

Electron beam dynamics in storage rings

Synchrotron radiation and its effect on electron dynamics

Lecture 1: Synchrotron radiation

Lecture 2: Undulators and Wigglers

Lecture 3: Electron dynamics-I

Lecture 4: Electron dynamics-II

Contents

Introduction

properties of synchrotron radiation
synchrotron light sources

Lienard-Wiechert potentials

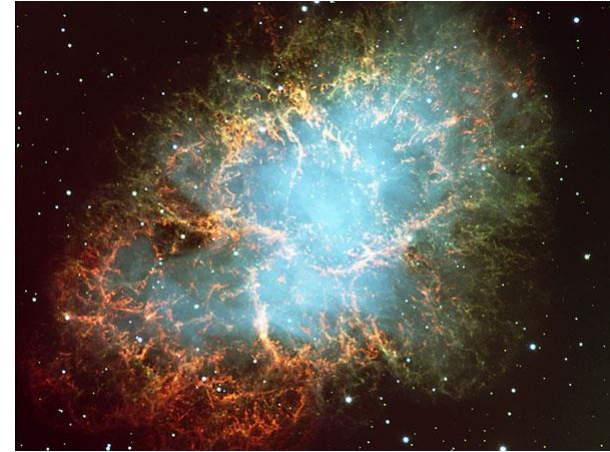
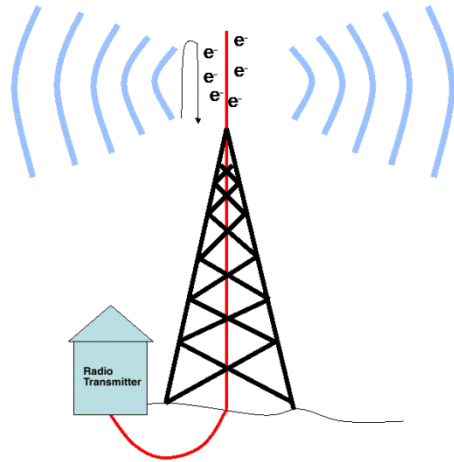
Angular distribution of power radiated by accelerated particles
non-relativistic motion: Larmor's formula
relativistic motion
velocity \perp acceleration: synchrotron radiation

Angular and frequency distribution of energy radiated:
the radiation integral
radiation integral for bending magnet radiation

Radiation from undulators and wigglers

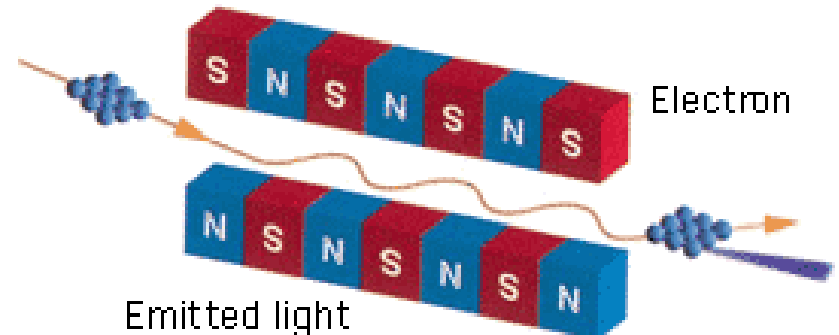
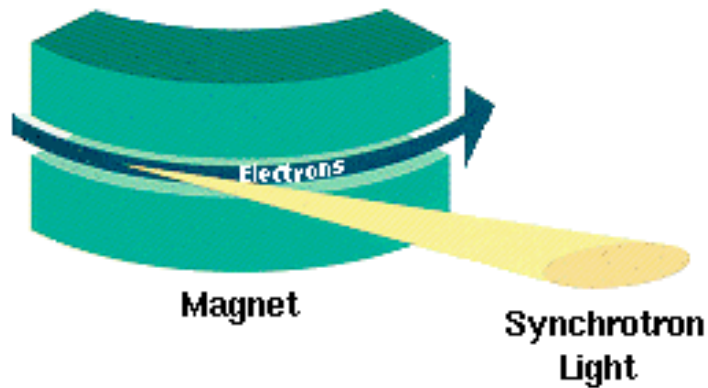
What is synchrotron radiation

Electromagnetic radiation is emitted by charged particles when accelerated



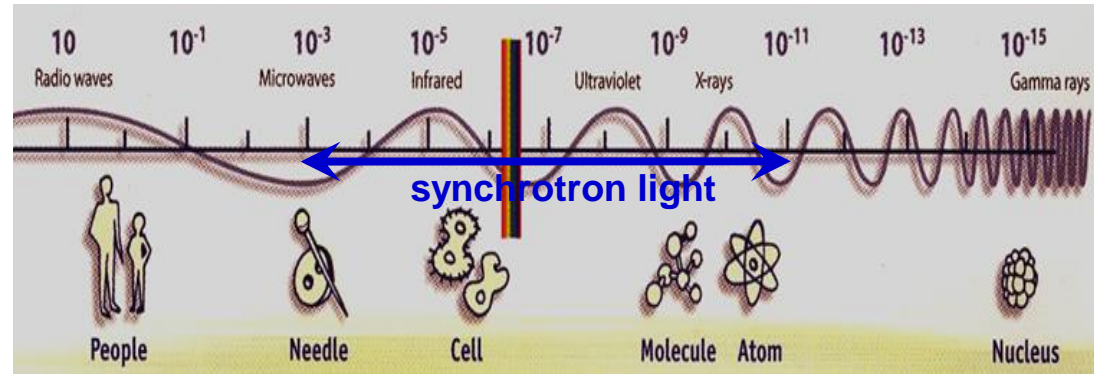
The electromagnetic radiation emitted when the charged particles are accelerated radially ($v \perp a$) is called **synchrotron radiation**

It is produced in the synchrotron radiation sources using bending magnets undulators and wigglers



Synchrotron radiation sources properties

Broad Spectrum which covers from microwaves to hard X-rays: the user can select the wavelength required for experiment;



High Flux: high intensity photon beam, allows rapid experiments or use of weakly scattering crystals;

$$\text{Flux} = \text{Photons} / (\text{s} \cdot \text{BW})$$

High Brilliance (Spectral Brightness): highly collimated photon beam generated by a small divergence and small size source (partial coherence);

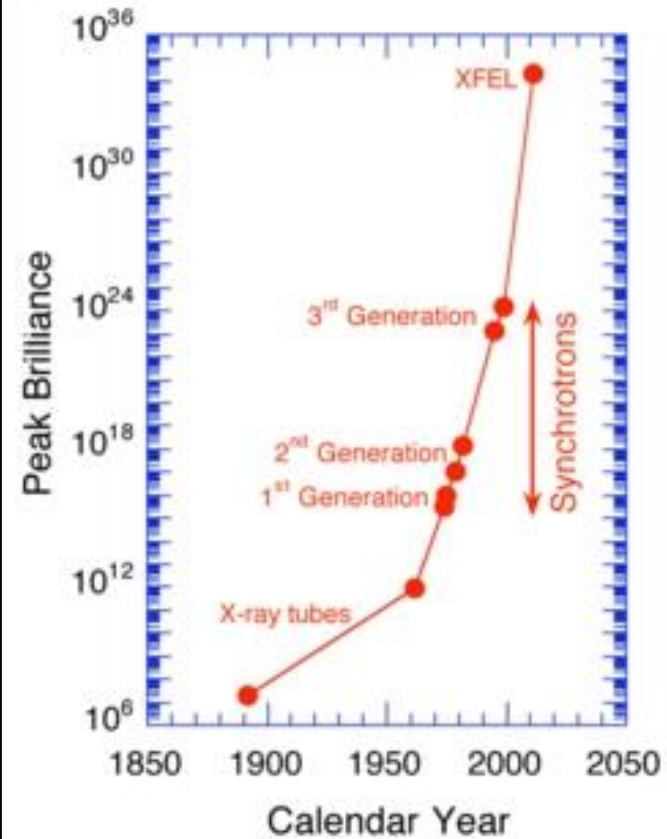
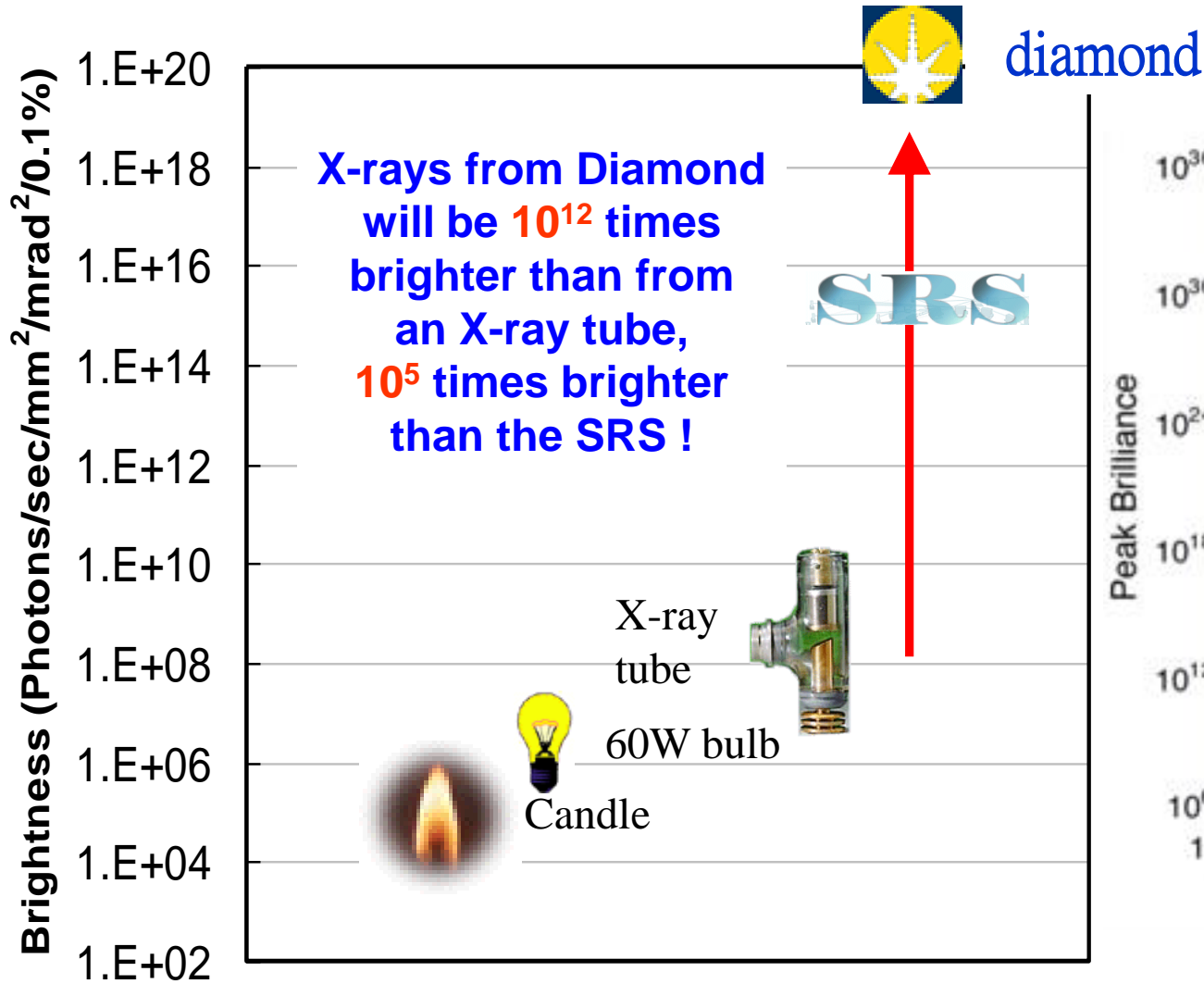
$$\text{Brilliance} = \text{Photons} / (\text{s} \cdot \text{mm}^2 \cdot \text{mrad}^2 \cdot \text{BW})$$

