Transverse Dynamics II

JAI Accelerator Physics Course
Michaelmas Term 2017

Dr. Suzie Sheehy
Royal Society University Research Fellow
University of Oxford

Acknowledgements

These lectures have been produced with the advice and some content from Ted Wilson, whose book is the main text for this course.

Also see:
S. Y. Lee, Accelerator Physics

R. Bartmann course in Advanced Accelerator Physics
http://lin12.triumf.ca/text/PHYS555B207-2014/

Reminder: last lecture

- Reminder: relativity
- Magnetic rigidity
- Transverse dynamics in a cyclotron
- AVF cyclotrons
- Synchrotrons - weak focusing
- Magnet types and multipoles
- Synchrotrons - strong focusing

Contents

- Equations of motion in transverse co-ordinates
- Check Solution of Hill's equation
- Transfer matrices
- Stability and AG focusing
- Physical meaning of tune and beta

Transverse Motion

Hamiltonian for particle motion

$$H = e\phi + c[m^2c^2 + (\vec{P} - e\vec{A})^2]^{1/2}$$

$\vec{P} = \vec{p} + e\vec{A}$ is the canonical momentum

$\vec{p}$ is the mechanical momentum

Hamilton's equations of motion

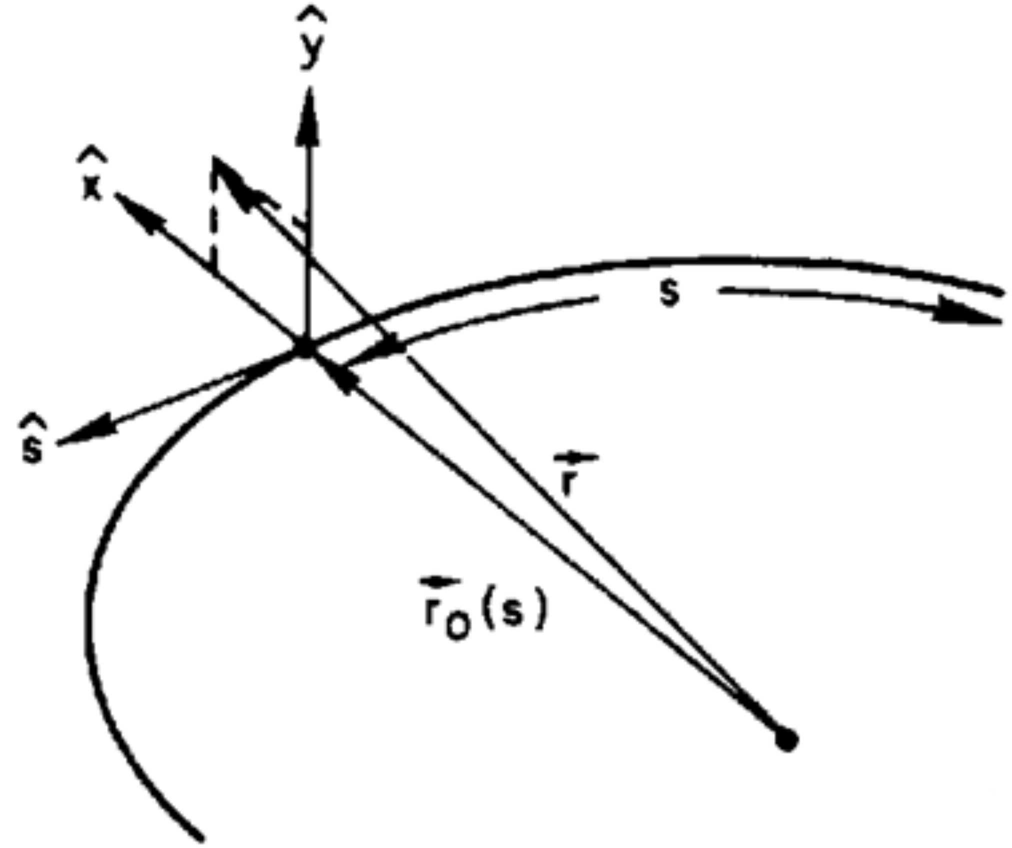
$$\dot{x} = \frac{\partial H}{\partial P_x}, \dot{P}_x = -\frac{\partial H}{\partial x}, etc...$$

(nb. dot denotes derivative wrt time)

nb. For much of the early part of this lecture I follow S. Y. Lee, Accelerator Physics, Chapter 2.

In accelerator physics we ask: "What are the particles' generalized coordinates when they reach a certain point in space?"

- *First, we convert to 'Frenet-Serret' co-ordinate system*



Particle motion is described with respect to a **reference orbit** in the non-inertial frame (x, y, s). This co-ordinate system is known as *Frenet-Serret*

- *First, we convert to 'Frenet-Serret' co-ordinate system*

$\hat{s}(s) = \frac{d\vec{r}_0(s)}{ds}$ Tangent unit vector to closed orbit

$\hat{x}(s) = -\rho(s)\frac{d\hat{s}(s)}{ds}$ Unit vector perpendicular to tangent vector

$\hat{y}(s) = \hat{x}(s) \times \hat{y}(s)$ Third unit vector…

Particle trajectory: $\vec{r}(s) = \vec{r}_0(s) + x\hat{x}(s) + y\hat{y}(s)$

nb. the reference frame moves WITH the particle

We perform a canonical transformation using the generating function:

$$F_3(\vec{P};x,s,y) = -\vec{P}.[\vec{r_0}(s) + x\hat{x}(s) + y\hat{y}(s)]$$

(note: P is momentum in cartesian system)

To obtain the Hamiltonian:

$$H = e\phi + c[m^2c^2 + \frac{(p_s - eA_s)^2}{(1 + x/\rho)^2} + (p_x - eA_x)^2 + (p_y - eA_y)^2]^{1/2}$$

- *Next, we change the independent variable from t to s*

The new conjugate phase space variables are $x, p_x; y, p_y; t, -H$

And the new Hamiltonian (s-dependent) is $\tilde{H} = -p_s$

$$\boxed{\tilde{H} = -(1 + x/\rho)\left[\frac{(H - e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]^{1/2} - eA_s}$$

Which is time-independent (if also ϕ, A are time-independent)

Expanding the Hamiltonian to second order in p_x, p_y

$$\tilde{H} \approx -p(1 + x/\rho) + \frac{1 + x/\rho}{2p}\left[(p_x - eA_x)^2 + (p_y - eA_y)^2\right]^{1/2} - eA_s$$

$H - e\phi = E$ is the total particle energy

$p = \sqrt{E^2/c^2 - m^2c^2}$ is the total particle momentum

Getting to Hill's equation (1)

Hamilton's equations of motion* are:

$$x' = \frac{\partial \tilde{H}}{\partial p_x} \qquad p_x' = -\frac{\partial \tilde{H}}{\partial x} \qquad\qquad y' = \frac{\partial \tilde{H}}{\partial p_y} \qquad p_y' = -\frac{\partial \tilde{H}}{\partial y}$$

With transverse magnetic fields we showed last time scaled & in (x,s,y) :

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

$$B_x = -\frac{1}{(1+x/\rho)}\frac{\partial A_s}{\partial y} \qquad B_y = -\frac{1}{(1+x/\rho)}\frac{\partial A_s}{\partial x}$$

Betatron equations of motion become: (neglect higher order terms)

$$x'' - \frac{\rho + x}{\rho^2} = \frac{B_y}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2 \qquad\qquad y'' = -\frac{B_x}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2$$

*neglecting synchrotron motion

Getting to Hill's equation (2)

So we have these equations:

$$x'' - \frac{\rho + x}{\rho^2} = \frac{B_y}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2 \qquad y'' = -\frac{B_x}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2$$

Expand the B field to first order in x,y:

$$B_y = -B_0 + \frac{\partial B_y}{\partial x}x \qquad B_x = \frac{\partial B_y}{\partial x}y$$

$$\frac{B_0}{B\rho} = \frac{1}{\rho}$$ ie. dipole field defines the closed orbit

$$x'' + K_x(s)x = 0 \qquad K_x = 1/\rho^2 - K_1(s)$$

$$y'' + K_y(s)y = 0 \qquad K_y = K_1(s)$$

$$K_1(s) = \frac{1}{B\rho}\frac{\partial B_1}{\partial x}$$

nb. in a quadrupole $K_x = -K_y$

Hill's Equation

Hill's equation is a linearised equation of motion describing particle oscillations:

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Question: we have ended up with linear equations of motion because we took 2nd order Hamiltonian only! What would happen if we took the full Hamiltonian?

