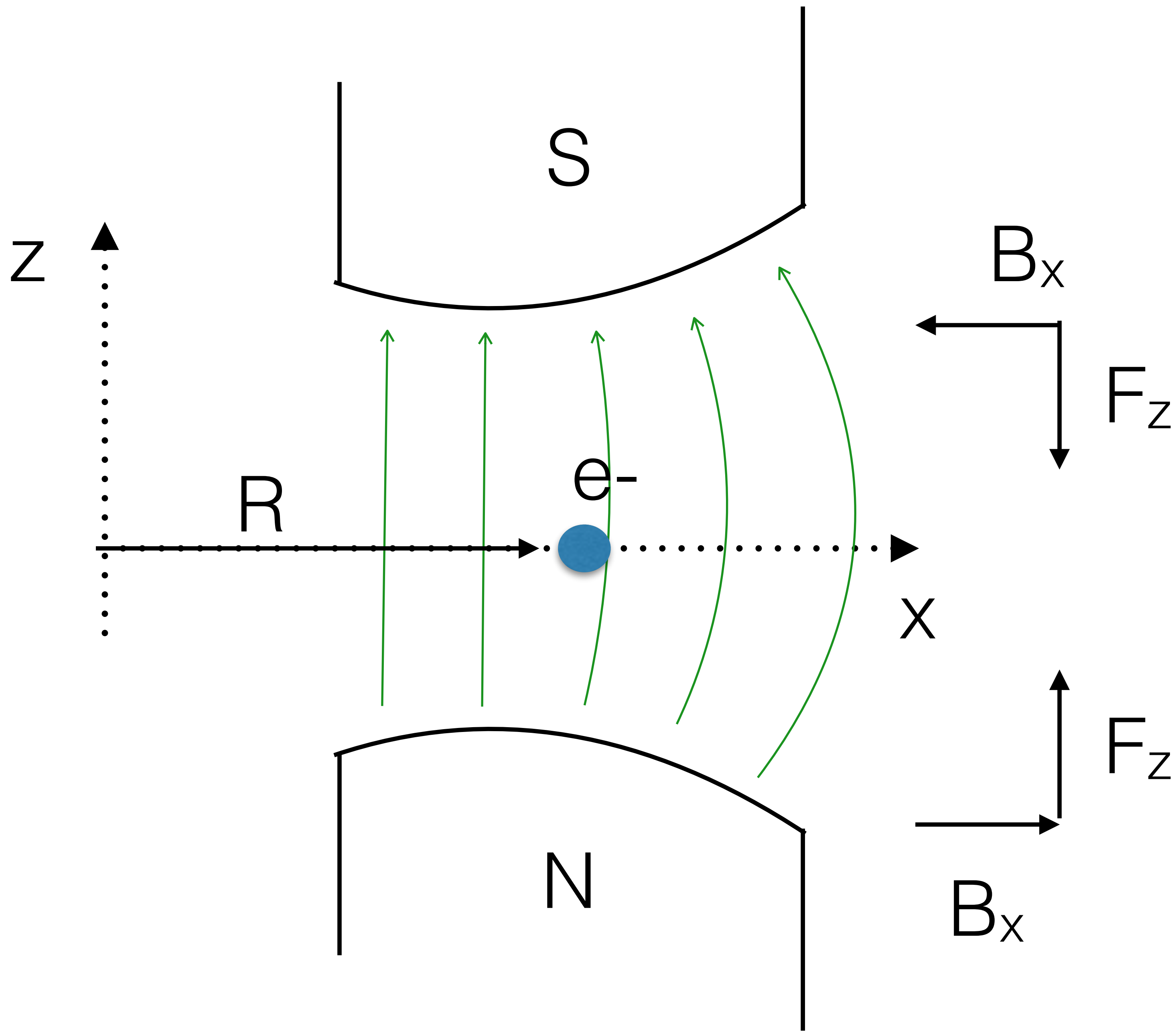
$$\sin(\theta / 2) = \frac{B(t)L}{2(B(t)\rho)} \qquad \theta \approx \frac{B(t)L}{p(t)/q}$$



Weak Focusing: Synchrotrons

- In vertical direction

For focusing in the vertical plane, we need a horizontal field component



$$\nabla \times \vec{B} = 0$$

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial r}$$

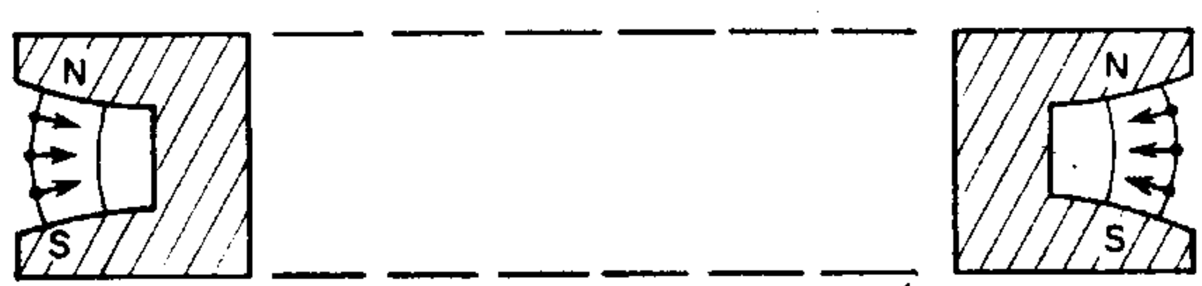
Require $\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial r} < 0$

so $n > 0$

i.e. decreasing field with radius

For both planes: $0 < n < 1$

The Cosmotron, 3.3 GeV p+, BNL



- Vertical focusing comes from the curvature of the field lines when the field falls off with radius (positive n-value)

- Horizontal focussing from the curvature of the path, sometimes called ‘body focusing’

- The negative field gradient defocuses horizontally and must not be so strong as to cancel the path curvature effect

Question

What do you think would happen if every other magnet was reversed in field gradient direction?

A. Nothing
B. Focusing would be weaker (i.e. cancels out)
C. Focusing would be stronger (i.e. adds somehow)

(From last lecture: this is re. longitudinal motion)