High Stability: submicron source stability

Polarisation: both linear and circular (with IDs)

Pulsed Time Structure: pulsed length down to tens of picoseconds allows the resolution of process on the same time scale

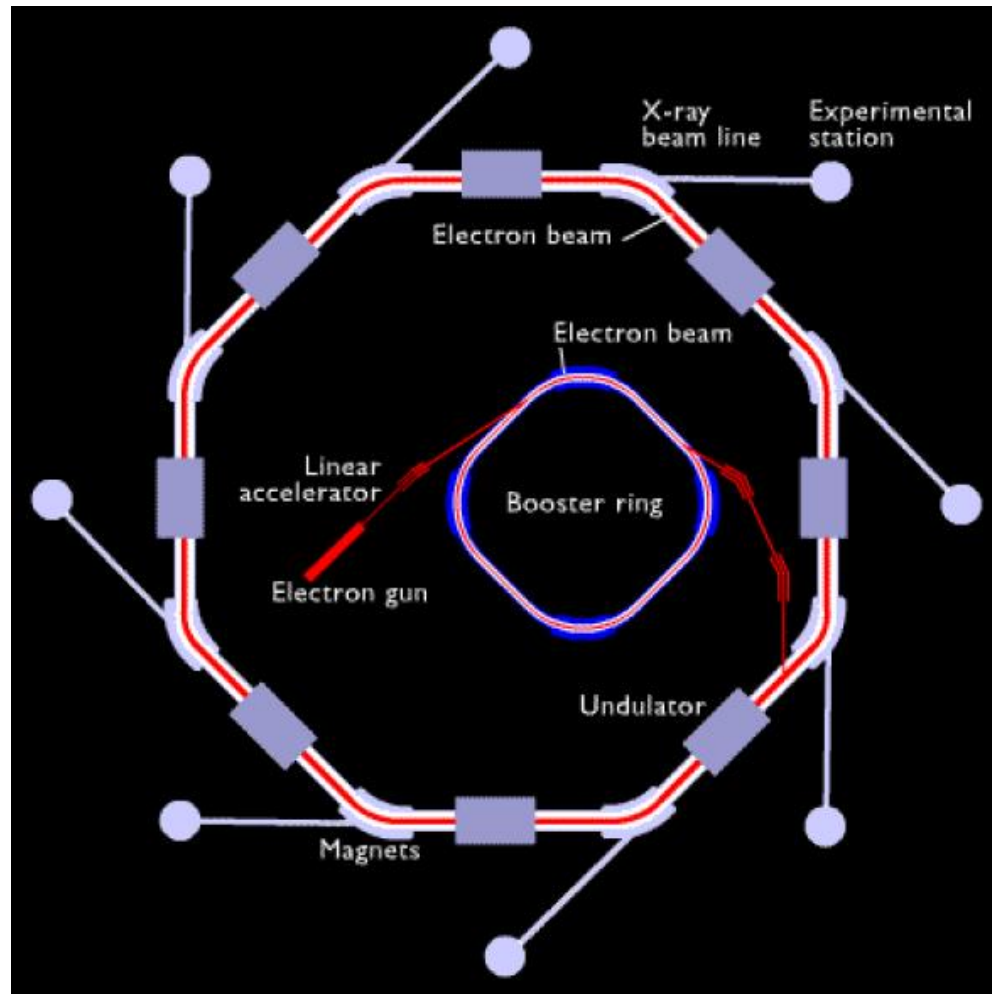
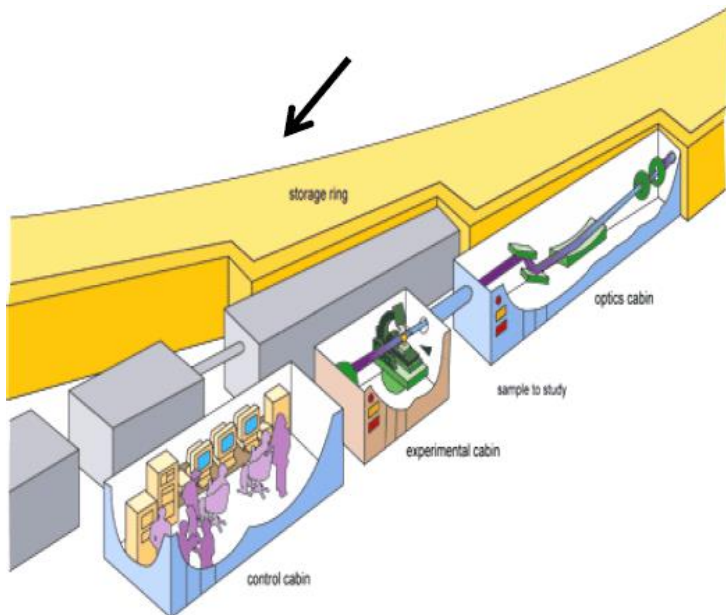
Peak Brilliance



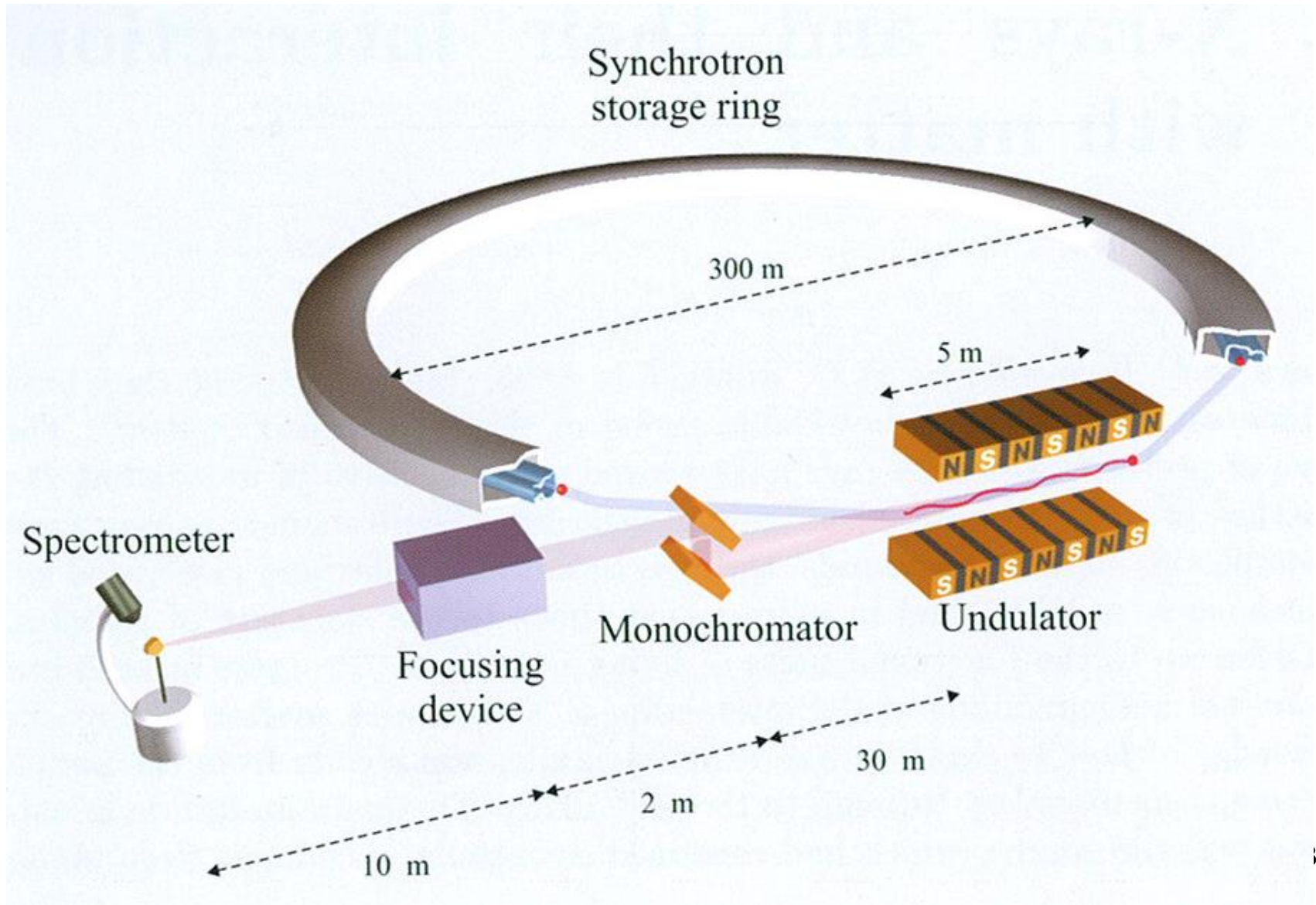
Layout of a synchrotron radiation source (I)

