Electron beam dynamics in storage rings

Synchrotron radiation
and its effect on electron dynamics

Lecture 1: Synchrotron radiation

Lecture 2: Undulators and Wigglers

Lecture 3: Electron dynamics-I

Lecture 4: Electron dynamics-II

Outline

Short recap on synchrotron radiation

Radiation damping of synchrotron oscillation

direct computation of damped longitudinal motion

Radiation damping of vertical betatron oscillations

modification of the vertical invariant of betatron oscillations

Radiation damping of horizontal betatron oscillations

modification of the horizontal invariant of betatron oscillations

Damping partition number and Robinson theorem

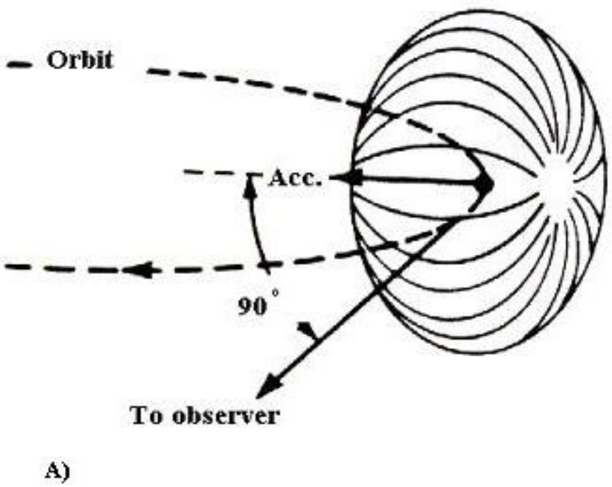
Modification of damping rates: the damping ring example

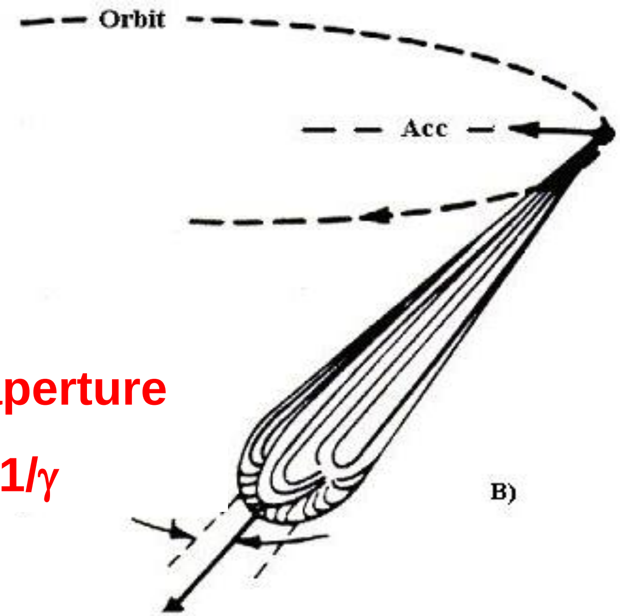
Radiation Integrals

Synchrotron radiation in a storage ring

Assuming $\bar{\beta} \perp \dot{\bar{\beta}}$ **and substituting the acceleration field**

$$\frac{d^2P}{d\Omega} = \frac{e^2}{(4\pi)^2\varepsilon_0 c^2}\frac{\left|\bar{n}\times\left[(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}\right]\right|^2}{(1-\bar{n}\cdot\bar{\beta})^5} = \frac{e^2\left|\dot{\bar{\beta}}\right|^2}{(4\pi)^2\varepsilon_0 c}\frac{1}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$





cone aperture

~ 1/γ

When the electron velocity approaches the speed of light the emission pattern is sharply collimated forward

Basic formulae for synchrotron radiation

Total instantaneous power radiated by one electron

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\left|\dot{\bar{\beta}}\right|^2 \gamma^4 = \frac{e^2}{6\pi\varepsilon_0 c}\left|\dot{\bar{\beta}}\right|^2 \frac{E}{E_o^4}^4 = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{d\bar{p}}{dt}\right|^2 \gamma^2 = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^2}{\rho^2} = \frac{e^2}{6\pi\varepsilon_0 m^4 c^5} E^2 B^2$$

Energy Loss per turn (per electron)

$$U_0 = \int P dt = PT_b = P\frac{2\pi\rho}{c} = \frac{e^2}{3\varepsilon_0}\frac{\gamma^4}{\rho} \qquad U_0(keV) = 88.46\frac{E(GeV)^4}{\rho(m)}$$

Power radiated by a beam of average current I_b

$$N_{tot} = \frac{I_b \cdot T_{rev}}{e} \qquad P(kW) = \frac{e\gamma^4}{3\varepsilon_0\rho} I_b = 88.46\frac{E(GeV)^4 I(A)}{\rho(m)}$$

The RF system will replace the energy lost by synchrotron radiation

Effects of synchrotron radiation on electron beam dynamics

The electrons radiate energy: the equations of motion have a dissipative term (non conservative system) and Liouville's theorem does not apply;