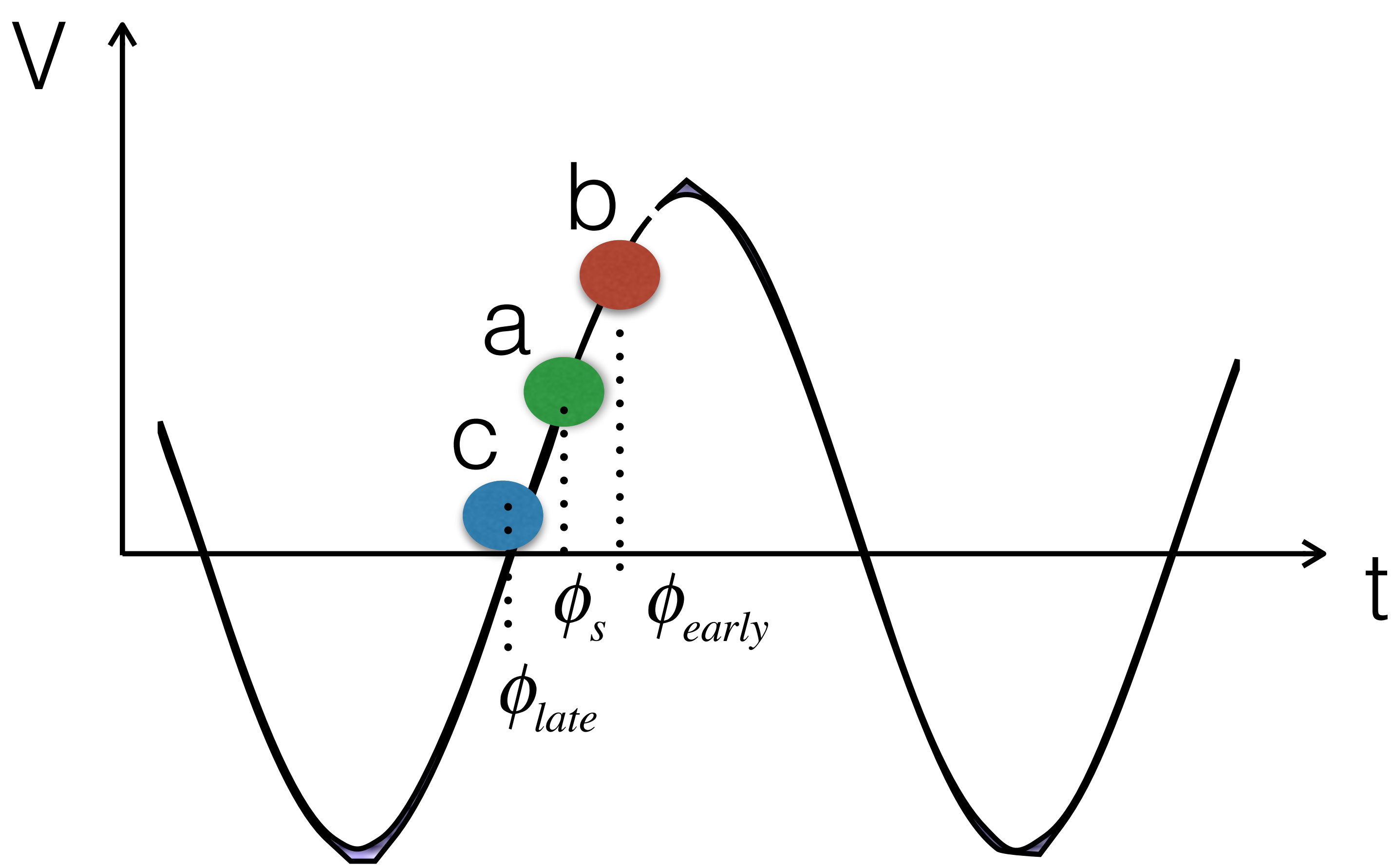
Phase stability

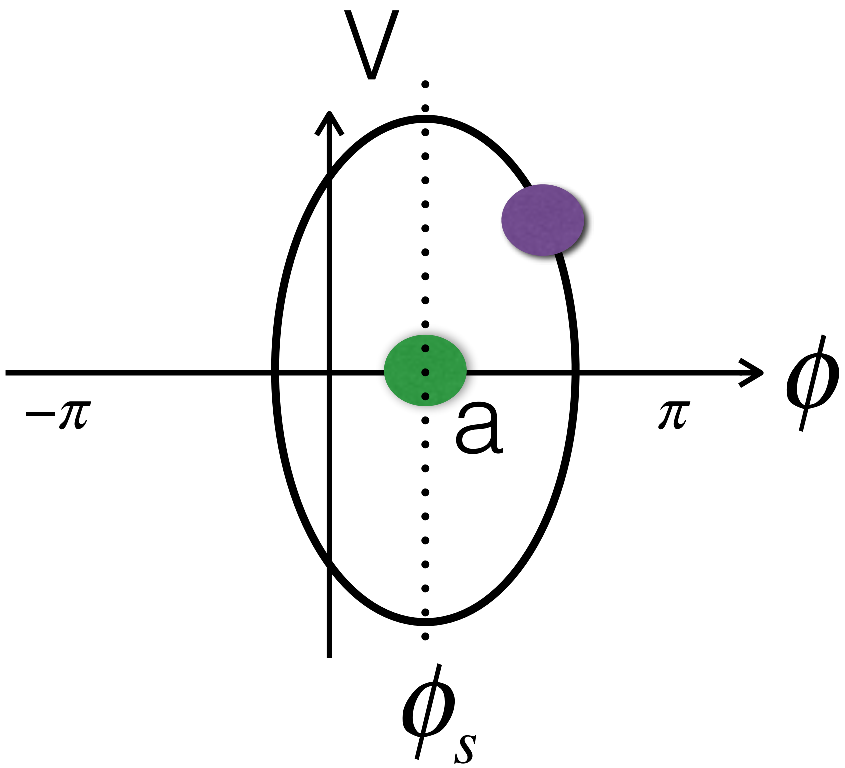
a - synchronous

b - arrives early, sees higher voltage, goes to larger orbit -> arrives later next time

c - arrives late, sees lower voltage, goes to smaller orbit -> arrives earlier next time

$$V = V_0 \sin(2\pi f_a + \phi_s)$$





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Magnetic Fields

- Maxwell's equations, time independent, no sources, so:

$$\vec{J} = 0$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\nabla \times \vec{B} = 0$$

$$\nabla \cdot \vec{B} = 0$$

- Consider a constant vertical field B_z, and

$$B_y + iB_x = C_n (x + iy)^{n-1}$$

- n is an integer > 0, C is a complex number
- (real part understood)

Now apply $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ to each side of $B_y + iB_x = C_n(x+iy)^{n-1}$

LHS:

$$= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} + i\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$= \left[\nabla \times \vec{B}\right]_z + i\nabla \cdot \vec{B}$$ Where we know B_z is constant.

RHS:

$$= (n-1)(x+iy)^{n-2} + i^2(n-1)(x+iy)^{n-2} = 0$$

$\therefore \quad \nabla \times \vec{B} = 0$ and $\nabla \cdot \vec{B} = 0$

So we find that as expected, the field $B_y + iB_x = C_n(x+iy)^{n-1}$ satisfies Maxwell's equations in free space