Electrons are generated and accelerated in a linac, further accelerated to the required energy in a booster and injected and stored in the storage ring

The circulating electrons emit an intense beam of synchrotron radiation which is sent down the beamline



Layout of a synchrotron radiation source (II)



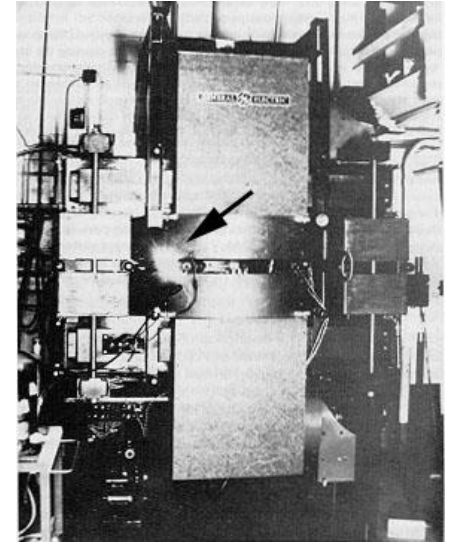
A brief history of storage ring synchrotron radiation sources

- **First observation:**

1947, General Electric, 70 MeV synchrotron

- **First user experiments:**

1956, Cornell, 320 MeV synchrotron



- **1st generation light sources:** machine built for High Energy Physics or other purposes used parasitically for synchrotron radiation
- **2nd generation light sources:** purpose built synchrotron light sources, SRS at Daresbury was the first dedicated machine (1981 – 2008)
- **3rd generation light sources:** optimised for high brilliance with low emittance and Insertion Devices; ESRF, Diamond, ...
- **4th generation light sources:** photoinjectors LINAC based Free Electron Laser sources; FLASH (DESY), LCLS (SLAC), ...
- **(5th generation light sources?):** FELS driven by LPWA...very speculative

3rd generation storage ring light sources

1992	ESRF , France (EU)	6 GeV
	ALS , US	1.5-1.9 GeV
1993	TLS , Taiwan	1.5 GeV
1994	ELETTRA , Italy	2.4 GeV
	PLS , Korea	2 GeV
	MAX II , Sweden	1.5 GeV
1996	APS , US	7 GeV
	LNLS , Brazil	1.35 GeV
1997	Spring-8 , Japan	8 GeV
1998	BESSY II , Germany	1.9 GeV
2000	ANKA , Germany	2.5 GeV
	SLS , Switzerland	2.4 GeV
2004	SPEAR3 , US	3 GeV
	CLS , Canada	2.9 GeV
2006:	SOLEIL , France	2.8 GeV
	DIAMOND , UK	3 GeV
	ASP , Australia	3 GeV
	MAX III , Sweden	700 MeV
	Indus-II , India	2.5 GeV
2008	SSRF , China	3.4 GeV
2011	ALBA , Spain	3 GeV



3rd generation storage ring light sources

recently commissioned

2014	NLSL-II , US	3 GeV
	TPS , Taiwan	3 GeV
2016	MAX-IV , Sweden	1.5-3 GeV
	SOLARIS , Poland	1.5 GeV
2017	SESAME , Jordan	2.5 GeV



major upgrades

2019	ESRF-II , France	6 GeV
> 2020	Spring8-II , Japan	6 GeV
	APS-U , US	6 GeV
	ALS-U , US	1.9 GeV
	HEPS , China	6 GeV



many others....

Diamond Aerial views

June 2003



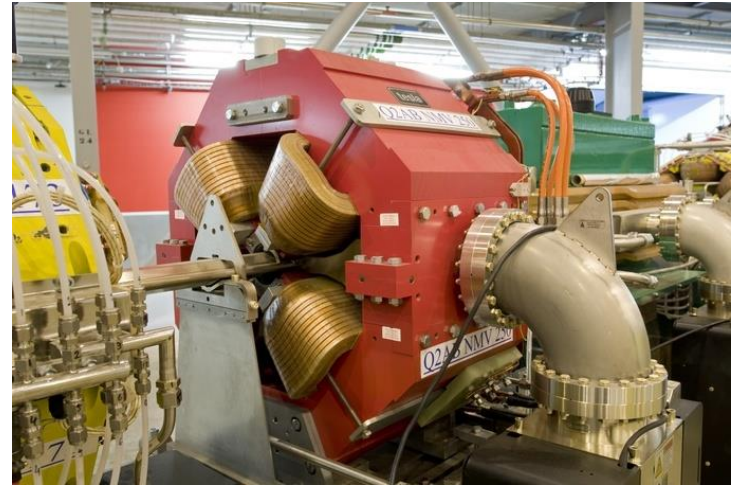
Oct 2006

Main components of a storage ring

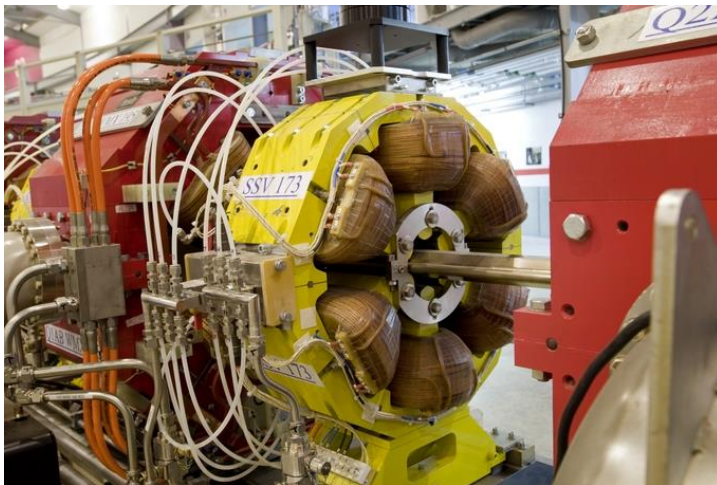
Dipole magnets to bend the electrons



Quadrupole magnets to focus the electrons



Sextupole magnets to focus off-energy electrons (mainly)

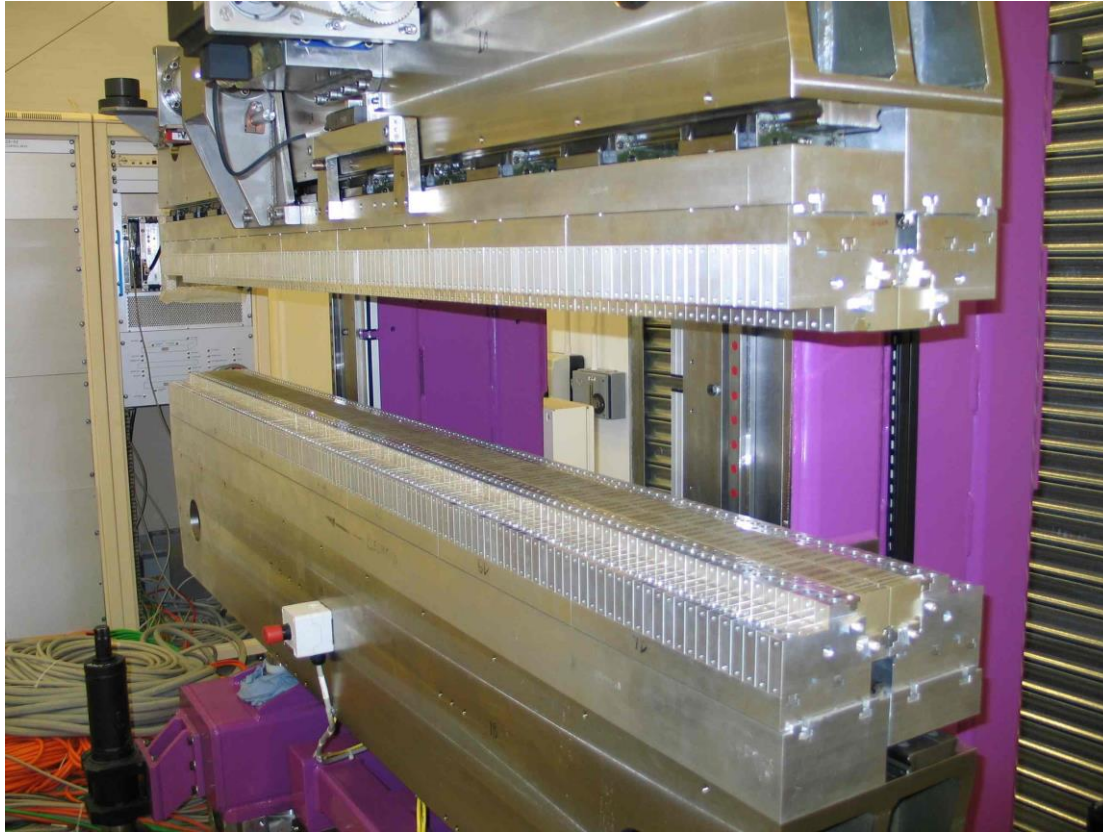


RF cavities to replace energy losses due to the emission of synchrotron radiation