The emission of radiation leads to damping of the betatron and synchrotron oscillations

Radiation is not emitted continuously but in individual photons. The emission time the energy emitted are random variables with a known distribution (from the theory of synchrotron radiation: see spectral angular distribution of the energy radiated)

This randomness introduces fluctuations which tend to increase the betatron and synchrotron oscillations

Damping and growth reach an equilibrium in an electron synchrotron. This equilibrium defines the characteristics of the electron beam (e.g. emittance, energy spread, bunch size, etc)

Effects of synchrotron radiation on electron beam dynamics

We will now look at the effect of radiation damping on the three planes of motion

We will use two equivalent formalisms:

damping from the equations of motion in phase space

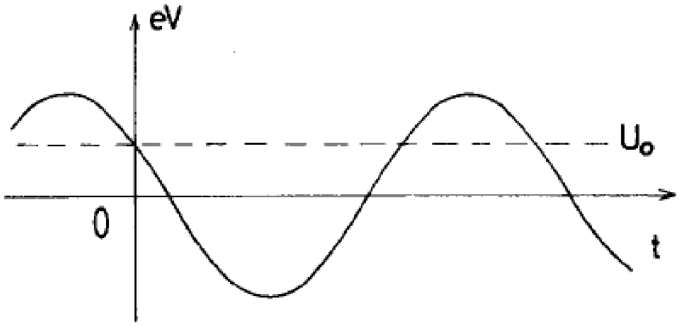
damping as a change in the Courant-Snyder invariant

The system is non-conservative hence the Courant-Snyder invariant – i.e. the area of the ellipse in phase space, is no longer a constant of motion

We will then consider the effect of radiation quantum excitation on the three planes of motion (next lecture)

We will use the formalism of the change of the Courant-Snyder invariant

From the lecture on longitudinal motion



A particle in an RF cavity changes energy according to the phase of the RF field found in the cavity

$$\Delta E = eV(t) = eV_o \sin(\omega_{RF} t + \varphi_s)$$

A particle can lose energy because of synchrotron radiation, interaction with the vacuum pipe, etc. Assume that for each turn the energy losses are U_0

The synchronous particle is the particle that arrives at the RF cavity when the voltage is such that it compensate exactly the average energy losses U_0

$$\Delta E = U_0 = eV_0 \sin(\varphi_s)$$

Negative RF slope ensure stability for α > 0 (above transition)
Veksler 1944 MacMillan 1945: the principle of phase stability

We describe the longitudinal dynamics in terms of the variables (ε, τ)

energy deviation ε w.r.t the synchronous particle

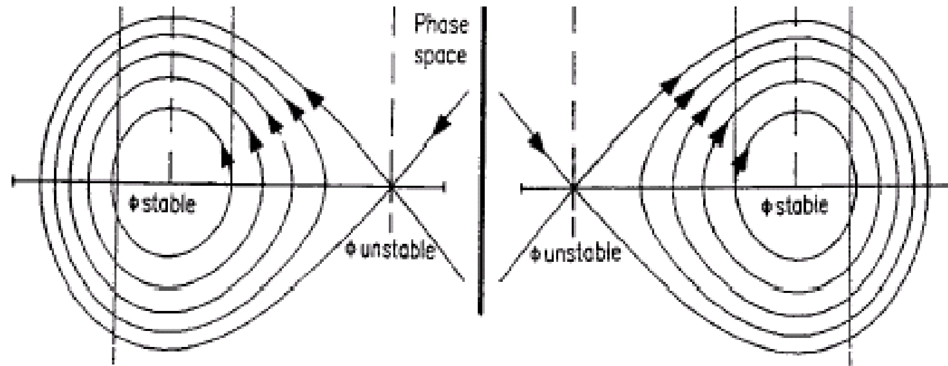
and τ time delay w.r.t. the synchronous particle

RF buckets recap.

Equations for the RF bucket

$$\varepsilon' = \frac{qV_0}{L}\left[\sin(\phi_s + \omega\tau) - \sin\phi_s\right]$$

$$\tau' = -\frac{\alpha_c}{E_s}\varepsilon$$



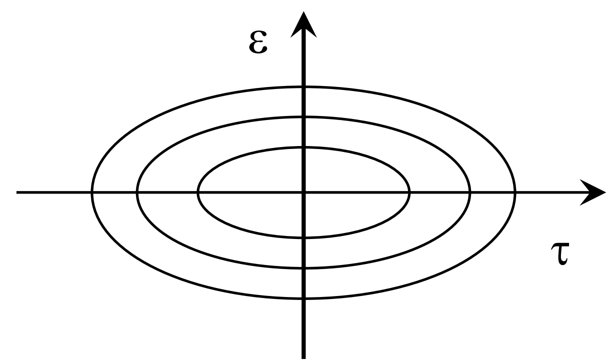
α < 0 below transition α > 0 above transition