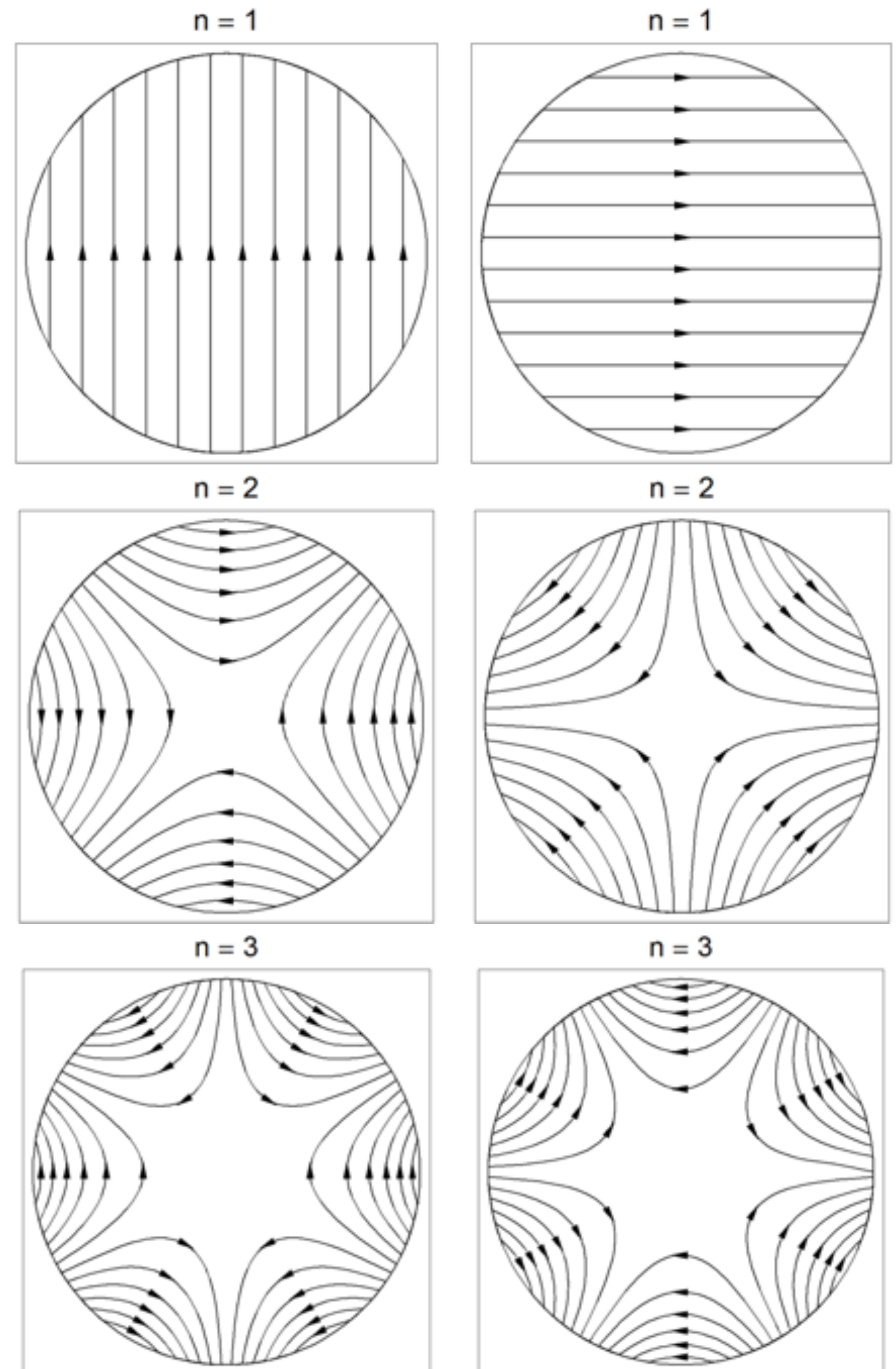
Multipole fields

In the usual notation:

$$B_y + iB_x = B_{ref} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{ref}} \right)^{n-1}$$

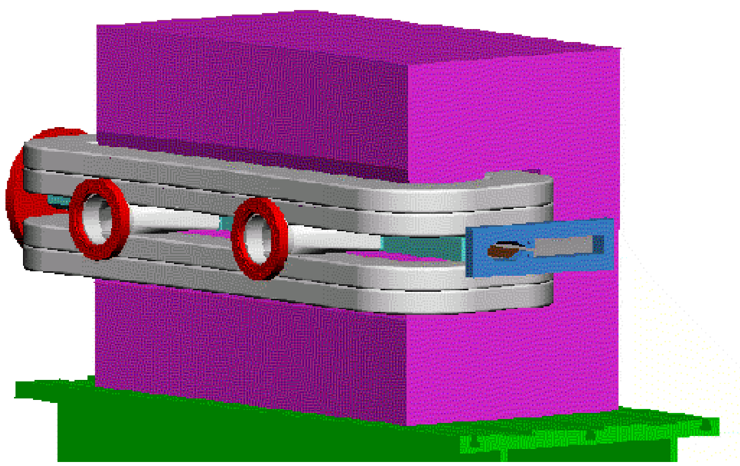
b_n are "normal multipole coefficients" (LEFT)
and a_n are "skew multipole coefficients" (RIGHT)
'ref' means some reference value

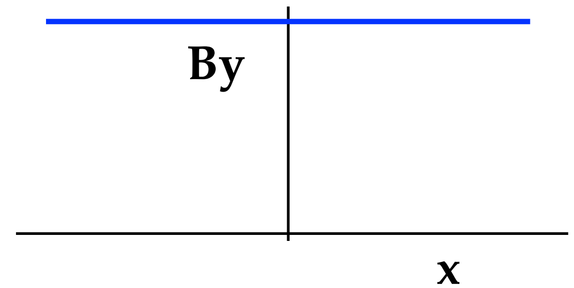
n=1, dipole field
n=2, quadrupole field
n=3, sextupole field



Images: A. Wolski, https://cds.cern.ch/record/1333874

Multipole Magnets





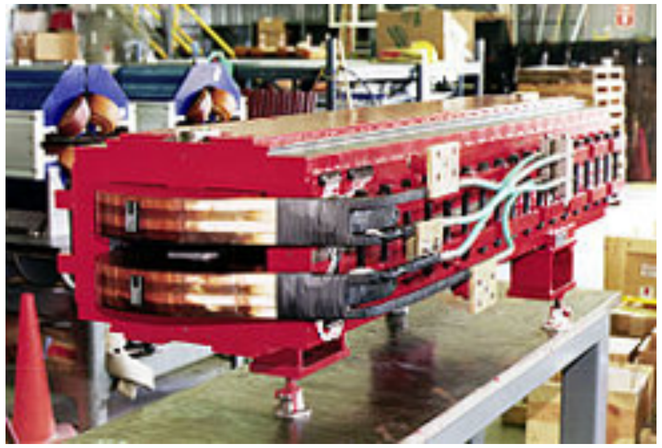
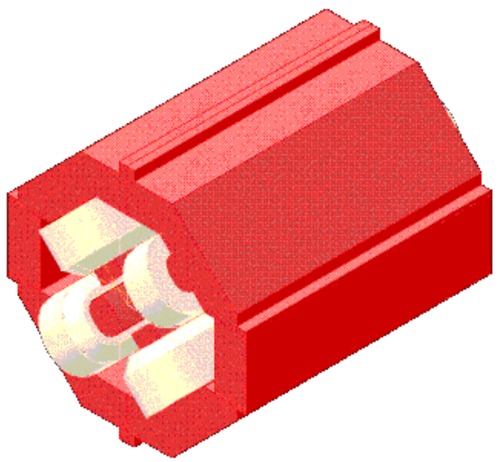
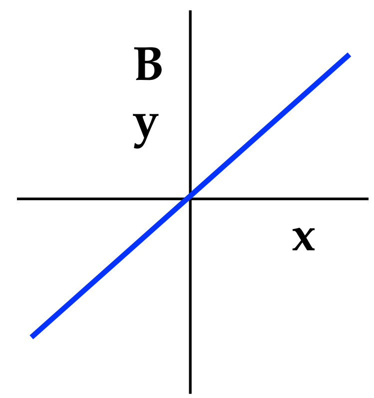