Main components of a storage ring

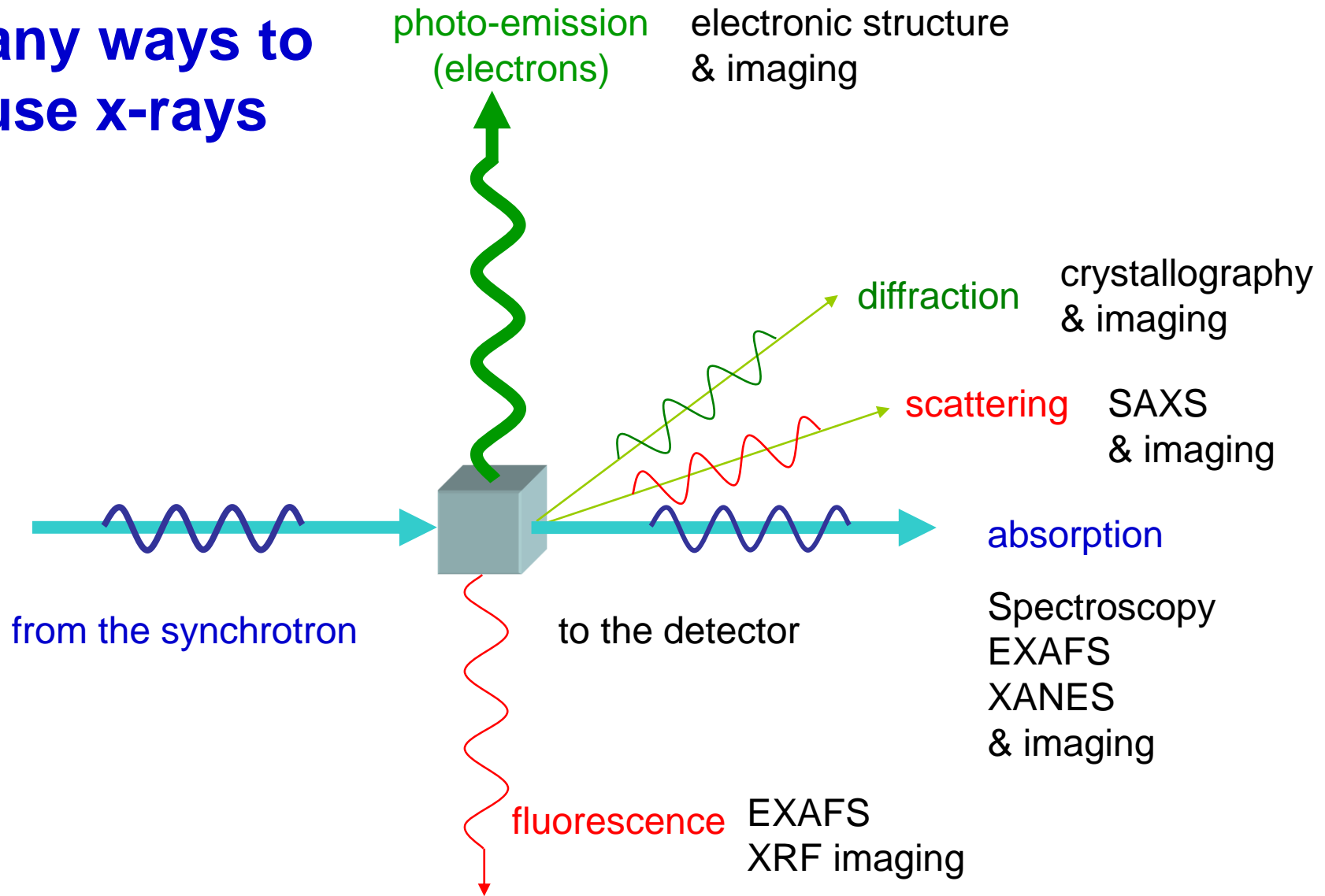
Insertion devices (undulators) to generate high brilliance radiation



Insertion devices (wiggler) to reach high photon energies



Many ways to use x-rays



Life science examples: DNA and myoglobin

Franklin and Gosling used a X-ray tube:

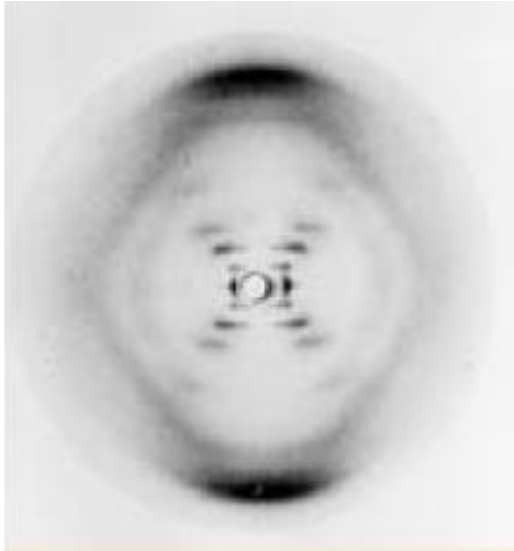
Brilliance was 10^8 (ph/sec/mm²/mrad²/0.1BW)

Exposure times of 1 day were typical (10^5 sec)

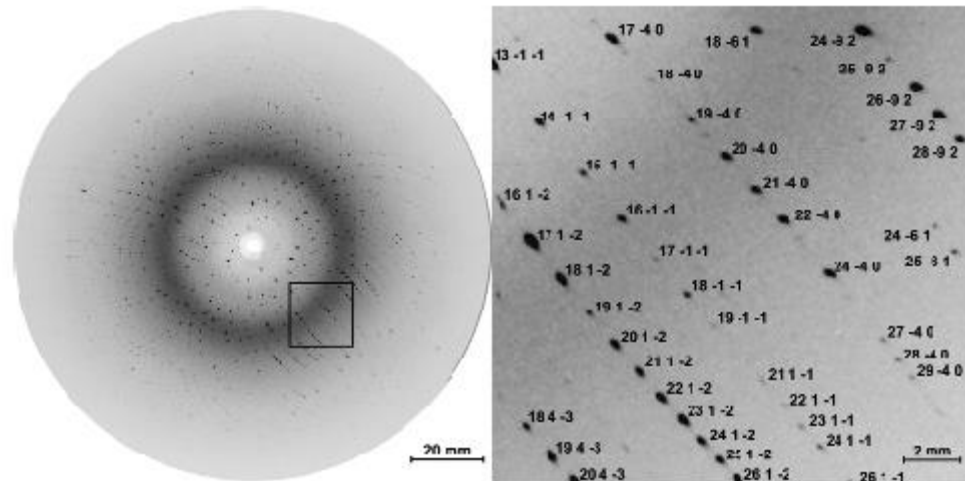
e.g. Diamond provides a brilliance of 10^{20}

100 ns exposure would be sufficient

Nowadays pump probe experiment in life science are performed using 100 ps pulses from storage ring light sources: e.g. ESRF myoglobin in action



Photograph 51
Franklin-Gosling
DNA (form B)
1952

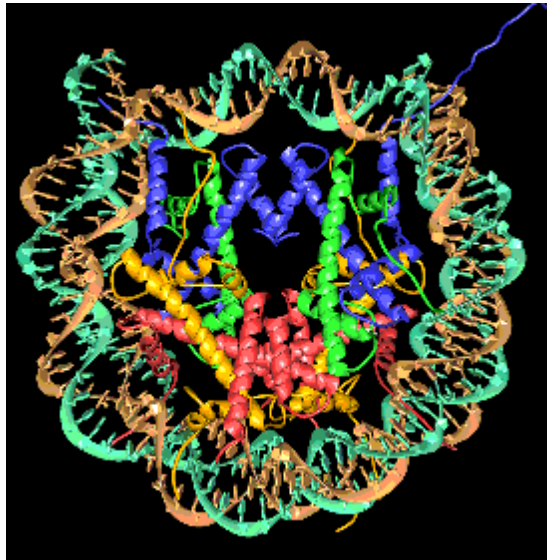


Applications

Medicine, Biology, Chemistry, Material Science, Environmental Science and more

Biology

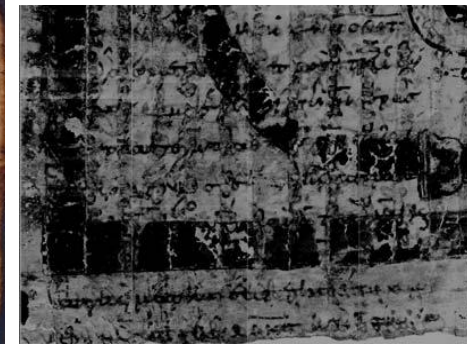
Reconstruction of the 3D structure of a nucleosome with a resolution of 0.2 nm



The collection of precise information on the molecular structure of chromosomes and their components can improve the knowledge of how the genetic code of DNA is maintained and reproduced

Archeology

A synchrotron X-ray beam at the SSRL facility illuminated an obscured work erased, written over and even painted over of the ancient mathematical genius Archimedes, born 287 B.C. in Sicily.



X-ray fluorescence imaging revealed the hidden text by revealing the iron contained in the ink used by a 10th century scribe. This x-ray image shows the lower left corner of the page.

Lienard-Wiechert potentials (I)

We want to compute the em field generated by a charged particle in motion on a given trajectory $\bar{\mathbf{x}} = \bar{\mathbf{r}}(t)$

The charge density and current distribution of a single particle read

$$\rho(\bar{\mathbf{x}}, t) = q\delta^{(3)}(\bar{\mathbf{x}} - \bar{\mathbf{r}}(t)) \quad \bar{\mathbf{J}}(\bar{\mathbf{x}}, t) = q\bar{\mathbf{v}}(t)\delta^{(3)}(\bar{\mathbf{x}} - \bar{\mathbf{r}}(t))$$

We have to solve Maxwell equations driven by such time varying charge density and current distribution.