Where k changes along the path, and $B_1(s) = \partial B_y / \partial x$

$$k_x(s) = \frac{1}{\rho^2} - \frac{B_1(s)}{B\rho} \qquad k_y(s) = \frac{B_1(s)}{B\rho}$$

evaluated at the closed orbit

Focusing functions are periodic over length L , ie. $K_{x,y}(s+L) = K_{x,y}(s)$

nb. In a quadrupole: $k_x(s) = -\dfrac{B_1(s)}{B\rho}$

Following similar notation to S. Y. Lee, Accelerator Physics, pp.41

E. D. Courant and H. S. Snyder, "Theory of the alternating-gradient synchrotron," Annals of Physics, vol. 3, no. 1, pp. 1–48, 1958.

Let's check if the following solves Hill's equation… $x'' + kx = 0$

$$x = \sqrt{\beta(s)\varepsilon}\cos(\phi(s) + \phi_0)$$

Substitute $w = \sqrt{\beta}$ $\phi = \phi(s) + \phi_0$

& differentiate…

we impose this…

$$x' = \sqrt{\varepsilon}\left\{ w'(s)\cos\phi - \frac{d\phi}{ds} w(s)\sin\phi \right\}$$

nb. we need: $\frac{d\phi}{ds} = \frac{1}{\beta(s)} = \frac{1}{w^2(s)}$

Differentiate again…

$$x'' = \sqrt{\varepsilon}\left\{ w''(s)\cos\phi \underbrace{- \frac{w'(s)}{w^2(s)}\sin\phi + \frac{w'(s)}{w^2(s)}\sin\phi}_{=0} - \frac{1}{w^3}\cos\phi \right\}$$

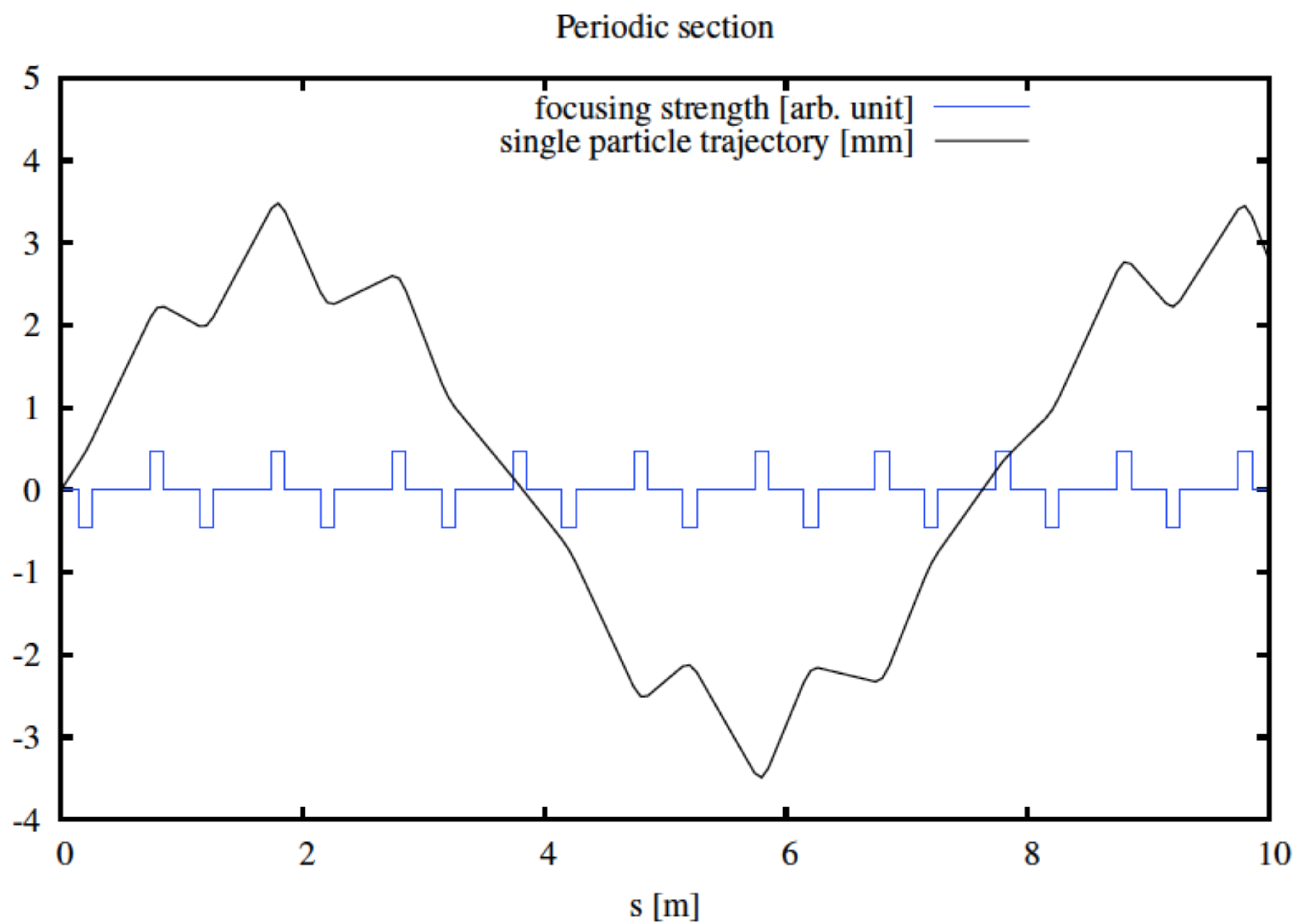
Sub into Hill's…

$$\sqrt{\varepsilon}\left\{ w''(s)\cos\phi - \frac{1}{w^3}\cos\phi \right\} + kw\sqrt{\varepsilon}\cos\phi = 0$$