Image: Wikimedia commons





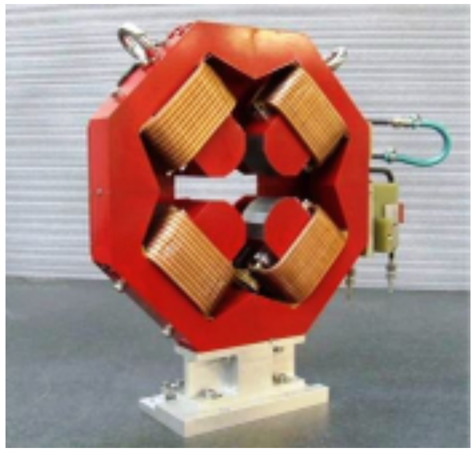
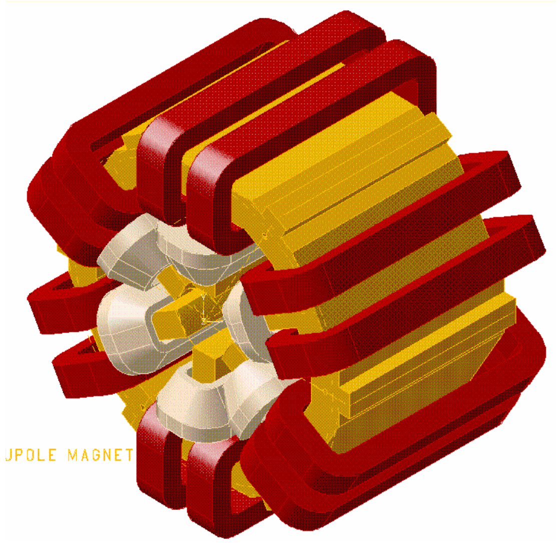
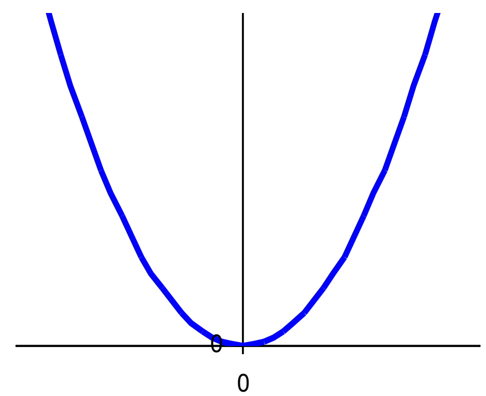


Image: STFC





Images: Ted Wilson, JAI Course 2012

Image: Danfysik

Combined function magnets

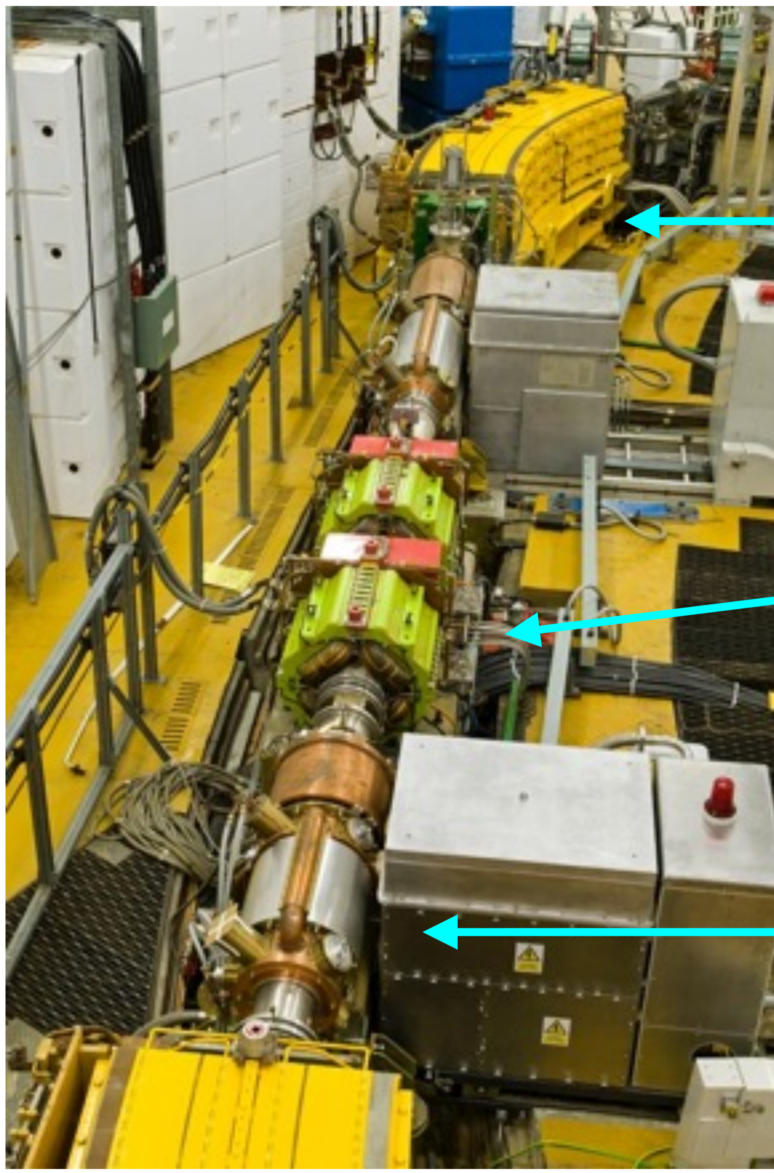
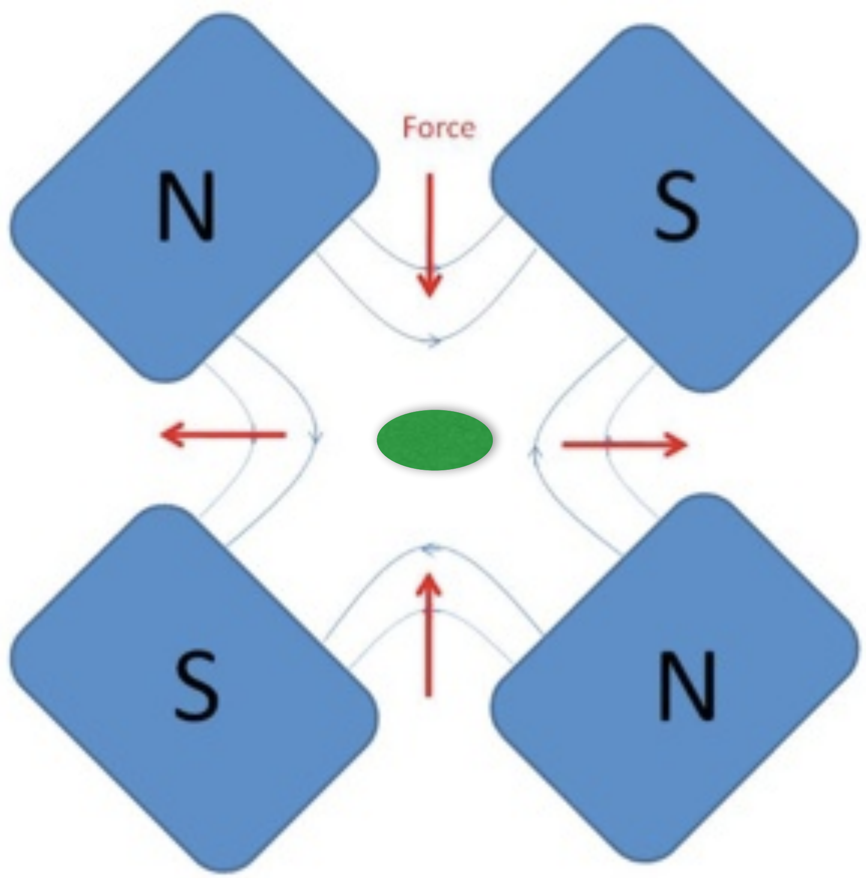
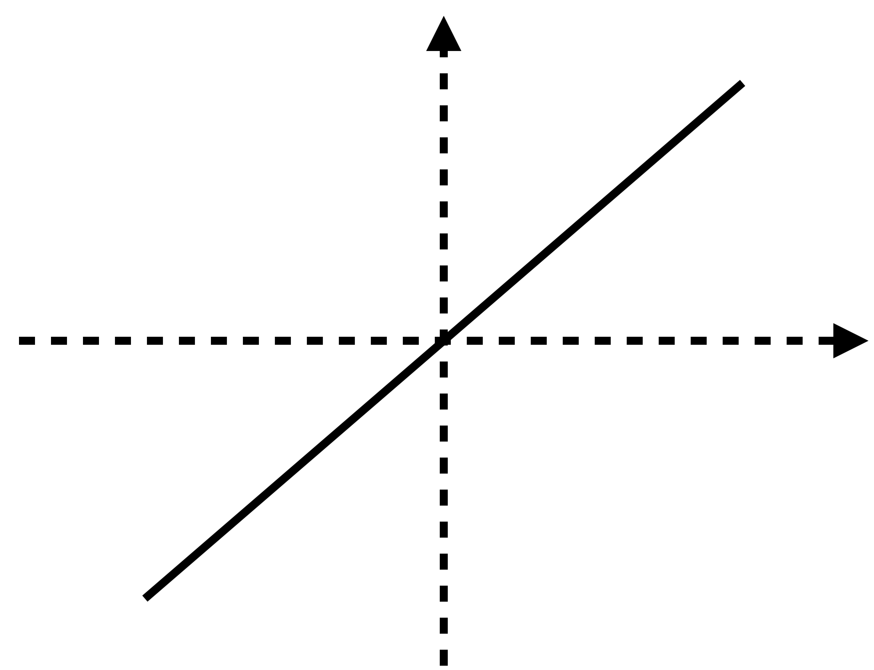