The general expression for the wave equation for the em potentials (in the Lorentz gauge) reads

$$\bar{\nabla}^2 \bar{\phi} - \frac{1}{c^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad \bar{\nabla}^2 \bar{\mathbf{A}} - \frac{1}{c^2} \frac{\partial^2 \bar{\mathbf{A}}}{\partial t^2} = -\mu_0 \bar{\mathbf{J}}$$

Lienard-Wiechert potentials (II)

The general solutions for the wave equation driven by a time varying charge and current density read (in the Lorentz gauge) [Jackson Chap. 6]

$$\Phi(\bar{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\bar{x}' \int dt' \frac{\rho(\bar{x}', t')}{|\bar{x} - \bar{x}'|} \delta\left(t' + \frac{|\bar{x} - \bar{x}'|}{c} - t\right) \quad \bar{A}(\bar{x}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\bar{x}' \int dt' \frac{\bar{J}(\bar{x}', t')}{|\bar{x} - \bar{x}'|} \delta\left(t' + \frac{|\bar{x} - \bar{x}'|}{c} - t\right)$$

Integrating the Dirac delta in time we are left with

$$\Phi(\bar{x}, t) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\bar{x}', t_{\text{ret}})}{|\bar{x} - \bar{x}'|} d^3\bar{x}' \quad \bar{A}(\bar{x}, t) = \frac{\mu_0}{4\pi} \iiint_V \frac{\bar{J}(\bar{x}', t_{\text{ret}})}{|\bar{x} - \bar{x}'|} d^3\bar{x}'$$

where ret means retarded $t_{\text{ret}} = t - \frac{|\bar{x}(t) - \bar{x}(t_{\text{ret}})|}{c}$ (see next slide)

Now we use the charge density and current distribution of a single particle

$$\rho(\bar{x}, t) = q\delta^{(3)}(\bar{x} - \bar{r}(t)) \quad \bar{J}(\bar{x}, t) = q\bar{v}(t)\delta^{(3)}(\bar{x} - \bar{r}(t))$$

Lienard-Wiechert potentials (III)

Substituting we get

$$\Phi(\bar{\mathbf{x}}, t) = \frac{q}{4\pi\epsilon_0} \iiint_V \frac{\delta^{(3)}[\bar{\mathbf{x}}' - \bar{\mathbf{r}}(t_{\text{ret}})]}{|\bar{\mathbf{x}} - \bar{\mathbf{x}}'|} d^3\bar{\mathbf{x}}' \quad \bar{\mathbf{A}}(\bar{\mathbf{x}}, t) = \frac{q\mu_0}{4\pi} \iiint_V \frac{\bar{\mathbf{v}}(t_{\text{ret}})[\bar{\mathbf{x}}' - \bar{\mathbf{r}}(t_{\text{ret}})]}{|\bar{\mathbf{x}} - \bar{\mathbf{x}}'|} d^3\bar{\mathbf{x}}'$$

Using again the properties of the Dirac deltas we can integrate and obtain the Lienard-Wiechert potentials

$$\Phi(\bar{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0} \left[\frac{e}{(1 - \bar{\boldsymbol{\beta}} \cdot \bar{\mathbf{n}})R} \right]_{\text{ret}} \quad \bar{\mathbf{A}}(\bar{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c} \left[\frac{e\bar{\boldsymbol{\beta}}}{(1 - \bar{\boldsymbol{\beta}} \cdot \bar{\mathbf{n}})R} \right]_{\text{ret}}$$

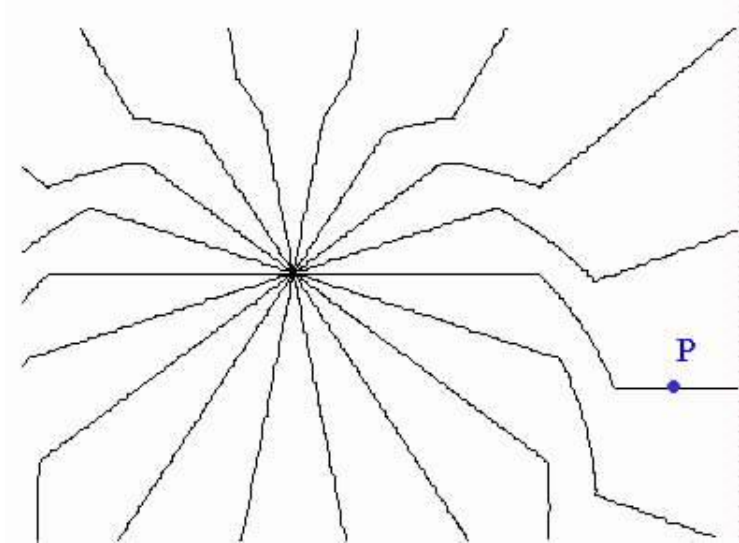
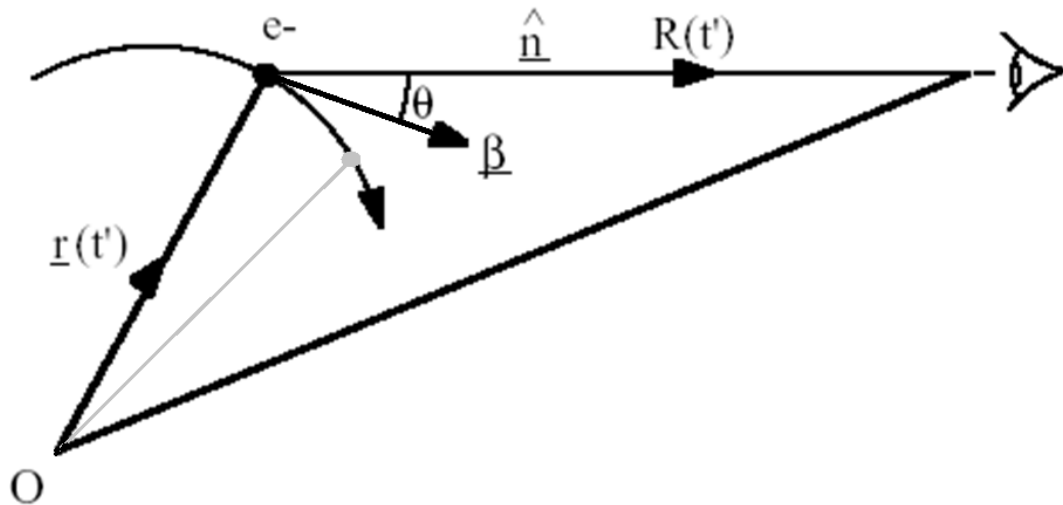
These are the potentials of the em fields generated by the charged particle in motion.

The trajectory itself is determined by external electric and magnetic fields

Lienard-Wiechert Potentials (IV)

[]_{ret} means computed at time t'

$$t = t' + \frac{R(t')}{c}$$



Potentials and fields at position x at time t are determined by the characteristic of the electron motion at a time t'

$t - t'$ is the time it takes for the em radiation to travel the distance $R(t')$

i.e. grey is the position of the electron at time t

Liendard-Wiechert Potentials (II)

The electric and magnetic fields generated by the moving charge are computed from the potentials

$$\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \qquad \bar{B} = \nabla \times \bar{A}$$

and are called Liendard-Wiechert fields

$$\bar{E}(\bar{x}, t) = \frac{e}{4\pi\epsilon_0} \left[\frac{\bar{n} - \bar{\beta}}{\gamma^2 (1 - \bar{\beta} \cdot \bar{n})^3 R^2} \right]_{ret} + \frac{e}{4\pi\epsilon_0 c} \left[\frac{\bar{n} \times (\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}}}{(1 - \bar{\beta} \cdot \bar{n})^3 R} \right]_{ret} \qquad \bar{B}(\bar{x}, t) = \frac{1}{c} [\bar{n} \times \bar{E}]_{rit}$$

velocity field
acceleration field
 $\propto \frac{1}{R}$
 $\vec{E} \perp \bar{B} \perp \hat{n}$

Power radiated by a particle on a surface is the flux of the Poynting vector

$$\bar{S} = \frac{1}{\mu_0} \bar{E} \times \bar{B} \qquad \Phi_{\Sigma}(\bar{S})(t) = \iint_{\Sigma} \bar{S}(\bar{x}, t) \cdot \bar{n} d\Sigma$$

Angular distribution of radiated power [see Jackson]

$$\frac{d^2 P}{d\Omega} = (\bar{S} \cdot \bar{n})(1 - \bar{n} \cdot \bar{\beta}) R^2 \qquad \text{radiation emitted by the particle}$$

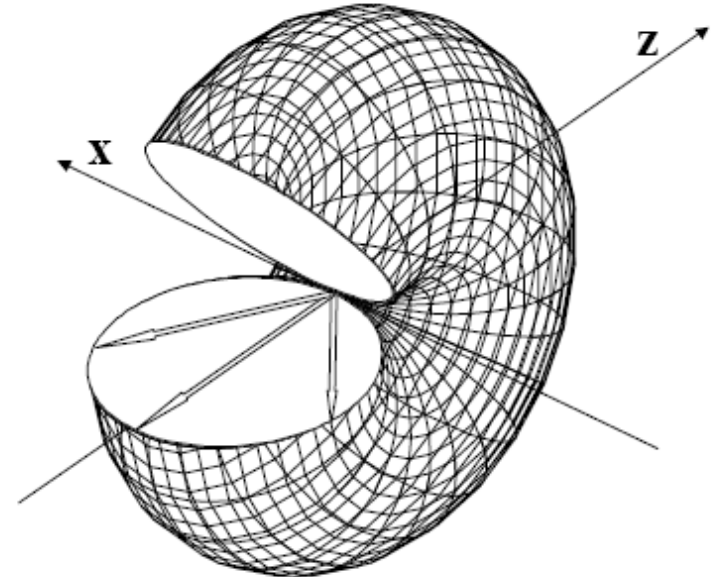
Angular distribution of radiated power: non relativistic motion

Assuming $\bar{\beta} \approx \bar{0}$ and substituting the acceleration field

$$\bar{E}_{acc}(\bar{x}, t) = \frac{e}{4\pi\epsilon_0 c} \left[\frac{\bar{n} \times (\bar{n} \times \dot{\bar{\beta}})}{R} \right]_{ret}$$

$$\frac{d^2 P}{d\Omega} = \frac{1}{\mu_0 c} |R \bar{E}_{acc}|^2 = \frac{e^2}{(4\pi)^2 \epsilon_0 c} |\bar{n} \times (\bar{n} \times \dot{\bar{\beta}})|^2$$

$$\frac{d^2 P}{d\Omega} = \frac{e^2}{(4\pi)^2 \epsilon_0 c} |\dot{\bar{\beta}}|^2 \sin^2 \theta$$