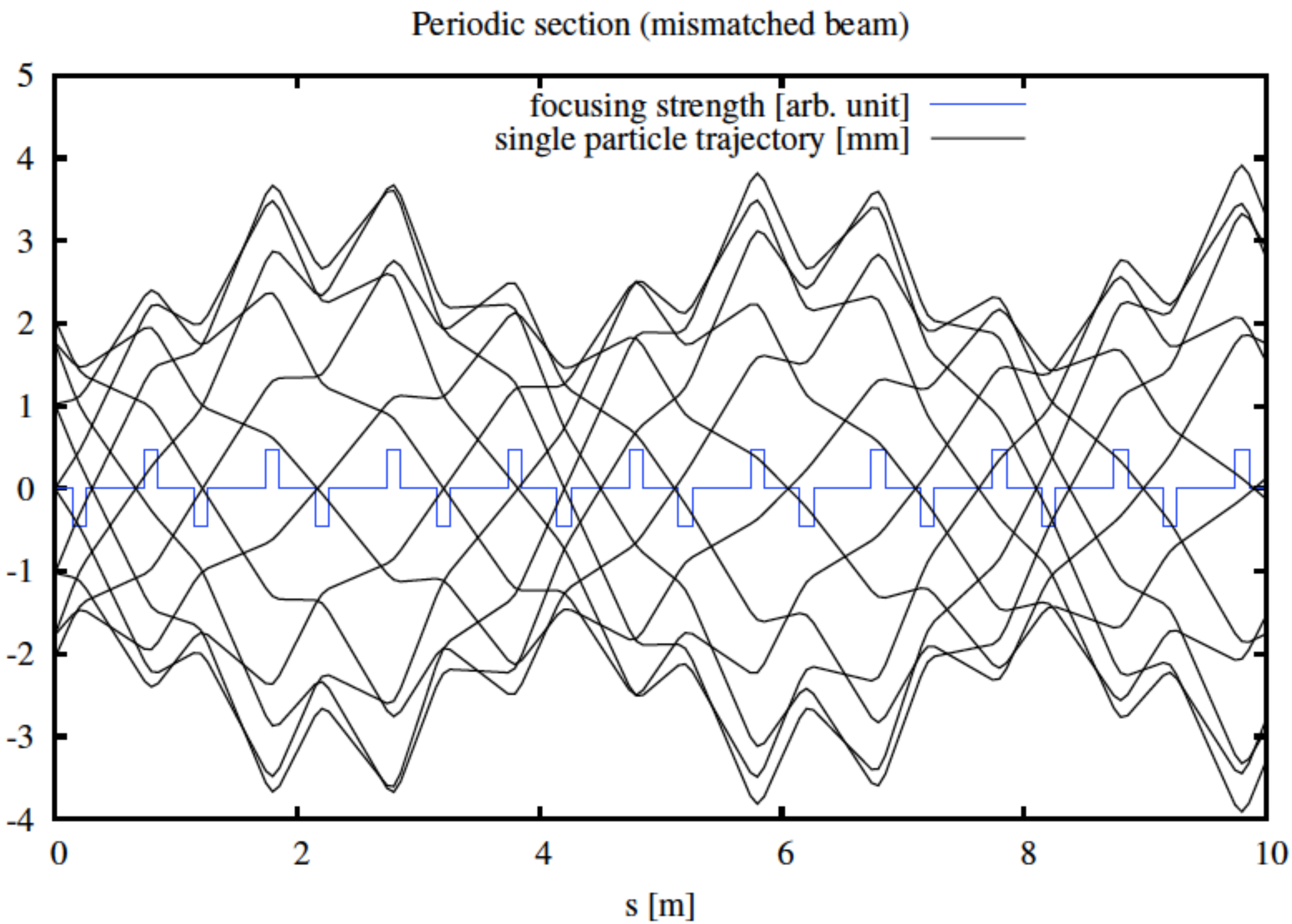
gives… $w''(s) - \frac{1}{w^3} + kw = 0$

'envelope equation' $\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + k\beta^2 = 1$