Image courtesy of ISIS, STFC

Quadrupole focusing



$$B_y = gx$$



$$k = \frac{g}{p/q}$$

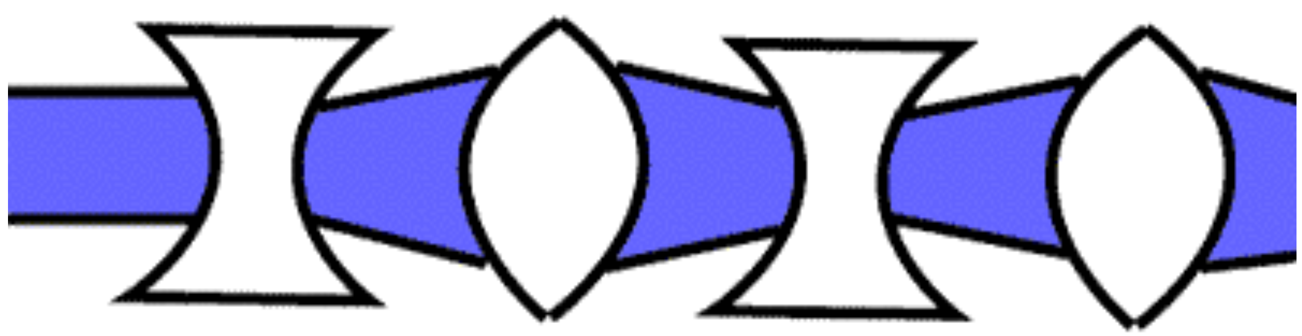
'normalised gradient' of quad

$$\frac{1}{f} = \frac{L(dB(t)/dx)}{p(t)/q}$$

'focal length'

Synchrotrons - Alternating Gradient

- An issue: for large R the deviations from ideal orbit get very large. This meant large aperture and expensive magnets.
- Greater focusing was needed in both horizontal and vertical…
- *“What if some of the magnets in the cosmotron were reversed?”*



E. Courant realised that the focusing would be STRONGER & the magnets could be SMALLER!

- 1952: Courant, Livingston, Snyder publish about strong focusing

- 1954: Wilson et al. build first synchrotron with strong focusing for 1.1MeV electrons at Cornell, 4cm beam pipe height, only 16 Tons of magnets.

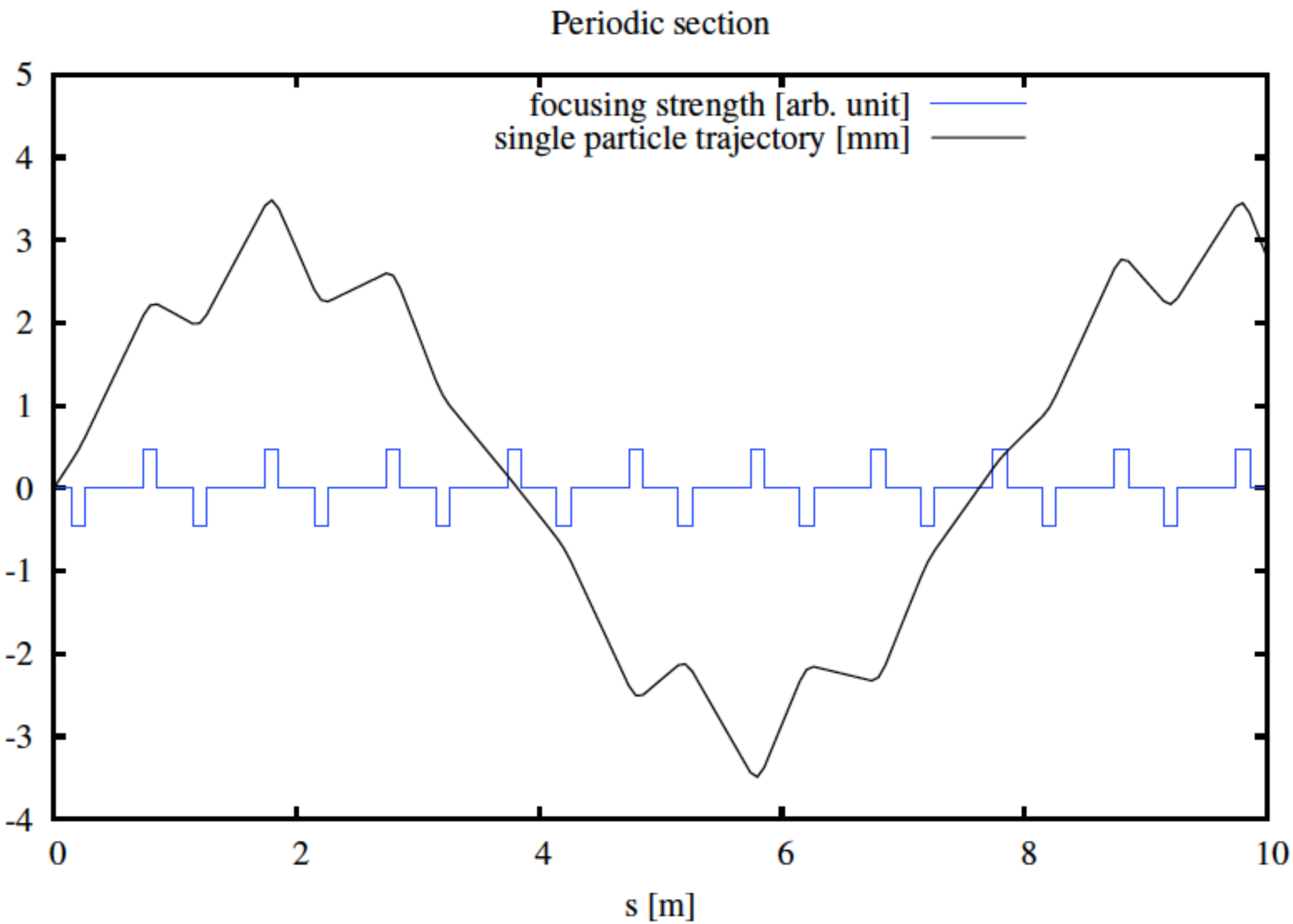
- 1959: CERN builds the PS for 28GeV after proposing a 5GeV weak focusing accelerator for the same cost (PS is still in use)