θ is the angle between the acceleration and the observation direction

Integrating over the angles gives the total radiated power

$$P = \frac{e^2}{6\pi\epsilon_0 c} |\dot{\bar{\beta}}|^2$$

Larmor's formula

Angular distribution of radiated power: relativistic motion

Substituting the acceleration field

$$\frac{d^2P}{d\Omega} = \frac{1}{\mu_0 c} \left| R \bar{E}_{acc} \right|^2 = \frac{e^2}{(4\pi)^2 \epsilon_0 c} \frac{\left| \bar{n} \times \left[(\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}} \right] \right|^2}{(1 - \bar{n} \cdot \bar{\beta})^5}$$

emission is peaked in the direction of the velocity

The pattern depends on the details of velocity and acceleration but it is dominated by the denominator

Total radiated power: computed either by integration over the angles or by relativistic transformation of the 4-acceleration in Larmor's formula

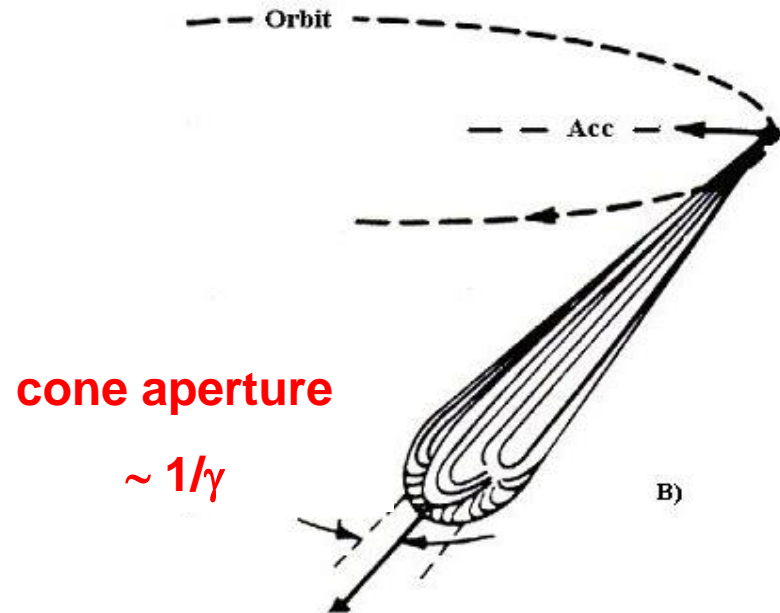
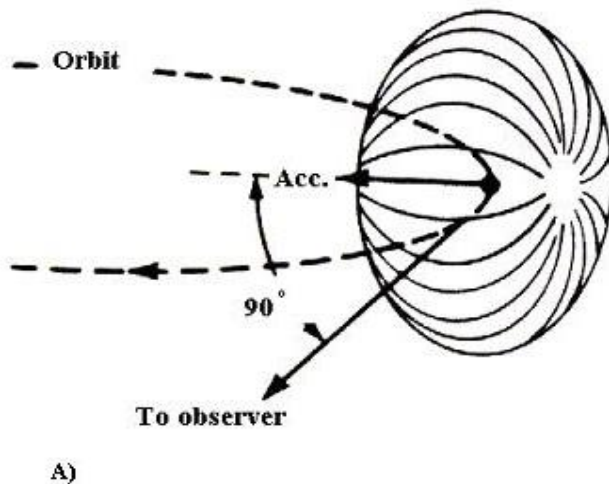
$$P = \frac{e^2}{6\pi\epsilon_0 c} \gamma^6 \left[(\dot{\bar{\beta}})^2 - (\bar{\beta} \times \dot{\bar{\beta}})^2 \right]$$

Relativistic generalization of
Larmor's formula

velocity \perp acceleration: synchrotron radiation

Assuming $\vec{\beta} \perp \dot{\vec{\beta}}$ and substituting the acceleration field

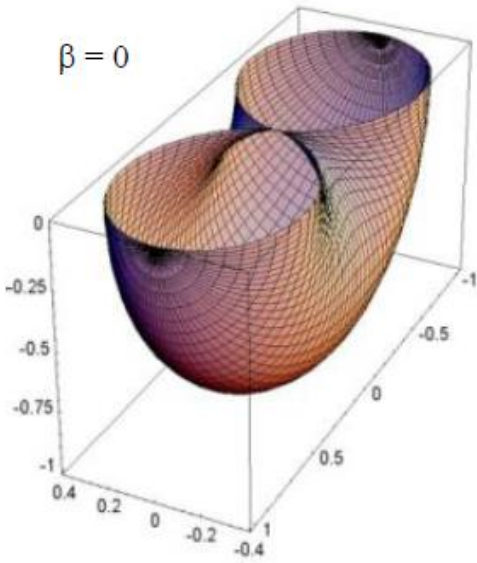
$$\frac{d^2 P}{d\Omega} = \frac{e^2}{(4\pi)^2 \epsilon_0 c^2} \frac{|\vec{n} \times [(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}]|^2}{(1 - \vec{n} \cdot \vec{\beta})^5} = \frac{e^2 |\dot{\vec{\beta}}|^2}{(4\pi)^2 \epsilon_0 c} \frac{1}{(1 - \beta \cos \theta)^3} \left[1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right]$$



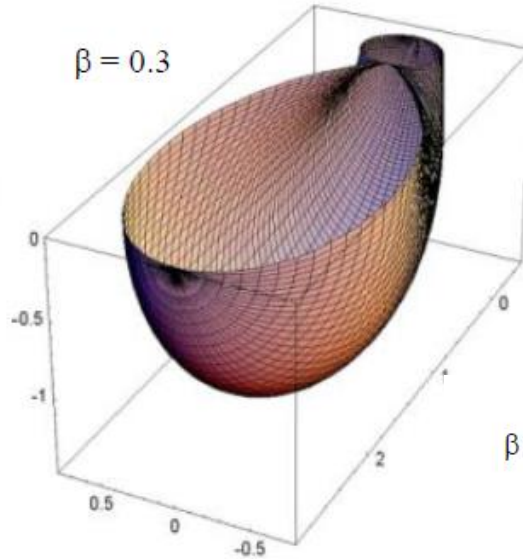
When the electron velocity approaches the speed of light the emission pattern is sharply collimated forward

velocity \perp acceleration: synchrotron radiation

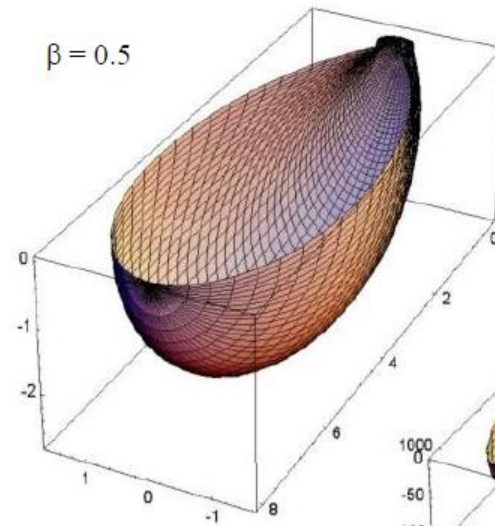
$\beta = 0$



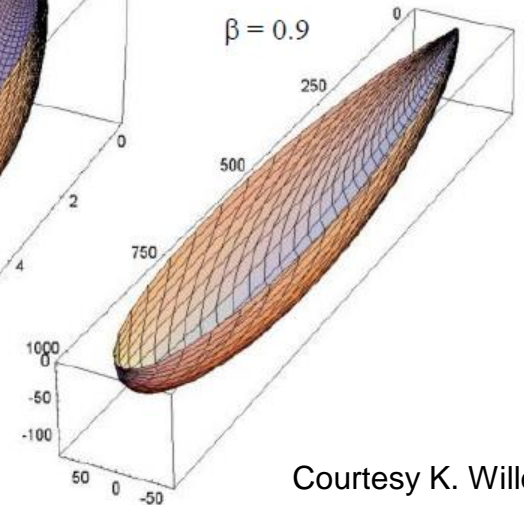
$\beta = 0.3$



$\beta = 0.5$



$\beta = 0.9$



Courtesy K. Wille

Total radiated power via synchrotron radiation

Integrating over the whole solid angle we obtain the total instantaneous power radiated by one electron

$$P = \frac{e^2}{6\pi\epsilon_0 c} \left| \dot{\beta} \right|^2 \gamma^4 = \frac{e^2}{6\pi\epsilon_0 c} \left| \dot{\beta} \right|^2 \frac{E^4}{E_0^4} = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left| \frac{d\bar{p}}{dt} \right|^2 \gamma^2 = \frac{e^2 c}{6\pi\epsilon_0} \frac{\gamma^4}{\rho^2} = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2$$

- **Strong dependence $1/m^4$ on the rest mass**
- **proportional to $1/\rho^2$ (ρ is the bending radius)**
- **proportional to B^2 (B is the magnetic field of the bending dipole)**

The radiation power emitted by an electron beam in a storage ring is very high.

The surface of the vacuum chamber hit by synchrotron radiation must be cooled.