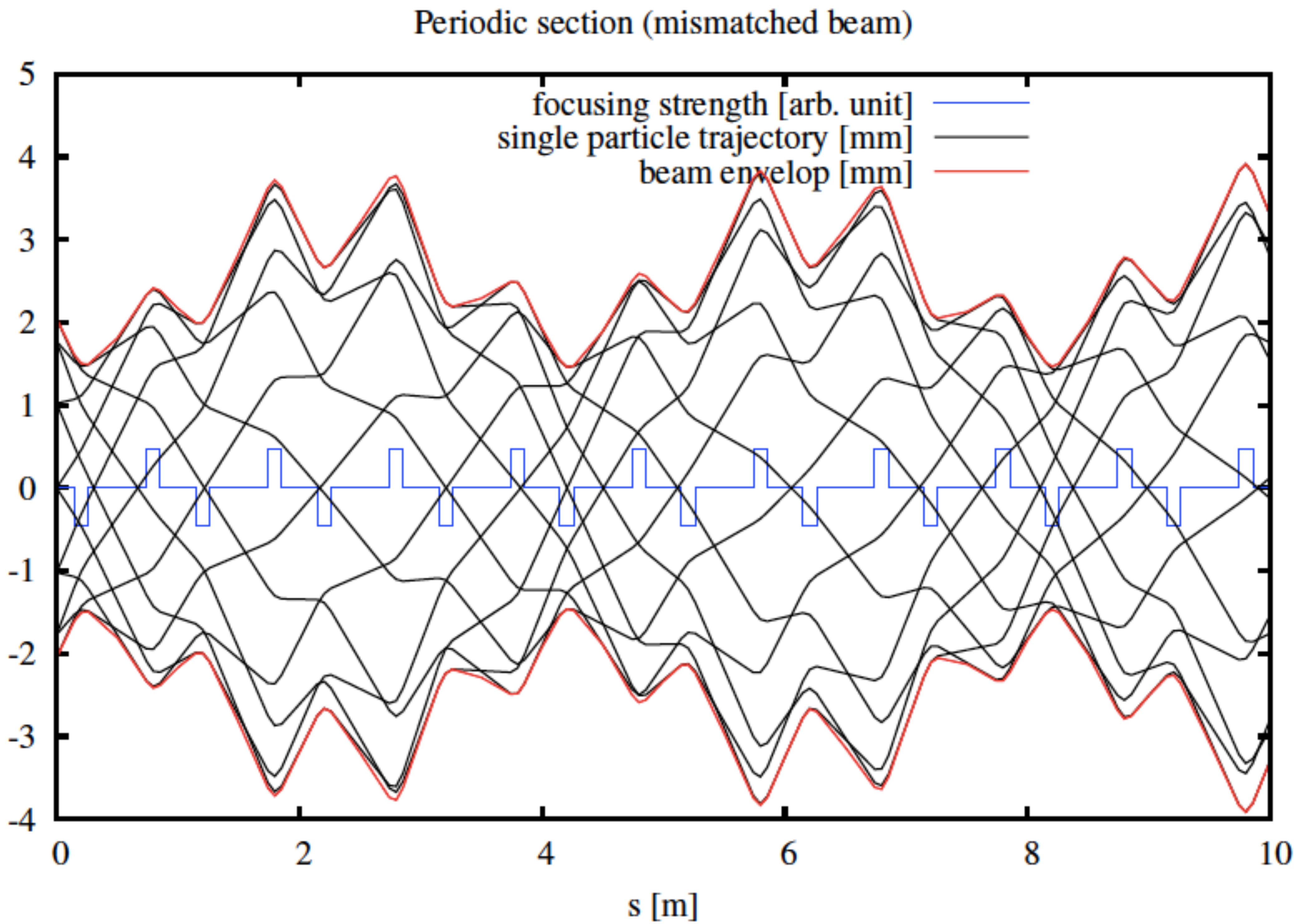
Particle in AG focusing



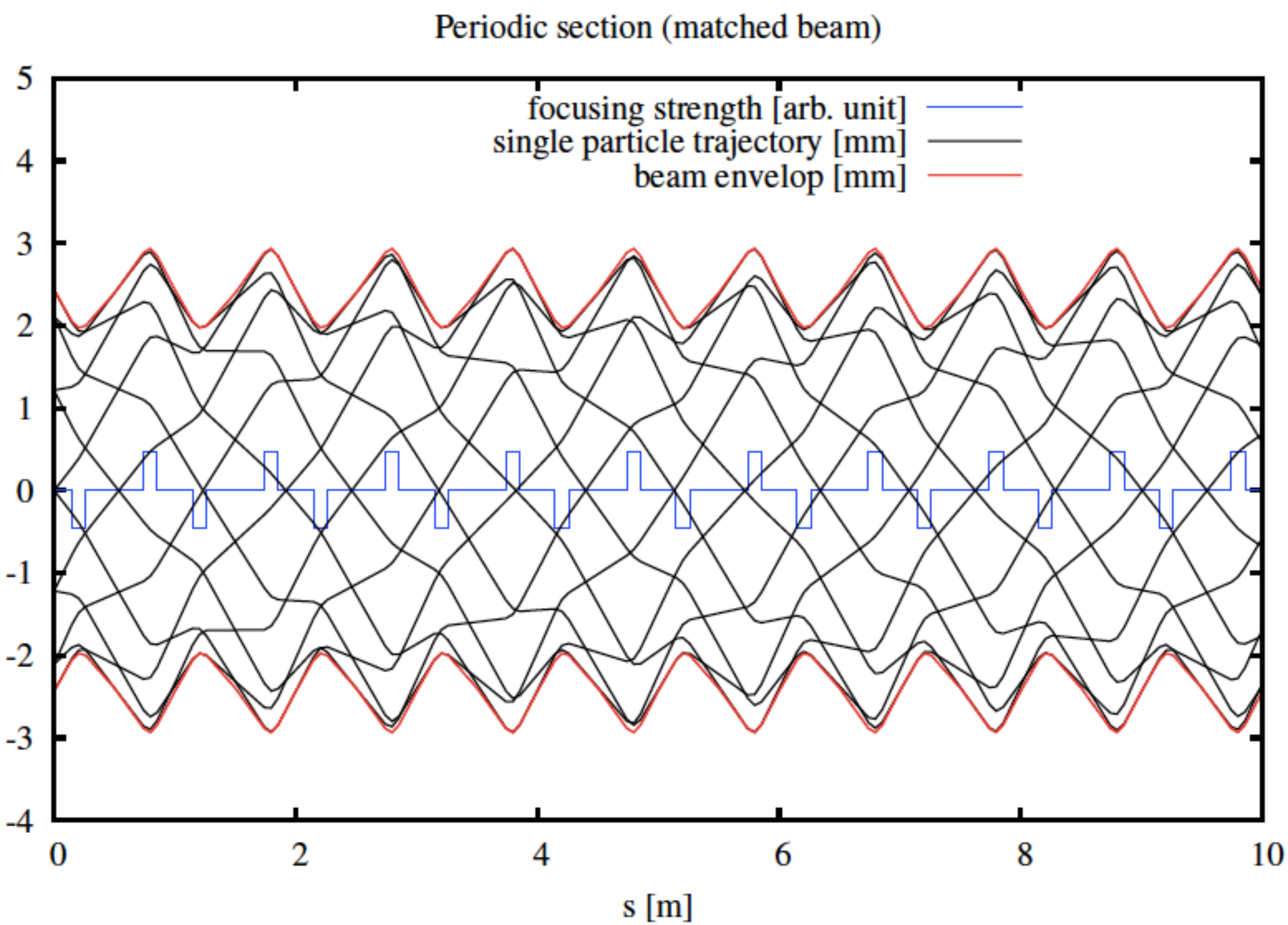
Particle in AG focusing



Particle in AG focusing



Particle in AG focusing



Solution of Hill's equation

$$x = \sqrt{\beta(s)}\sqrt{\varepsilon}\sin[\phi(s) + \phi_0]$$

initial phase

phase

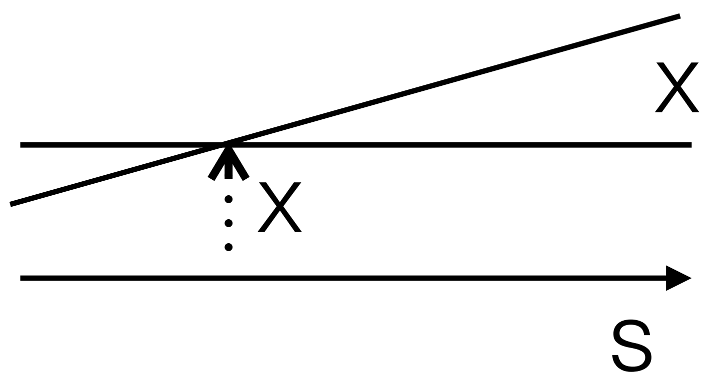
betatron function
property of the machine
(not the beam)

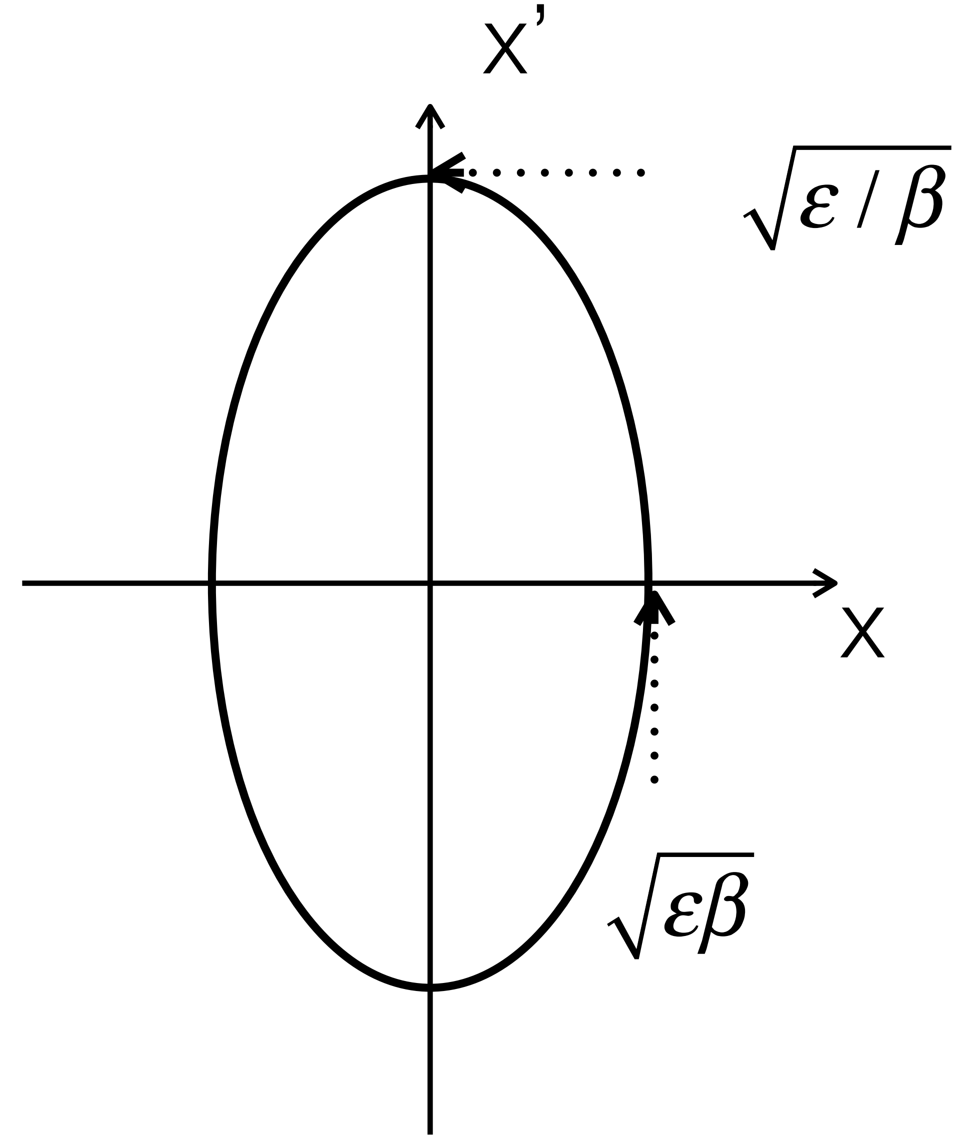
emittance
(property of beam)

phase advance 'tune' $\phi = \int \frac{ds}{\beta(s)}$

Because transverse oscillations in accelerators were theoretically studied by Kerst and Serber (Physical Review, 60, 53 (1941)) for the first time in betatrons, transverse oscillations in accelerators are known generically as betatron oscillations

Transverse ‘phase space’ ellipse





Ellipse area:

$$A = \pi\sqrt{\varepsilon\beta}\sqrt{\varepsilon / \beta} = \pi\varepsilon$$

Ellipse can change shape but not area!
Emittance is conserved. (cf. ’Louiville’s theorem’)

Transfer matrices

Express solution in matrix form…

$$\vec{x}(s) = M(s \mid s_0)\vec{x}(s_0) \qquad \vec{x}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Where M is the 'transfer matrix'.

$$\frac{d\phi(s)}{ds} = \frac{1}{w^2}$$

We already know (because we showed)

$$x = w\sqrt{\varepsilon}\cos(\phi(s) + \phi_0)$$

reminder…

$$\frac{d(\cos(f(x))}{dx} = -\sin(f(x))\frac{df(x)}{dx}$$

Take derivative for x'…

$$x' = w'\sqrt{\varepsilon}\cos(\phi(s) + \phi_0) - \frac{\sqrt{\varepsilon}}{w}\sin(\phi(s) + \phi_0)$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