Historical note: Nicholas Christofilos

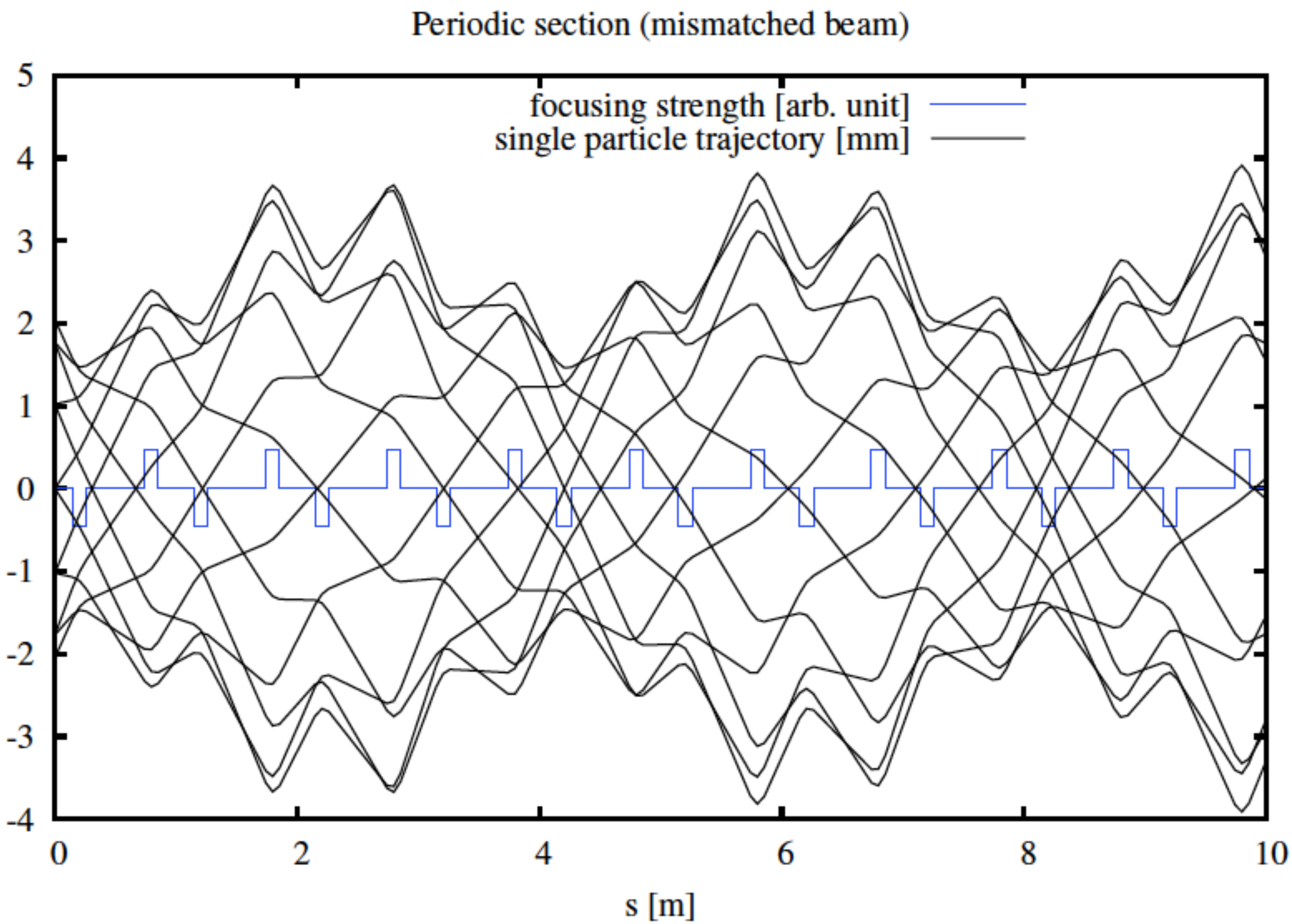
Greek physicist, had the AG idea in 1949, opted to patent it instead of publishing. He is often forgotten in physics books…