Aide-memoire for stable motion: above transition the head goes up in energy, below transition the head goes down in energy

Linearised equations for the motion in the RF bucket: the phase space trajectories become ellipses

$$\varepsilon' = \frac{e}{T_0}\frac{dV}{d\tau}\tau$$

$$\tau' = -\frac{\alpha_c}{E_s}\varepsilon$$



$$\omega_s^2 = \frac{\alpha_c e\dot{V}}{T_0 E_0}$$ **angular synchrotron frequency**

Radiation damping: Longitudinal plane (I)

In presence of synchrotron radiation losses, with energy loss per turn U_0, the RF fields will compensate the loss per turn and the synchronous phase will be such that

$$U_0 = eV_0 \sin(\varphi_s)$$

The energy loss per turn U_0 depends on energy E. The rate of change of the energy will be given by two terms

$$\frac{\Delta E}{T_0} = \frac{eV(t) - U_0(E)}{T_0}$$

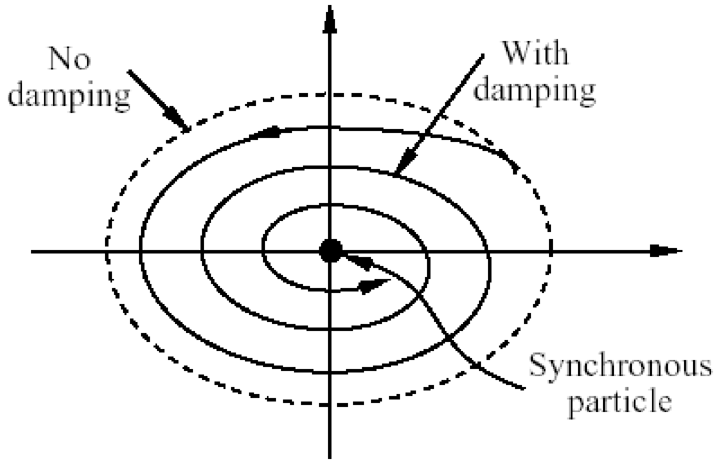
Assuming ΔE << E and τ << T_0 we can expand

$$\frac{d\varepsilon}{dt} = \frac{\left(U_0(0) + e\dfrac{dV}{d\tau}\tau\right) - \left(U_0(0) + \dfrac{dU_0}{dE}\varepsilon\right)}{T_0} = \frac{e}{T_o}\frac{dV}{d\tau}\tau - \frac{1}{T_0}\frac{dU_0}{dE}\varepsilon$$

$$\frac{d\tau}{dt} = -\alpha_c \frac{\varepsilon}{E_s}$$

additional term responsible for damping

Radiation damping: Longitudinal plane (II)



The derivative $\dfrac{dU_0}{dE} \quad (>0)$

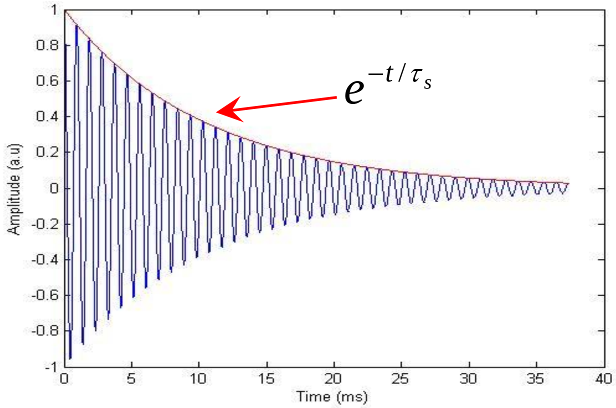
is responsible for the damping of the longitudinal oscillations

Combining the two equations for (ε, τ) in a single second order differential equation

$$\frac{d^2\varepsilon}{dt^2} + \frac{2}{\tau_s}\frac{d\varepsilon}{dt} + \omega_s^2\varepsilon = 0 \quad \longrightarrow \quad \varepsilon = Ae^{-t/\tau_s}\sin\left(\sqrt{\omega_s^2 - \frac{4}{\tau_s^2}}\,t + \varphi\right)$$

$$\omega_s^2 = \frac{\alpha e\dot{V}}{T_0 E_0}$$ **angular synchrotron frequency**

$$\frac{1}{\tau_s} = \frac{1}{2T_0}\frac{dU_0}{dE}$$ **longitudinal damping time**