Energy loss via synchrotron radiation emission in a storage ring

In the time T_b spent in the bendings
the particle loses the energy U_0

$$U_0 = \int P dt = P T_b = P \frac{2\pi\rho}{c} = \frac{e^2}{3\epsilon_0} \frac{\gamma^4}{\rho}$$

i.e. **Energy Loss per turn** (per electron)

$$U_0 (keV) = \frac{e^2 \gamma^4}{3\epsilon_0 \rho} = 88.46 \frac{E(GeV)^4}{\rho(m)}$$

Power radiated by a beam of average current I_b : this power loss has to be compensated by the RF system

$$N_{tot} = \frac{I_b \cdot T_{rev}}{e}$$

$$P(kW) = \frac{e\gamma^4}{3\epsilon_0 \rho} I_b = 88.46 \frac{E(GeV)^4 I(A)}{\rho(m)}$$

Power radiated by a beam of average current I_b in a dipole of length L
(energy loss per second)

$$P(kW) = \frac{e\gamma^4}{6\pi\epsilon_0 \rho^2} L I_b = 14.08 \frac{L(m) I(A) E(GeV)^4}{\rho(m)^2}$$

The radiation integral (I)

The energy received by an observer (per unit solid angle at the source) is

$$\frac{d^2W}{d\Omega} = \int_{-\infty}^{\infty} \frac{d^2P}{d\Omega} dt = c\epsilon_0 \int_{-\infty}^{\infty} |R\bar{E}(t)|^2 dt$$

Using the Fourier Transform we move to the frequency space

$$\frac{d^2W}{d\Omega} = 2c\epsilon_0 \int_0^{\infty} |R\bar{E}(\omega)|^2 d\omega$$

Angular and frequency distribution of the energy received by an observer

$$\frac{d^3W}{d\Omega d\omega} = 2\epsilon_0 c R^2 \left| \hat{E}(\omega) \right|^2$$

Neglecting the velocity fields and assuming the observer in the far field:
n constant, R constant

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{4\pi\epsilon_0 4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{\bar{n} \times [(\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}}]}{(1 - \bar{n} \cdot \bar{\beta})^2} e^{i\omega(t - \bar{n} \cdot \bar{r}(t)/c)} dt \right|^2$$

Radiation Integral

The radiation integral (II)

The radiation integral can be simplified to [see Jackson]

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi\epsilon_0 4\pi^2 c} \left| \int_{-\infty}^{\infty} \bar{n} \times (\bar{n} \times \bar{\beta}) e^{i\omega(t - \bar{n} \cdot \bar{r}(t)/c)} dt \right|^2$$

How to solve it?

- ✓ determine the particle motion $\bar{r}(t); \bar{\beta}(t); \dot{\bar{\beta}}(t)$
- ✓ compute the cross products and the phase factor
- ✓ integrate each component and take the vector square modulus

Calculations are generally quite lengthy: even for simple cases as for the radiation emitted by an electron in a bending magnet they require Airy integrals or the modified Bessel functions (available in MATLAB)

Radiation integral for synchrotron radiation

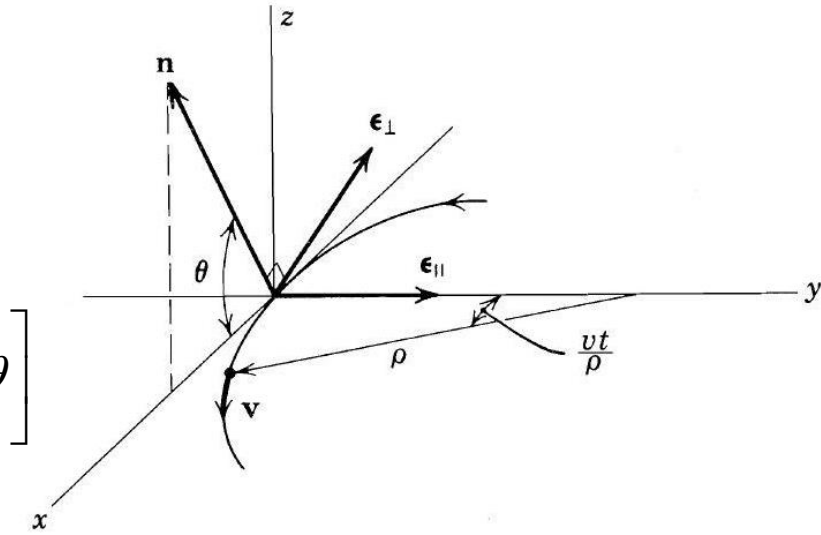
Trajectory of the arc of circumference [see Jackson]

$$\bar{r}(t) = \left(\rho \left(1 - \cos \frac{\beta c}{\rho} t \right), \sin \frac{\beta c}{\rho} t, 0 \right)$$

In the limit of small angles we compute

$$\bar{n} \times (\bar{n} \times \bar{\beta}) = \beta \left[-\bar{\epsilon}_{\parallel} \sin \left(\frac{\beta c t}{\rho} \right) + \bar{\epsilon}_{\perp} \cos \left(\frac{\beta c t}{\rho} \right) \sin \theta \right]$$

$$\omega \left(t - \frac{\bar{n} \cdot \bar{r}(t)}{c} \right) = \omega \left[t - \frac{\rho}{c} \sin \left(\frac{\beta c t}{\rho} \right) \cos \theta \right]$$



Substituting into the radiation integral and introducing $\xi = \frac{\rho \omega}{3c\gamma^3} (1 + \gamma^2 \theta^2)^{3/2}$

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{16\pi^3 \epsilon_0 c} \left(\frac{2\omega\rho}{3c\gamma^2} \right)^2 (1 + \gamma^2 \theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} K_{1/3}^2(\xi) \right]$$

Critical frequency and critical angle

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{16\pi^3 \epsilon_0 c} \left(\frac{2\omega\rho}{3c\gamma^2} \right)^2 (1 + \gamma^2\theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1 + \gamma^2\theta^2} K_{1/3}^2(\xi) \right]$$

Using the properties of the modified Bessel function we observe that the radiation intensity is negligible for $\xi \gg 1$

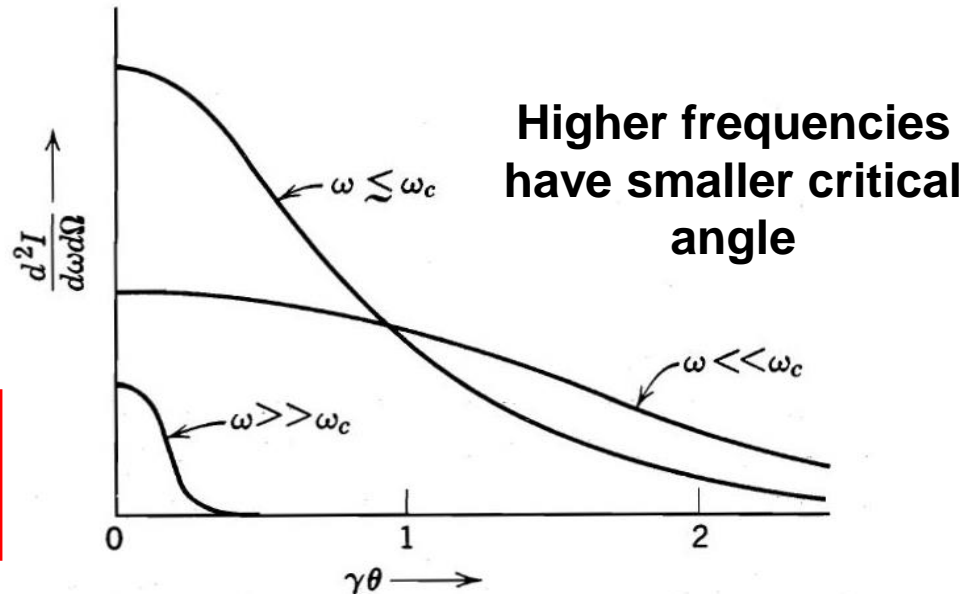
$$\xi = \frac{\omega\rho}{3c\gamma^3} (1 + \gamma^2\theta^2)^{3/2} \gg 1$$

Critical frequency

$$\omega_c = \frac{3}{2} \frac{c}{\rho} \gamma^3$$

Critical angle

$$\theta_c = \frac{1}{\gamma} \left(\frac{\omega_c}{\omega} \right)^{1/3}$$



For frequencies much larger than the critical frequency and angles much larger than the critical angle the synchrotron radiation emission is negligible

Frequency distribution of radiated energy

Integrating on all angles we get the frequency distribution of the energy radiated

$$\frac{dW}{d\omega} = \iint_{4\pi} \frac{d^3I}{d\omega d\Omega} d\Omega = \frac{\sqrt{3}e^2}{4\pi\epsilon_0 c} \gamma \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x) dx$$

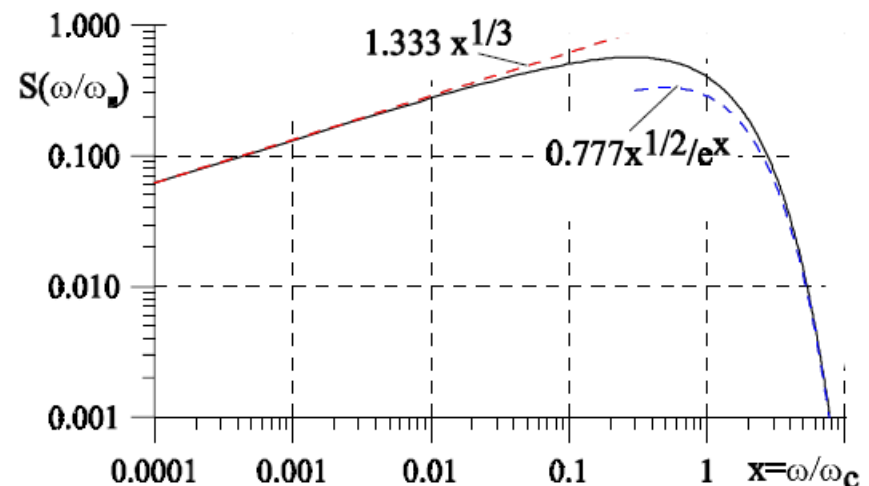
$$\frac{dW}{d\omega} \approx \frac{e^2}{4\pi\epsilon_0 c} \left(\frac{\omega\rho}{c} \right)^{1/3} \quad \omega \ll \omega_c$$

$$\frac{dW}{d\omega} \approx \sqrt{\frac{3\pi}{2}} \frac{e^2}{4\pi\epsilon_0 c} \gamma \left(\frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \quad \omega \gg \omega_c$$

often expressed in terms of the function $S(\xi)$ with $\xi = \omega/\omega_c$

$$S(\xi) = \frac{9\sqrt{3}}{8\pi} \xi \int_{\xi}^{\infty} K_{5/3}(x) dx \quad \int_0^{\infty} S(\xi) d\xi = 1$$

$$\frac{dW}{d\omega} = \frac{\sqrt{3}e^2\gamma}{4\pi\epsilon_0 c} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x) dx = \frac{2e^2\gamma}{9\epsilon_0 c} S(\xi)$$