http://www.google.com/patents?vid=2736799

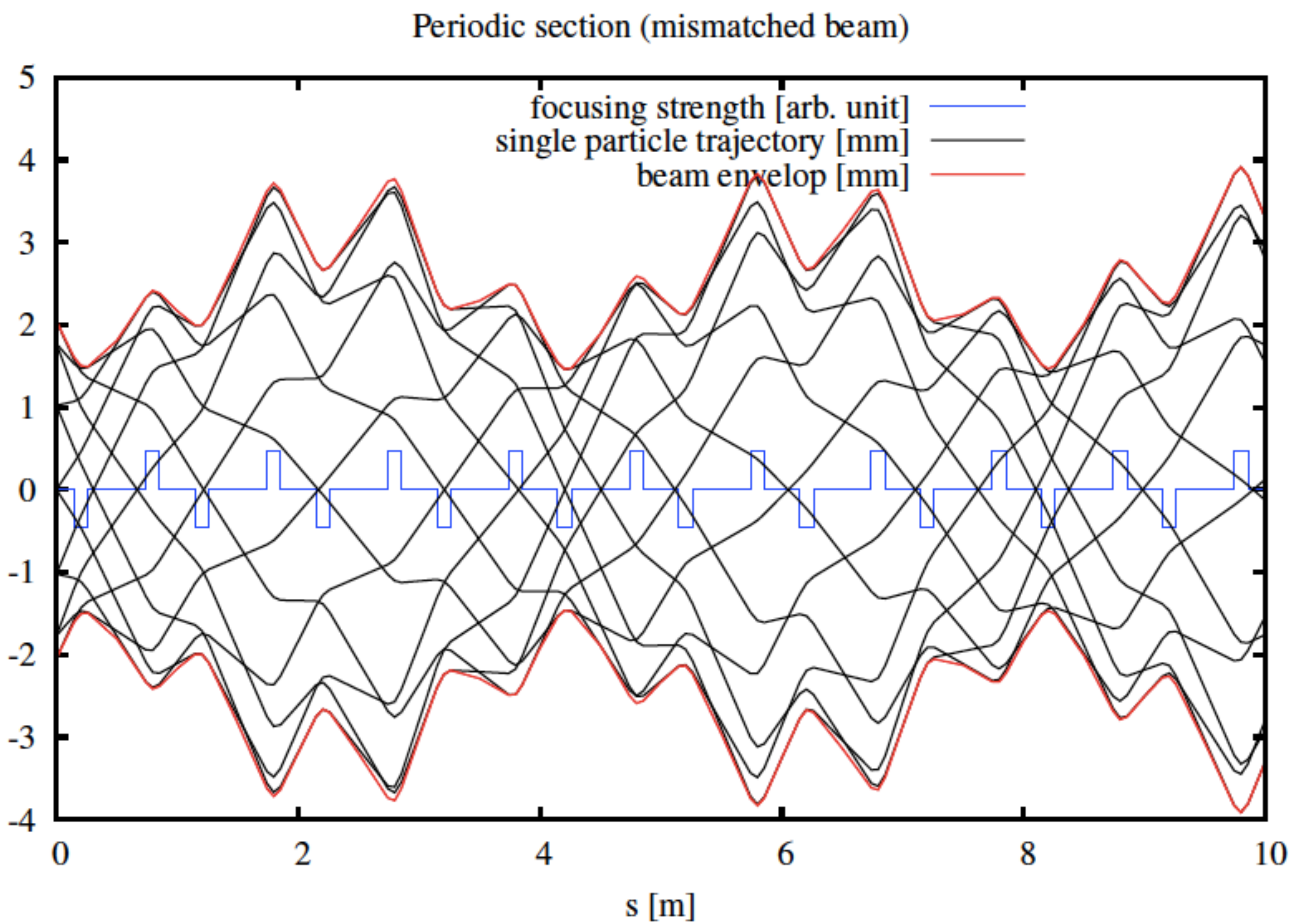
Particle in AG focusing



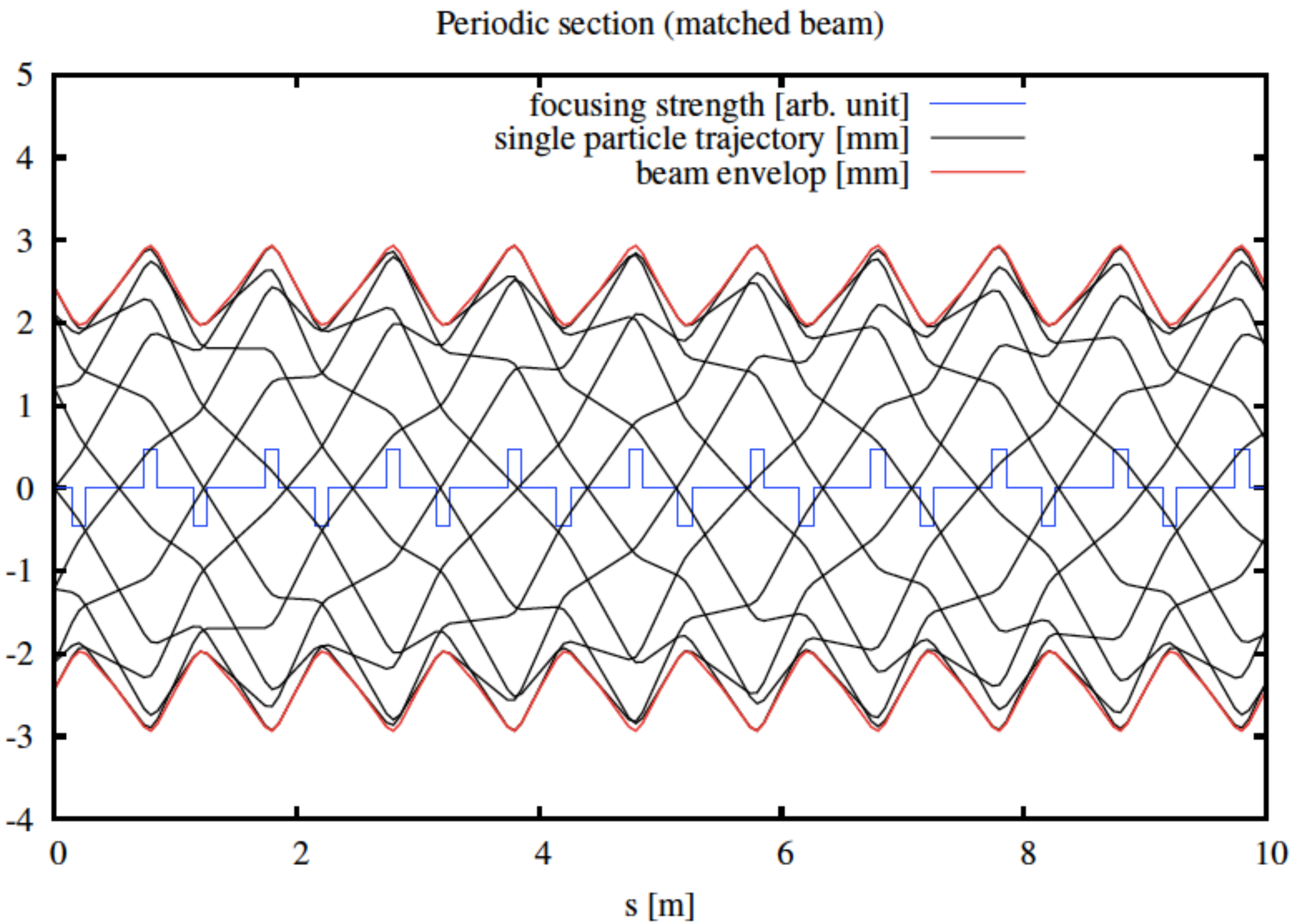
Particle in AG focusing



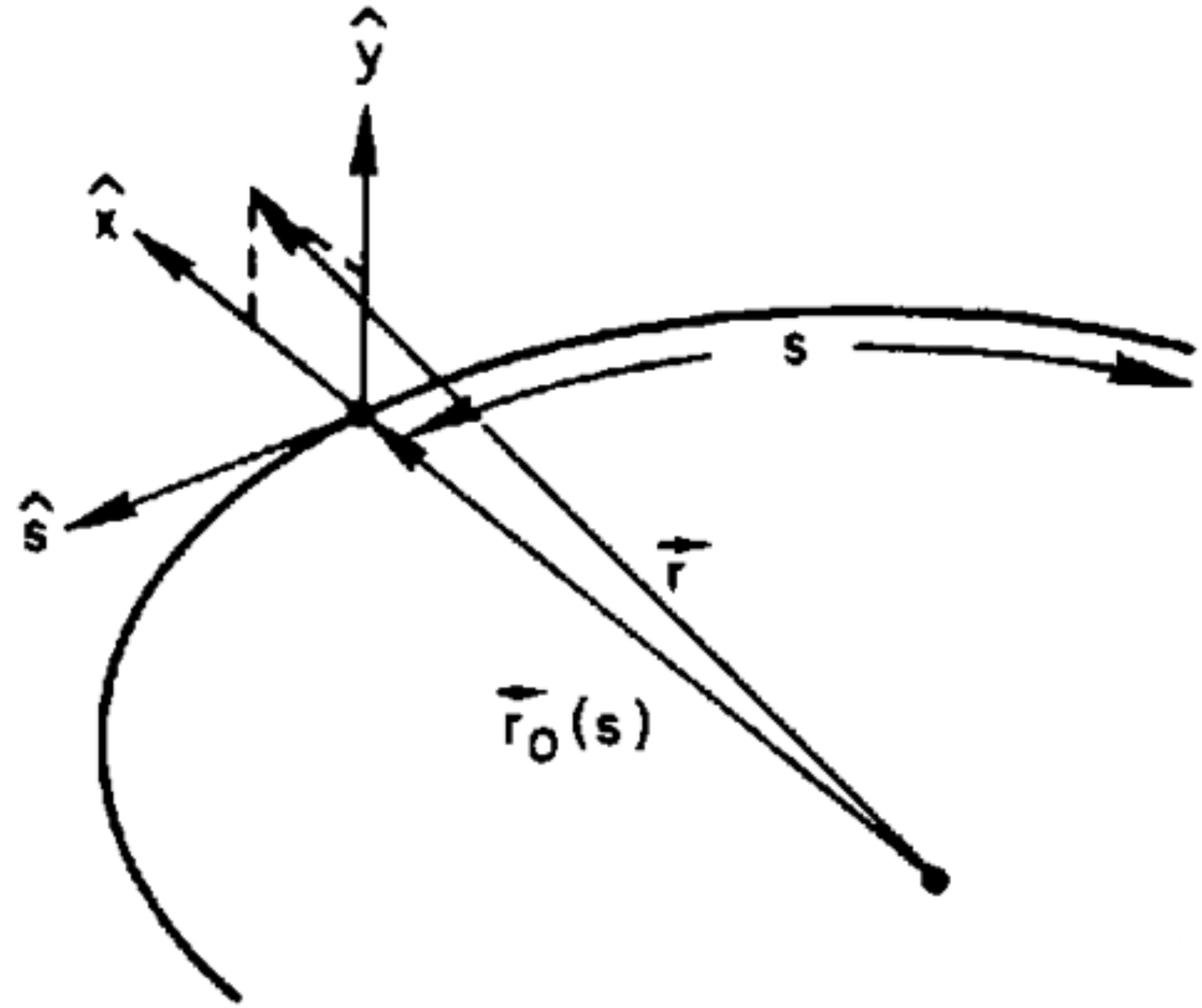
Particle in AG focusing



Particle in AG focusing



Transverse co-ordinates



Particle motion is described with respect to a **reference orbit** in the non-inertial frame (x, y, s). This co-ordinate system is known as *Frenet-Serret*

Hill's Equation (a first look)

Hill's equation is a linearised equation of motion describing particle oscillations:

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Where k changes along the path, and

$$k_x(s) = \frac{1}{\rho^2} - \frac{B_1(s)}{B\rho} \qquad k_y(s) = \frac{B_1(s)}{B\rho} \qquad B_1(s) = \partial B_y / \partial x$$

evaluated at the closed orbit

Focusing functions are periodic over length L , ie. $K_{x,y}(s+L) = K_{x,y}(s)$