Computation of dU_0/dE

We have to compute the dependence of U_0 on energy the E (or rather on the energy deviation ε)

$$U(\varepsilon) = \oint P dt$$

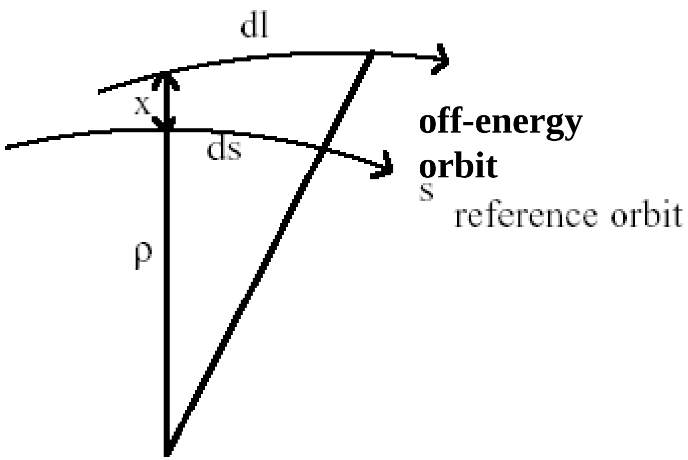
The energy loss per turn is the integral of the power radiated over the time spent in the bendings. Both depend on the energy of the particle.

$$U(\varepsilon) = \oint P(\varepsilon) dt = \frac{1}{c} \oint P(\varepsilon) dl = \frac{1}{c} \oint P(\varepsilon)\left(1 + \frac{x}{\rho}\right) ds$$

The time that an off-energy particle spends in the bending element dl is given by

$$dt = \frac{dl}{c} = \frac{1}{c}\left(1 + \frac{x}{\rho}\right) ds$$

This is an elementary geometric consideration on the arc length of the trajectory for different energies



Computation of dU_0/dE

Using the dispersion function

$$U(\varepsilon) = \frac{1}{c}\oint P\left(1 + \frac{D}{\rho}\frac{\varepsilon}{E_0}\right)ds \qquad U(0) = U_0$$

Computing the derivative w.r.t. ε at $\varepsilon = 0$ we get [Sands]

$$\frac{dU}{d\varepsilon} = \frac{1}{c}\oint\left(\frac{dP}{d\varepsilon} + \frac{D}{\rho}\frac{P}{E_0}\right)ds$$

To compute $dP/d\varepsilon$ we use the result obtained in the lecture on synchrotron radiation, whereby the instantaneous power emitted in a bending magnet with field B by a particle with energy E is given by

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\left|\dot{\vec{\beta}}\right|^2\gamma^4 = \frac{e^2}{6\pi\varepsilon_0 c}\left|\dot{\vec{\beta}}\right|^2\frac{E}{E_o^4}^4 = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left|\frac{d\vec{p}}{dt}\right|^2\gamma^2 = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^4}{\rho^2} = \frac{e^4}{6\pi\varepsilon_0 m^4 c^5}E^2 B^2$$

Watch out! There is an implicit dependence of ρ or B on E. Off energy particles have different curvatures ρ or can experience different B if B varies with x

Computation of dU_0/dE

and since P is proportional to E^2B^2 we can write [Sands]

$$\frac{dP}{d\varepsilon} = \frac{2P_0}{E_0} + \frac{2P_0}{B_0}\frac{dB}{d\varepsilon} = \frac{2P_0}{E_0} + \frac{2P_0}{B_0}\frac{D}{E_0}\frac{dB}{dx}$$

check this as an exercise !

we get

$$\frac{dU}{d\varepsilon} = \frac{1}{c}\oint\left(\frac{2P_0}{E_0} + \frac{2P_0}{B_0}\frac{dB}{dx}\frac{D}{E_0} + \frac{P_0 D}{\rho E_0}\right)ds$$

and using

$$k\rho = \frac{1}{B_0}\frac{dB}{dx} \qquad U_0 = \frac{1}{c}\oint P_0 ds$$

We have the final result

$$\frac{dU(\varepsilon)}{d\varepsilon} = \frac{2U_0}{E_0} + \frac{1}{cE_0}\oint P_0 D\left(\frac{1}{\rho} - 2k\rho\right)ds$$

Radiation damping: Longitudinal plane (V)

Longitudinal damping time

$$\frac{1}{\tau_\varepsilon} = \frac{1}{2T_0}\frac{dU}{dE} = \frac{1}{2T_0}\frac{U_0}{E_0}(2+\wp)$$

$$\wp = \frac{\oint \frac{D}{\rho}\left(\frac{1}{\rho^2} - 2k\rho\right)ds}{\oint \frac{1}{\rho^2}ds}$$

$\wp$ depends only on the magnetic lattice; typically it is a small positive quantity

$$\tau_\varepsilon = \frac{2T_0 E_0}{U_0(2+\wp)} \approx \frac{E_0 T_0}{U_0}$$

τ_ε is approximately the time it takes an electron to radiate all its energy (with constant energy loss U_0 per turn)

For separated function magnets with constant dipole field:

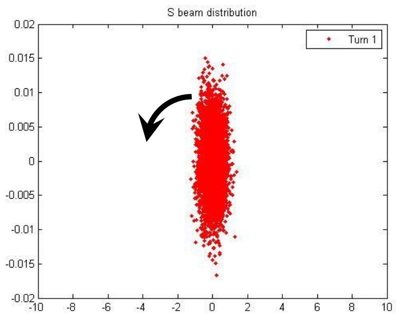
$$\frac{1}{\tau_\varepsilon} = \frac{U_0}{2E_0T_0}\left(2 + \frac{\alpha R}{\rho}\right) \qquad \frac{1}{\tau_\varepsilon} \propto \frac{\gamma^3}{\rho R}$$

Tracking example: longitudinal plane

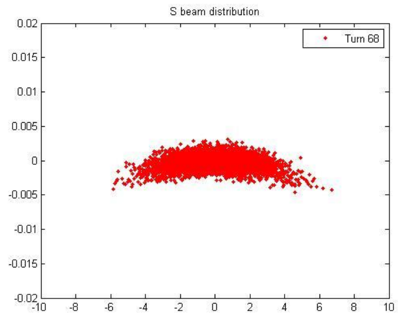
Consider a storage ring with a synchrotron tune of 0.0037 (273 turns);

and a radiation damping of 6000 turns:

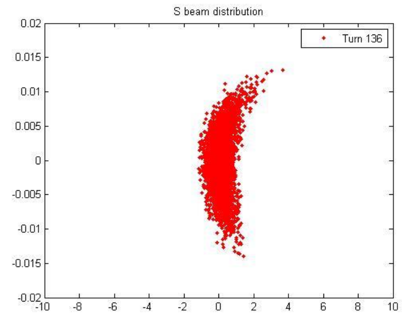
start



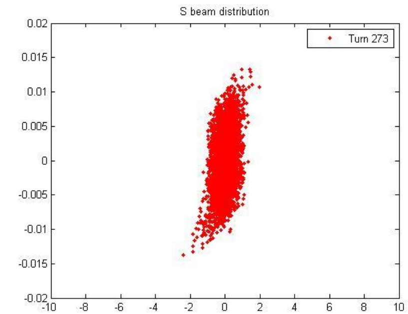
¼ of synch period



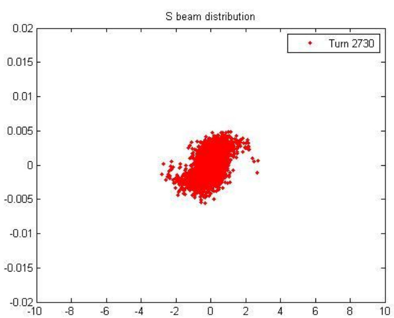
½ of synch period



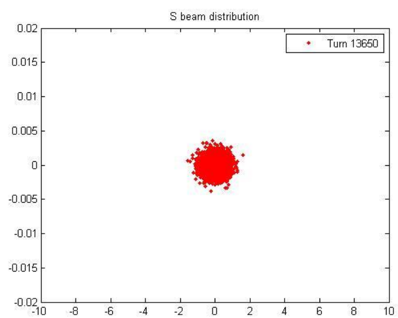
1 synch period



10 synch periods



50 synch periods

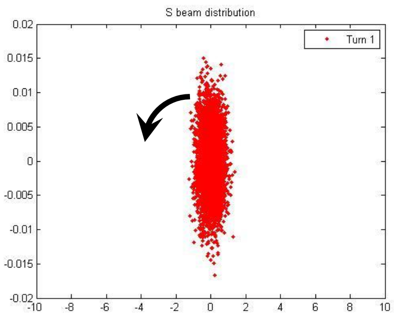


After 50 synchrotron periods (~2 radiation damping time) the longitudinal phase space distribution has almost reached the equilibrium and is matched to the RF bucket

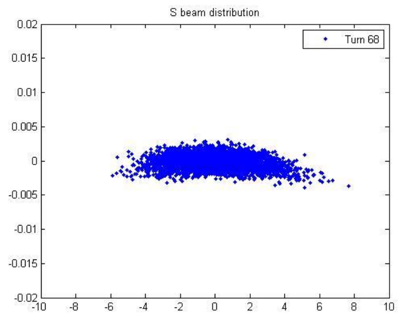
Tracking example: longitudinal plane

Consider a storage ring with a synchrotron tune of 0.0037 (273 turns); negligible radiation damping:

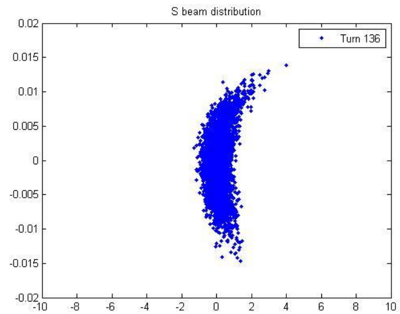
start



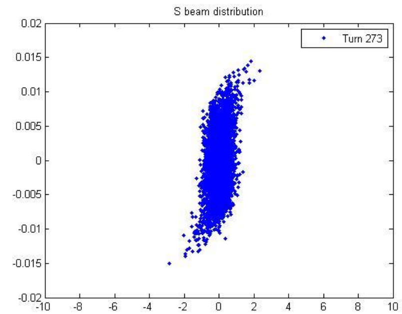
¼ of synch period



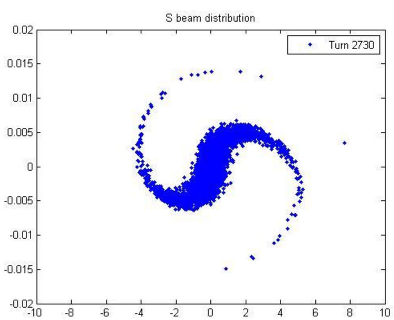
½ of synch period



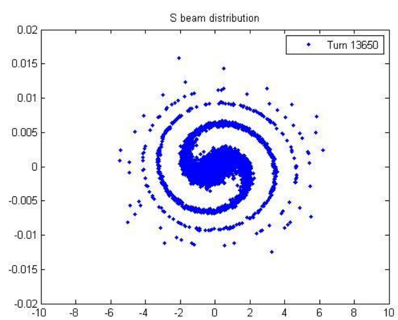
1 synch period



10 synch periods



50 synch periods



After 50 synchrotron periods the longitudinal phase space distribution is completely filamented (decoherence).

Any injection mismatch will blow up the beam

Transverse plane: vertical oscillations (I)

We now want to investigate the radiation damping in the vertical plane.

Because of radiation emission the motion in phase space is no longer Conservative and symplectic, i.e. the area of the ellipse defining the Courant-Snyder invariant is changing along one turn. We want to investigate this change.

We assume to simplify the calculations that we are in a section of the ring where (α_z = 0), then

$$z = A\cos(\phi(s) + \phi_0) \qquad z' = -\frac{A}{\beta}\sin(\phi(s) + \phi_0)$$

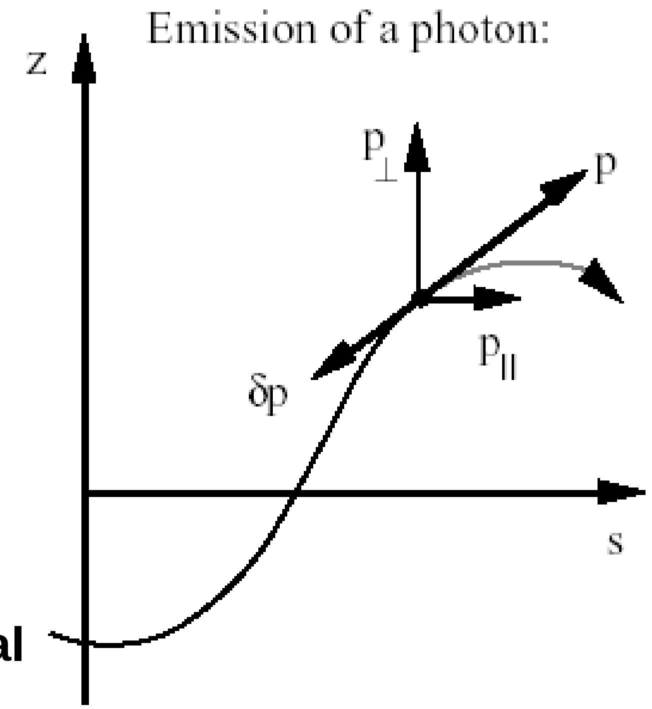
The ellipse in the vertical phase space is upright. The Courant-Snyder invariant reads

$$A^2 = z^2 + (\beta z')^2$$

Transverse plane: vertical oscillations (II)

Effect of the emission of a photon:

The photon is emitted in the direction of the momentum of the electron (remember the cone aperture is 1/γ)



The momentum p is changed in modulus by dp but it is not changed in direction

neither z nor z' change

and

the oscillation pattern is not affected since $D_z = 0$

(see later case where Dx $\neq$ 0 as for the horizontal plane)

Therefore the Courant-Snyder invariant does not change as result of the emission of a photon

… however the RF cavity must replenish the energy lost by the electron

Transverse plane: vertical oscillations (III)

In the RF cavity the particle sees a longitudinal accelerating field therefore only the longitudinal component is increased to restore the energy