Trace two rays…

‘cosine like’ $\phi = 0$ ‘sine like’ $\phi = \pi / 2$

$$x = w\sqrt{\varepsilon}\cos(\phi(s) + \phi_0)$$

$$x' = w'\sqrt{\varepsilon}\cos(\phi(s) + \phi_0) - \frac{\sqrt{\varepsilon}}{w}\sin(\phi(s) + \phi_0)$$

Yields 4 simultaneous equations so we can solve for a,b,c,d…

$$\mu = \phi_2 - \phi_1$$

$$M_{12} = \begin{pmatrix} \frac{w_2}{w_1}\cos\mu - w_2 w_1{}' \sin\mu & w_1 w_2 \sin\mu \\ -\frac{1 + w_1 w_1{}' w_2 w_2{}'}{w_1 w_2}\sin\mu - \left(\frac{w_1{}'}{w_2} - \frac{w_2{}'}{w_1}\right)\cos\mu & \frac{w_1}{w_2}\cos\mu + w_1 w_2{}' \sin\mu \end{pmatrix}$$

Simplify by considering a period or 'turn', and w's are equal.

$$M_{period} = \begin{pmatrix} \cos\mu - ww'\sin\mu & w^2\sin\mu \\ -\frac{1+w^2w'^2}{w^2}\sin\mu & \cos\mu + ww'\sin\mu \end{pmatrix}$$

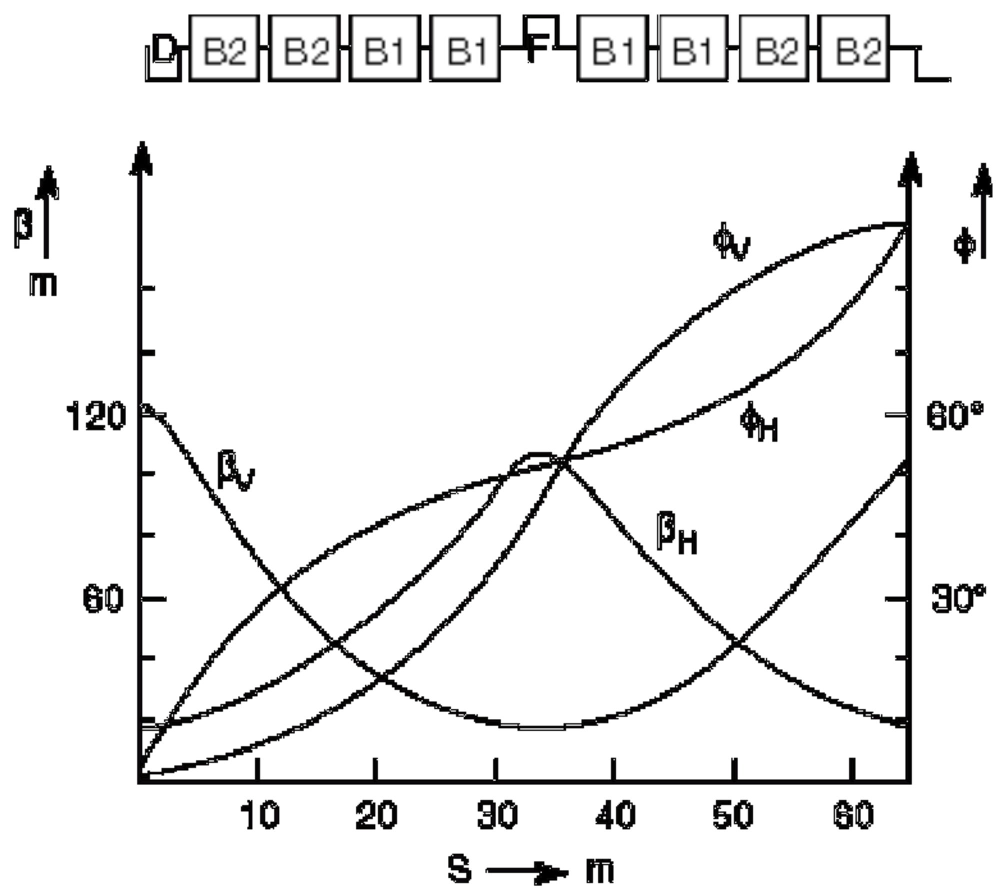
If we define the so-called 'Twiss' or 'Courant-Snyder' parameters:

$$\beta = w^2 \qquad \alpha = -\frac{1}{2}\beta' \qquad \gamma = \frac{1+\alpha}{\beta}$$

$$M_{period} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

(sorry that we are reusing symbols again… these are NOT the relativistic parameters)