Frequency distribution of radiated energy

It is possible to verify that the integral over the frequencies agrees with the previous expression for the total power radiated [Hubner]

$$P = \frac{U_0}{T_b} = \frac{1}{T_b} \int_0^\omega \frac{dW}{d\omega} d\omega = \frac{1}{T_b} \frac{2e^2 \gamma}{9\epsilon_0 c} \omega_c \int_0^\omega \xi d\xi \int_\xi^\infty K_{5/3}(x) dx = \frac{e^2 c}{6\epsilon_0 c} \frac{\gamma^4}{\rho^2}$$

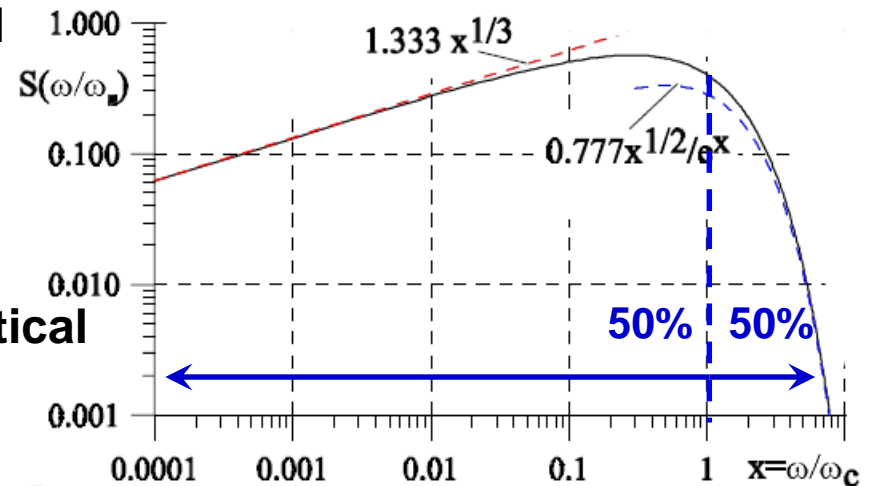
The frequency integral extended up to the critical frequency contains half of the total energy radiated, the peak occurs approximately at $0.3\omega_c$

It is also convenient to define the critical photon energy as

$$\epsilon_c = \hbar\omega_c = \frac{3}{2} \frac{\hbar c}{\rho} \gamma^3$$

For electrons, the **critical energy** in practical units reads

$$\epsilon_c [keV] = 2.218 \frac{E[GeV]^3}{\rho[m]} = 0.665 \cdot E[GeV]^2 \cdot B[T]$$



Polarisation of synchrotron radiation

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{16\pi^3 \epsilon_0 c} \left(\frac{2\omega\rho}{3c\gamma^2} \right)^2 (1 + \gamma^2\theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1 + \gamma^2\theta^2} K_{1/3}^2(\xi) \right]$$

**Polarisation in
the orbit plane**

**Polarisation orthogonal
to the orbit plane**

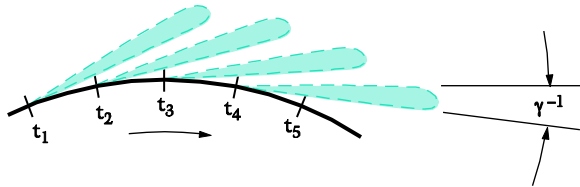
In the orbit plane $\theta = 0$, the polarisation is purely horizontal

Integrating on all frequencies we get the angular distribution of the energy radiated

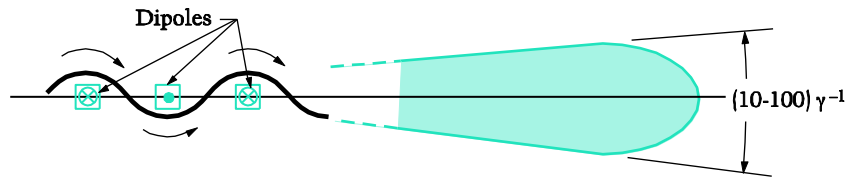
$$\frac{d^2W}{d\Omega} = \int_0^\infty \frac{d^3I}{d\omega d\Omega} d\omega = \frac{7e^2\gamma^5}{64\pi\epsilon_0\rho} \frac{1}{(1 + \gamma^2\theta^2)^{5/2}} \left[1 + \frac{5}{7} \frac{\gamma^2\theta^2}{1 + \gamma^2\theta^2} \right]$$

Integrating on all the angles we get a polarization on the plan of the orbit 7 times larger than on the plan perpendicular to the orbit

Synchrotron radiation from undulators and wigglers (more in lecture 16)



bending magnet - a "sweeping searchlight"



wiggler - incoherent superposition $K > 1$



undulator - coherent interference $K < 1$

Continuous spectrum characterized by $\varepsilon_c =$ critical energy

$$\varepsilon_c(\text{keV}) = 0.665 B(\text{T})E^2(\text{GeV})$$

eg: for $B = 1.4\text{T}$ $E = 3\text{GeV}$ $\varepsilon_c = 8.4\text{ keV}$

(bending magnet fields are usually lower $\sim 1 - 1.5\text{T}$)

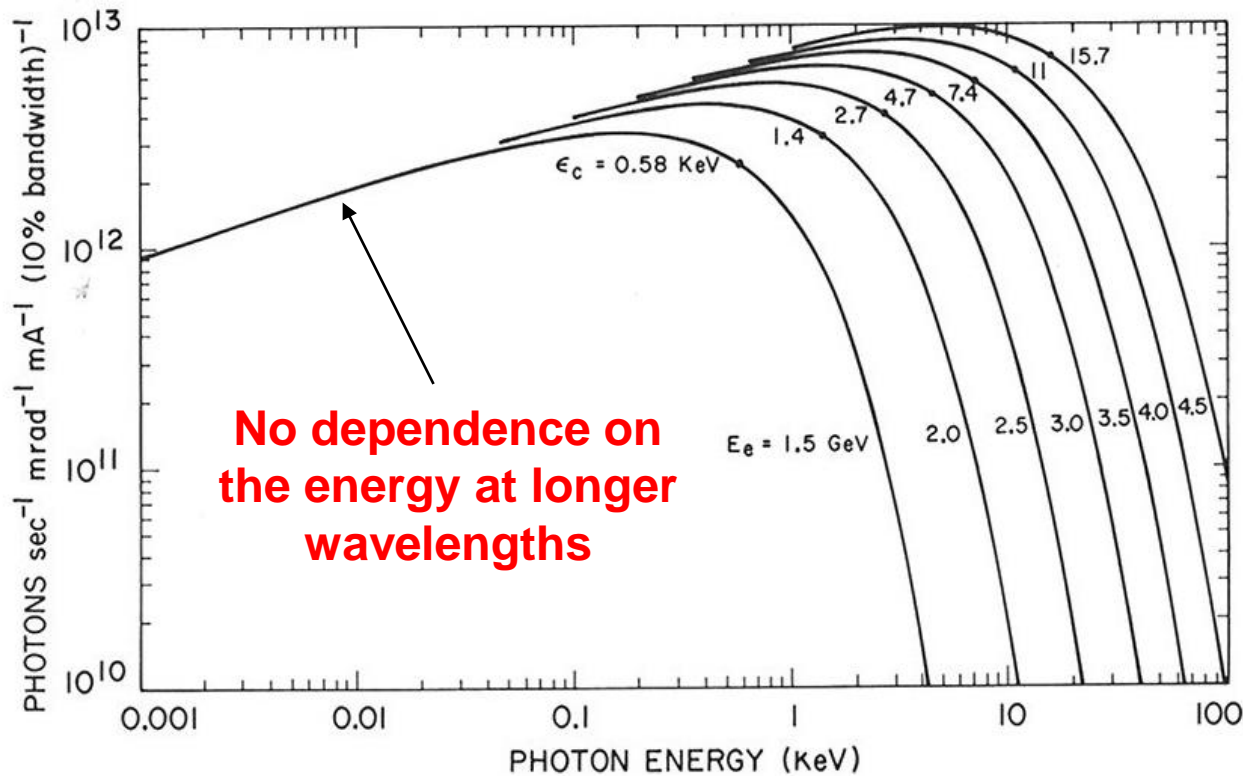
Quasi-monochromatic spectrum with peaks at lower energy than a wiggler

$$\lambda_n = \frac{\lambda_u}{2n\gamma^2} \left(1 + \frac{K^2}{2} \right) \approx \frac{\lambda_u}{n\gamma^2}$$

$$\varepsilon_n(\text{eV}) = 9.496 \frac{nE[\text{GeV}]^2}{\lambda_u[\text{m}] \left(1 + \frac{K^2}{2} \right)}$$

Synchrotron radiation emission as a function of beam the energy

Dependence of the frequency distribution of the energy radiated via synchrotron emission on the electron beam energy



Critical frequency

$$\omega_c = \frac{3}{2} \frac{c}{\rho} \gamma^3$$

Critical angle

$$\theta_c = \frac{1}{\gamma} \left(\frac{\omega_c}{\omega} \right)^{1/3}$$

Critical energy

$$\epsilon_c = \hbar \omega_c = \frac{3}{2} \frac{\hbar c}{\rho} \gamma^3$$

Summary

Accelerated charged particles emit electromagnetic radiation

Synchrotron radiation is stronger for light particles and is emitted by bending magnets in a narrow cone within a critical frequency

Undulators and wigglers enhance the synchrotron radiation emission

Synchrotron radiation has unique characteristics and many applications

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