nb. In a quadrupole: $k_x(s) = -\frac{B_1(s)}{B\rho}$

Following similar notation to S. Y. Lee, Accelerator Physics, pp.41

E. D. Courant and H. S. Snyder, "Theory of the alternating-gradient synchrotron," Annals of Physics, vol. 3, no. 1, pp. 1–48, 1958.

Solution of Hill's equation

(More next lecture…)

$$x = \sqrt{\beta(s)}\sqrt{\varepsilon}\sin[\phi(s) + \phi_0]$$

initial phase

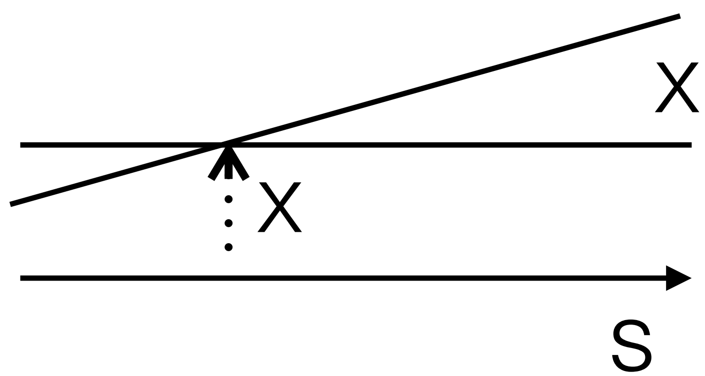
phase

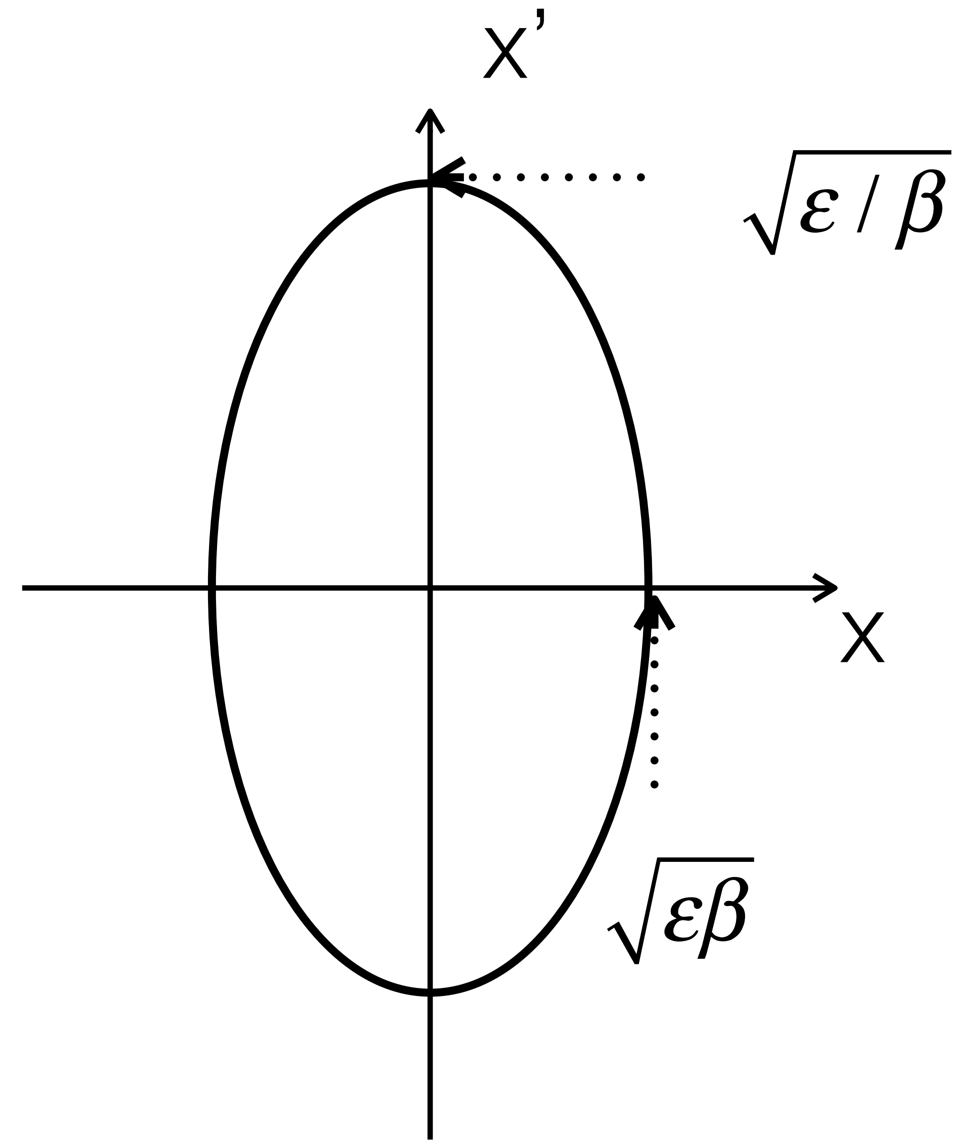
betatron function
property of the machine
(not the beam)

emittance
(property of beam)

phase advance 'tune' $\phi = \int \frac{ds}{\beta(s)}$

Transverse 'phase space' ellipse





Ellipse area:

$$A = \pi\sqrt{\varepsilon\beta}\sqrt{\varepsilon / \beta} = \pi\varepsilon$$

Ellipse can change shape but not area!
Emittance is conserved. (cf. 'Louiville's theorem')

Topics Covered

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