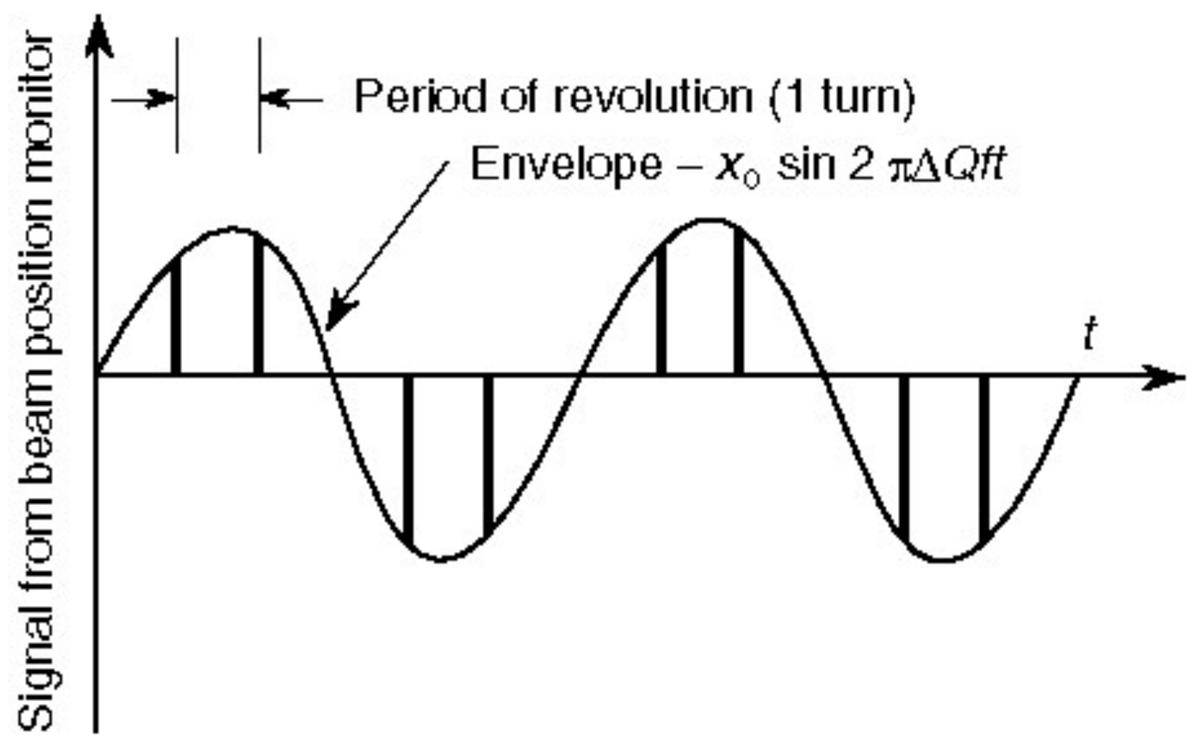
Evolution of beta in a lattice…



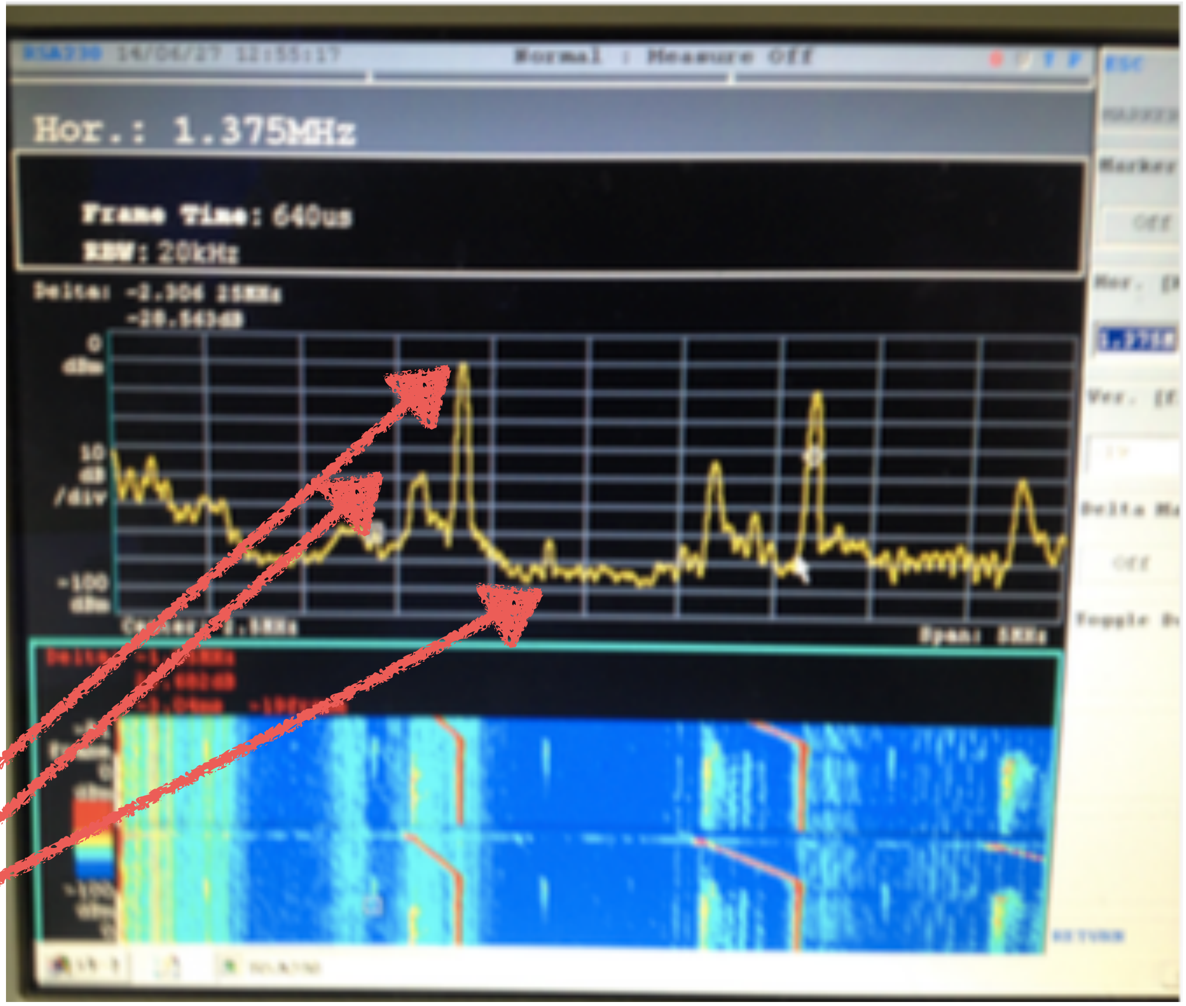
	LENGTH	ANGLE	K(V)	ALPHA(P)	BETA(H)	ALPHA(H)	MUH/2PI	BETA(V)	ALPHA(V)	MUV/2PI	AH/2	AV/2
01	3,085000	0,000000	-,015063	1,386440	104,884855	2,452160	,004571	19,011703	-,520345	,026571	65,715663	9,917560
2	,360000	0,000000	0,000000	1,374653	103,127965	2,428089	,005122	19,395014	-,544408	,029565	64,547513	10,017039
03	6,260000	,008445	0,000000	1,196124	75,348859	2,009521	,016433	28,828710	-,962519	,072198	64,004371	12,212911
4	,400000	0,000000	0,000000	1,186405	73,751941	1,982775	,017287	29,609417	-,989248	,074377	54,751341	12,376828
05	6,260000	,008445	0,000000	1,060742	51,548094	1,564207	,033474	44,610910	-1,407071	,101988	54,174091	15,192432
6	,390000	0,000000	0,000000	1,054559	50,338182	1,538130	,034692	45,718685	-1,433122	,103302	45,428681	15,379447
07	6,260000	,008445	0,000000	,981762	33,701223	1,119563	,058975	66,274961	-1,850527	,121441	44,905056	18,517478
8	,380000	0,000000	0,000000	,978948	32,860011	1,094184	,060793	67,691002	-1,875896	,122344	36,980337	18,713708
09	6,260000	,008445	0,000000	,959017	21,781569	,675586	,098381	93,787676	-2,292753	,134861	36,534921	22,028267
10	2,342700	0,000000	0,000000	,961460	18,983146	,518942	,116758	104,896272	-2,449038	,138621	30,069327	23,295624
11	3,085000	0,000000	,015037	1,034354	18,983068	-,518916	,143368	104,901620	2,447388	,143191	28,349412	23,716825
12	,350000	0,000000	0,000000	1,050730	19,354500	-,542318	,146275	103,198611	2,424067	,143726	28,638028	23,296218
13	6,260000	,008445	0,000000	1,370047	28,764399	-,960879	,189011	75,452122	2,007802	,165027	35,089639	23,106121
14	,380000	0,000000	0,000000	1,391035	29,504322	-,986287	,191088	73,935822	1,982463	,165836	35,546047	19,787412
15	6,260000	,008445	0,000000	1,763219	44,472640	-1,404847	,218731	51,724094	1,565610	,171975	43,750575	19,557880
16	,390000	0,000000	0,000000	1,788053	45,578591	-1,430924	,220109	50,513067	1,539589	,173189	44,298587	16,358398
17	6,260000	,008445	0,000000	2,213103	66,113699	-1,849484	,238298	33,849177	1,122280	,197377	53,470174	16,165762
18	,400000	0,000000	0,000000	2,241952	67,603985	-1,876229	,239281	32,962034	1,095579	,199283	54,079136	13,233307
19	6,260000	,008445	0,000000	2,719868	93,714254	-2,294790	,251780	21,859390	,677943	,236745	63,830251	13,058741
20	2,352700	0,000000	0,000000	2,909420	104,882261	-2,452099	,255558	19,038995	,520847	,255140	67,592709	10,634409
21	3,085000	0,000000	-,015063	2,946010	104,882266	2,452098	,260189	19,038106	-,520546	,281673	68,853088	9,924676
22	,360000	0,000000	0,000000	2,925443	103,125421	2,428027	,260680	19,421551	-,544579	,284653	67,665889	10,023890
23	6,260000	,008445	0,000000	2,594240	75,347037	2,009467	,271992	28,854181	-,962177	,327246	67,105194	12,218305
24	,400000	0,000000	0,000000	2,574765	73,750162	1,982722	,272846	29,634602	-,988874	,329424	57,546939	12,382087
25	6,260000	,008445	0,000000	2,296428	51,546933	1,564162	,289032	44,628208	-1,406185	,356957	56,950187	15,195377
26	,390000	0,000000	0,000000	2,280734	50,337057	1,538085	,290251	45,735180	-1,432204	,358331	47,899567	15,382238
27	6,260000	,008445	0,000000	2,055264	33,700612	1,119525	,314534	66,276862	-1,849098	,376466	47,356928	18,517744
28	,380000	0,000000	0,000000	2,043182	32,859428	1,094117	,316382	67,691805	-1,874435	,377369	39,127022	18,713817
29	6,260000	,008445	0,000000	1,870577	21,781395	,675557	,353941	93,766993	-2,290782	,389888	38,663082	22,025838
30	2,342700	0,000000	0,000000	1,815875	18,983101	,518917	,372318	104,865902	-2,446875	,393648	31,892336	23,292251
31	3,085000	0,000000	,015037	1,873603	18,983178	-,518943	,398928	104,862544	2,447912	,398220	30,027986	23,712598

How do we measure a 'tune'?

Measure the turn-by-turn oscillations of a bunch



Tune measurement example from
Kyoto University 150 MeV proton FFAG



Main frequency = revolution frequency
'Sideband' frequency gives the tune

Transfer matrices

$$\vec{x}(s) = M(s \mid s_0)\vec{x}(s_0) \qquad \vec{x}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

Where M is the 'transfer matrix'.

The effect of a succession of drifts & lenses can be found by multiplying their transfer matrices…

$$\vec{x}(s_n) = M_n(s_n \mid s_{n-1})\ldots M_3(s_3 \mid s_2)M_2(s_2 \mid s_1)M_1(s_1 \mid s_0)\vec{x}(s_0)$$

We could do this for a whole ring, but usually can exploit some symmetry (superperiod or cell)

AG focusing

Transfer matrix (x, x') for a quadrupole:

$$M(s,s_0)=\begin{cases}\begin{pmatrix}\cos\sqrt{K}l & \frac{1}{\sqrt{K}}\sin\sqrt{K}l \\ -\sqrt{K}\sin\sqrt{K}l & \cos\sqrt{K}l\end{pmatrix} & K>0, \\ \begin{pmatrix}1 & l \\ 0 & 1\end{pmatrix} & K=0, \\ \begin{pmatrix}\cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ -\sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l\end{pmatrix} & K<0.\end{cases}$$

Focusing quad

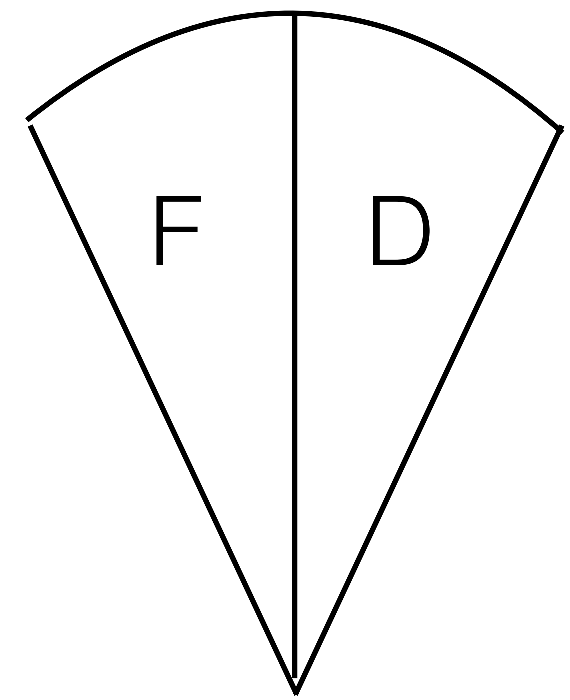
Drift space

Defocusing quad

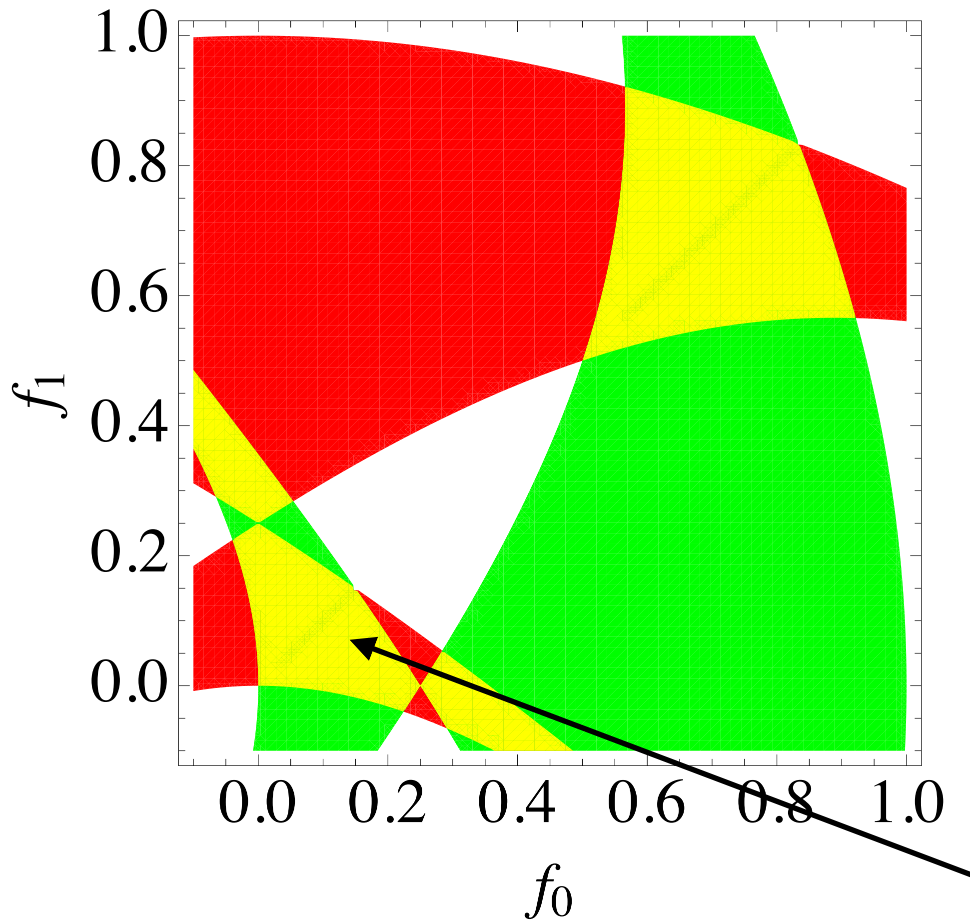
Stability: an example

This solution is 'stable' in periodic system when there is a real betatron phase advance or tune, such that:

$$|Tr(M)| \leq 2$$



So let's test this out…



$$f(\theta) = \begin{cases} f_0 + f_1 = const., & 0 < \theta < \frac{1}{2}\theta_0 \\ f_0 - f_1 = const., & \frac{1}{2}\theta_0 < \theta < \theta_0. \end{cases}$$

I parameterise it in terms of f, think of f as focal length

'stability region'

nb. no drift space & no edge focusing in this case

AG focusing: thin lens

For infinitesimally short lenses, we can recover most of the physics

$K(s) = \pm\delta(s)/f$ where f is the focal length.

In the 'thin lens' approximation, for a 'FODO' lattice:

$$M = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{d}{f} - \frac{d^2}{f^2} & 2d + \frac{d^2}{f} \\ -\frac{d}{f^2} & 1 + d/f \end{pmatrix}$$

Focusing & defocusing with a drift between doesn't cancel out.
This is what gives us 'alternating gradient' focusing

Equating these two:

$$M_{FODO} = \begin{pmatrix} 1 - \frac{d}{f} - \frac{d^2}{f^2} & 2d + \frac{d^2}{f} \\ -\frac{d}{f^2} & 1 + d/f \end{pmatrix} \qquad M_{period} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

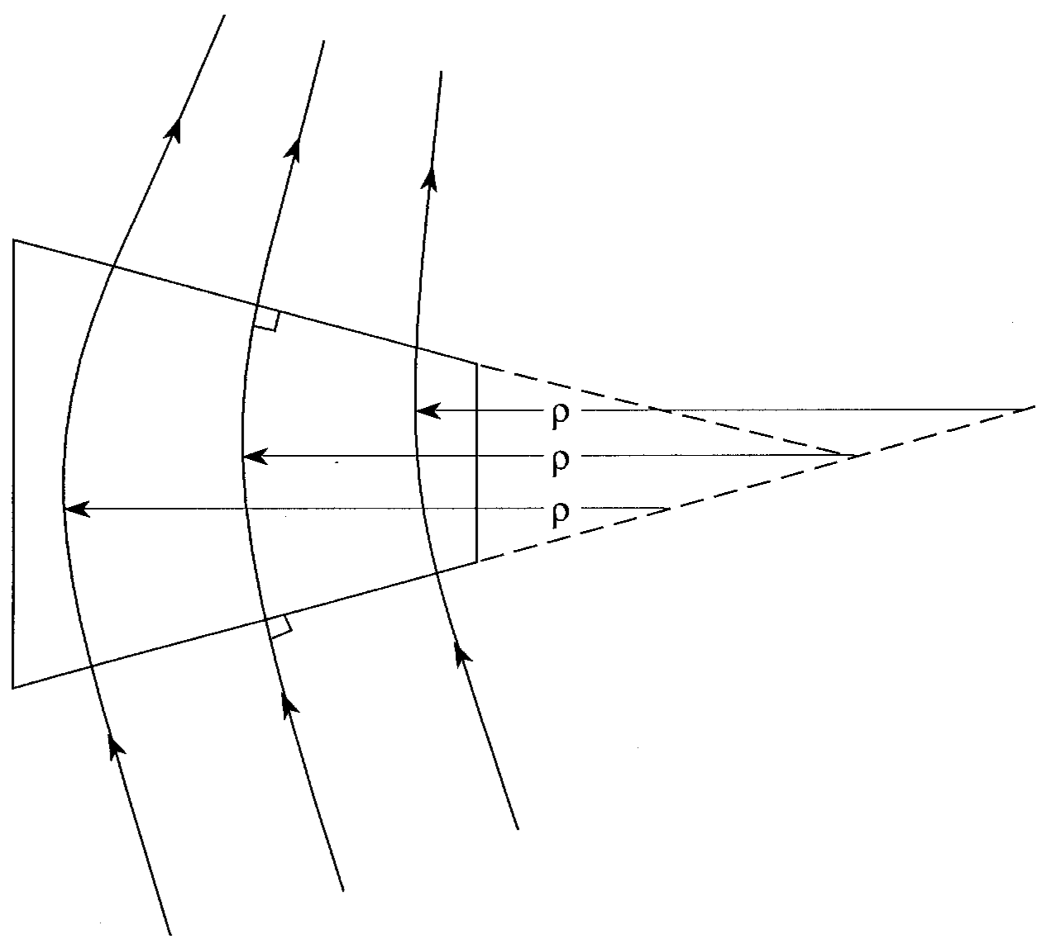
We find:

$$\cos\mu = \left(m_{11} + m_{22}\right)/2 = 1 - \frac{d^2}{2f^2}$$

Beta function $\beta_{CS} = 2d\frac{1+\sin(\mu/2)}{\sin\mu}$

Phase advance $\mu(s) = \int_0^s \frac{ds}{\beta(s)}$ 'tune' $\nu = \mu / 2\pi$

nb. sector magnet focusing



$$M_x = \begin{pmatrix} \cos\theta\ , & \rho\sin\theta \\ -\dfrac{1}{\rho}\sin\theta, & \cos\theta \end{pmatrix}$$

Summary

- Equations of motion in transverse co-ordinates
- Check Solution of Hill's equation
- Transfer matrices
- Stability and AG focusing
- Physical meaning of tune and beta

Note: Ted Wilson's cyclotron derivation

$w = \frac{q}{m_0} B_0$

Equation of motion in a cyclotron

- **Non relativistic**

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F} \qquad \frac{d(m\mathbf{v})}{dt} = q[\mathbf{v}\times\mathbf{B}]$$

- **Cartesian**

$$\frac{d(mv_x)}{dt} = \frac{d(m\dot{x})}{dt} = q[\dot{y}B_z - \dot{z}B_y]$$
$$\frac{d(mv_y)}{dt} = \frac{d(m\dot{y})}{dt} = q[\dot{z}B_x - \dot{x}B_z]$$
$$\frac{d(mv_z)}{dt} = \frac{d(m\dot{z})}{dt} = q[\dot{x}B_y - \dot{y}B_x]$$

- **Cylindrical**

$$\frac{d(m\dot{r})}{dt} - mr\dot{\theta}^2 = q[r\dot{\theta}B_z - \dot{z}B_\theta]$$
$$\frac{d(mr\dot{\theta})}{dt} + m\dot{r}\dot{\theta} = q[\dot{z}B_r - \dot{r}B_z]$$
$$\frac{d(m\dot{z})^2}{dt} = q[rB_\theta - r\dot{\theta}B_r]$$

Cyclotron orbit equation

- **For non-relativistic particles ($m = m_0$) and with an axial field $B_z = -B_0$**

$$m_0\left(\ddot{r} - r\dot{\theta}^2\right) = -qr\dot{\theta}B_z$$
$$m_0\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = q\dot{r}B_z$$
$$m_0\ddot{z} = 0$$

- **The solution is a closed circular trajectory which has radius**

$$R = \frac{p}{qB_z}$$

- **and an angular frequency**

$$\omega = \frac{q}{m_0}B_z$$

- **Take into account special relativity by**

$$m = m_0\gamma = m_0\frac{E}{E_0}$$

- **And increase B with g to stay synchronous!**

Cyclotron focusing – small deviations

- See earlier equation of motion

$$\frac{d\left(m\dot{r}\right)}{dt}+mr\dot{\theta}^{2}+q\left[r\dot{\theta}B_{z}-\dot{z}B_{\theta}\right]=0$$

- If all particles have the same velocity:

$$\rho\dot{\theta}=v_{0}=\dot{z}$$

$$\frac{d}{dt}\left(m\frac{d\rho}{dt}\right)+\frac{mv_{0}^{2}}{\rho}+ev_{0}B_{z}=0$$

- Change independent variable and substitute for small deviations

$$\frac{d}{dt}=v_{0}\frac{d}{ds}\ ,\qquad \Delta B_{z}=B_{z}-B_{0},\qquad \mathrm{x}=\rho-\rho_{0}$$

- Substitute

$$p_{0}=mv_{0}$$

- To give

$$\frac{1}{mv_{0}}\frac{d}{ds}\left(p_{0}\frac{dx}{ds}\right)+\frac{x}{\rho_{0}^{2}}+\frac{1}{\rho_{0}}\frac{\Delta B_{z}}{B_{0}}=0$$

Cyclotron focusing – field gradient

- From previous slide

$$\frac{1}{mv_{0}}\frac{d}{ds}\left(p_{0}\frac{dx}{ds}\right)+\frac{x}{\rho_{0}^{2}}+\frac{1}{\rho_{0}}\frac{\Delta B_{z}}{B_{0}}=0$$

- Taylor expansion of field about orbit

$$B_{z}=B_{0}+\frac{\partial B_{z}}{\partial x}x+\frac{1}{2!}\frac{\partial^{2}B_{z}}{\partial x^{2}}x^{2}+........$$

- Define field index (focusing gradient)

$$k=-\frac{1}{\left(B_{0}\rho_{0}\right)}\frac{\partial B_{z}}{\partial x}$$

- To give horizontal focusing

$$\frac{1}{p_{0}}\frac{d}{ds}\left(p_{0}\frac{dx}{ds}\right)+\left(\frac{1}{\rho^{2}}-k\right)x=0$$

Cyclotron focusing – betatron oscillations

- From previous slide - horizontal focusing:

$$\frac{1}{p_0}\frac{d}{ds}\left(p_0\frac{dx}{ds}\right)+\left(\frac{1}{\rho^2}-k\right)x=0$$

- Now Maxwell's

$$\nabla\times\mathbf{B}=0$$

- Determines $$\left(\frac{\partial B_x}{\partial z}\right)_0=\left(\frac{\partial B_z}{\partial x}\right)$$

- hence $B_x=-zB_0\rho_0 k$
- In vertical plane

$$\frac{1}{p_0}\frac{d}{ds}\left(p_0\frac{dz}{ds}\right)+kz=0$$

- Simple harmonic motion with a number of oscillations per turn:

$$Q_x=\sqrt{\frac{1}{\rho^2}-k},\qquad Q_z=\sqrt{k}$$

- These are "betatron" frequencies $\omega Q_x,\ \ \omega Q_y$
- Note vertical plane is unstable if $k>\frac{1}{\rho^2}$