The momentum variation is no longer parallel to the momentum

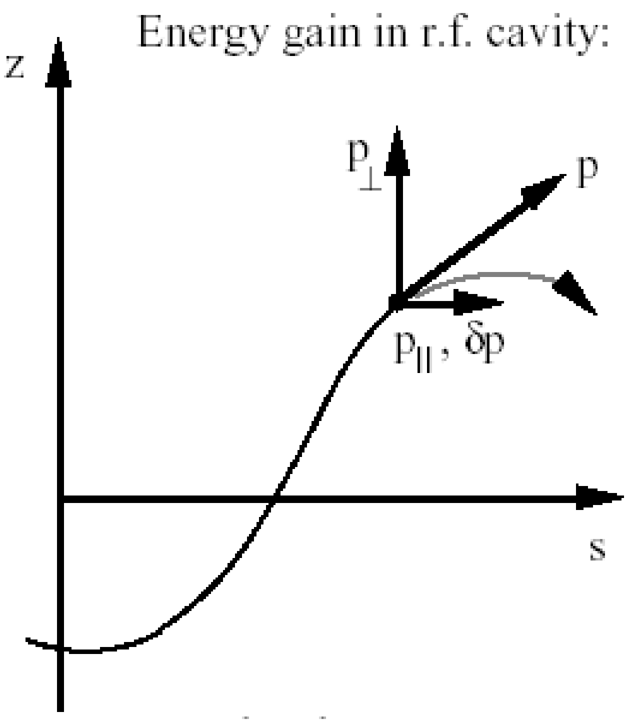
this leads to a reduction of the betatron oscillations amplitude

The angle $z' = \dfrac{P_{\perp}}{p_{\parallel}}$ **changes because**

$$z' + \delta z' = \frac{p_{\perp}}{p_{\parallel} + \delta p} \approx z'\left(1 - \frac{\delta p}{p}\right)$$

$$\delta z' = -z' \frac{\delta \varepsilon}{E_0}$$

δε gained in the RF cavity



Transverse plane: vertical oscillations (IV)

After the passage in the RF cavity the expression for the vertical invariant becomes

$$A^2 = z^2 + (\beta z')^2 \qquad (\mathrm{A} + \delta\mathrm{A})^2 = z^2 + [\beta(z' + \delta z')]^2$$

$$A\delta A = \beta^2 z' \, \delta \, z' = -\beta^2 z'^2 \frac{\delta\varepsilon}{E_0}$$

The change in the Courant-Snyder invariant depends on the angle z’ for this particular electron. Let us consider now all the electrons in the phase space travelling on the ellipse, and therefore having all the same invariant A

For each different z’ the change in the invariant will be different. However averaging over the electron phases, assuming a uniform distribution along the ellipse, we have

$$< z'^2 >_{e-} = \frac{A^2}{2\beta^2}$$ **and therefore** $$< \delta A >_{e-} = -\frac{A}{2}\frac{\delta\varepsilon}{E_0}$$

The average invariant decreases.

Transverse plane: vertical oscillations (V)

Let us consider now all the photons emitted in one turn. The total energy lost is

$$U_0 = \sum_C \varepsilon$$

The RF will replenish all the energy lost in one turn.

Summing the contributions $\delta\varepsilon$, we find that in one turn:

$$<< \delta A >_{e^-} >_{ph} = -\frac{A}{2}\frac{U_0}{E_0} \qquad \textbf{we write} \qquad \frac{\Delta A}{A} = -\frac{U_0}{2E_0} \qquad -\frac{1}{A}\frac{dA}{dt} = \frac{U_0}{2E_0T_0} = \frac{1}{\tau_z}$$

The average invariant decreases exponentially with a damping time τ_z
$\tau_z \approx$ half of longitudinal damping time always dependent on $1/\gamma^3$.

This derivation remains true for more general distribution of electron in phase space with invariant A (e.g Gaussian)

The synchrotron radiation emission combined with the compensation of the energy loss with the RF cavity causes the damping.

Transverse plane: vertical oscillations (VI)

The betatron oscillations are damped in presence of synchrotron radiation

$$z(t) = z_0 e^{-t/\tau_z} \sin\left(\omega_\beta t + \varphi\right)$$

Since the emittance of a bunch of particle is given by the average of the square of the betatron amplitude of the particles in the bunch taken over thebunch distribution in phase space

$$\varepsilon_z = \frac{< z^2 >}{\beta_z}$$

the emittance decays with a time constant which is half the radiation damping time

$$\varepsilon_z(t) = \varepsilon_z(0) \exp\left(-\frac{2t}{\tau_z}\right)$$

Transverse plane: horizontal oscillations (I)

The damping of the horizontal oscillation can be treated with the same formalism used for the vertical plane, e.g.

- **consider the electron travelling on an ellipse in phase space with invariant A**
- **compute the change in coordinates due to the emission of one photon**
- **compute the change of coordinates due to the passage in the RF**
- **averaging over all electron with the same invariant**
- **compute the change in the average invariant for all photons emitted in one turn**

The new and fundamental difference is that in the horizontal plane we do not neglect the dispersion, i.e. $D_x \neq 0$

The reference orbit changes when a quantum is emitted because of D_x in the bendings. The electron will oscillate around its new off-energy orbit. In details:

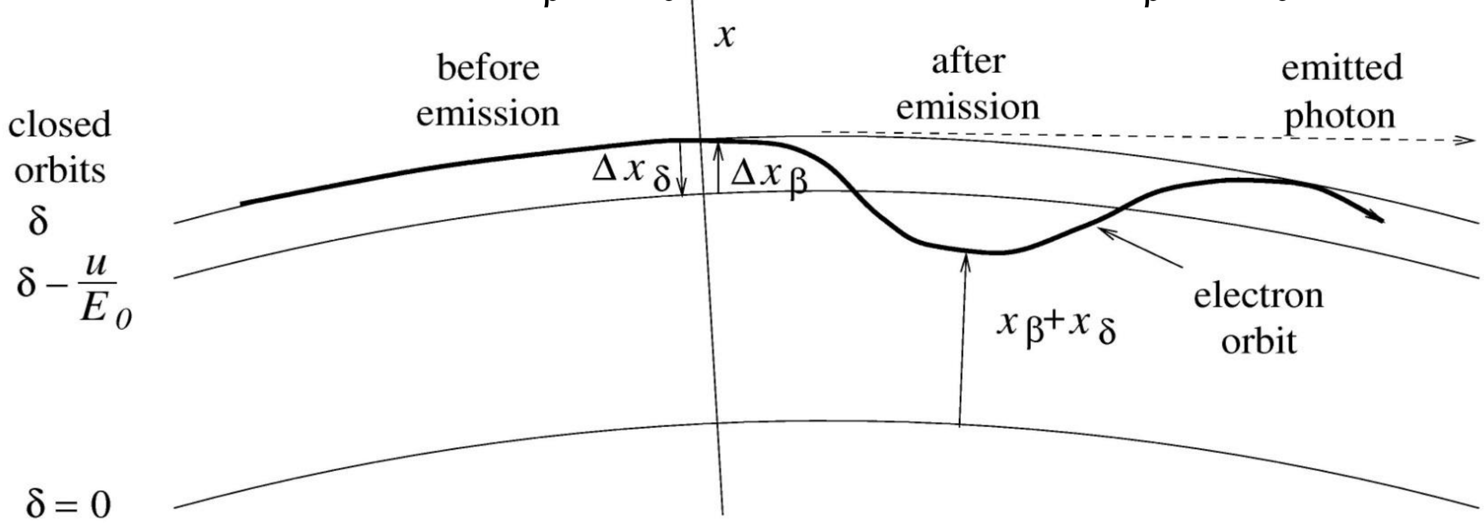
Transverse plane: horizontal oscillations (II)

After the emission of a photon, the physical position and the angle of the electron do not change. However they must be referenced to a new orbit:

This is the off-energy orbit corresponding to the new energy of the electron

With respect to the off-energy orbit, the emission of a photon appears as an offset (and an angle)

$$\delta x = 0,\ \delta x' = 0 \text{ but } \delta x_\beta + \delta x_\delta = 0 \text{ (and likewise } \delta x'_\beta + \delta x'_\delta = 0)$$



Transverse plane: horizontal oscillations (III)

We follow the same line as done for the vertical plane. The equations of motion in the horizontal plane (α_x = 0) are

$$x = A\cos(\phi(s) + \phi_0) \qquad x' = -\frac{A}{\beta}\sin(\phi(s) + \phi_0)$$

Invariant in the horizontal plane

$$A^2 = x^2 + (\beta x')^2$$

After the photon emission position and angle do not change but with respect to the new (off energy) orbit

$$x_\beta = x + \delta x_\beta \qquad x'_\beta = x' + \delta x'_\beta$$

and we has said that

$$\delta x_\beta = -\delta x_\varepsilon = -D\frac{\delta\varepsilon}{E_0} \qquad \textbf{and similarly} \qquad \delta x'_\beta = -\delta x'_\varepsilon = -D'\frac{\delta\varepsilon}{E_0}$$

The new invariant in the horizontal plane (with respect to the new orbit) reads

$$(A + \delta A)^2 = (x + \delta x_\beta)^2 + [\beta(x' + \delta x'_\beta)^2]$$

Transverse plane: horizontal oscillations (IV)

The change in the Courant-Snyder invariant due to δx_β and $\delta x'_\beta$ to first order in $\delta\varepsilon$ reads

$$A\delta A = x_\beta \delta x_\beta + \beta^2 x_\beta' \delta x_\beta' = -(D x_\beta + \beta^2 D' x_\beta')\frac{\delta\varepsilon}{E_0}$$

As before the change in the Courant Snyder invariant depends on the specific

betatron coordinates x_β and x'_β of the electron .

We want to average of all possible electron in an ellipse with the same Courant- Snyder invariant and get

$$< A\delta A >_{e^-} = -D\frac{< x_\beta \delta\varepsilon >_{e^-}}{E_0} + \beta^2 D' \frac{< x_\beta' \delta\varepsilon >_{e^-}}{E_0}$$

If for each photon emission the quantity $\delta\varepsilon$ is independent on x_β and x'_β, then averaging the previous expression over the phases of the betatron oscillations would give zero.

However, in the horizontal plane $\delta\varepsilon$ depends on x_β in two ways [Sands]

Transverse plane: horizontal oscillations (V)

Let us compute the dependence of the energy $\delta\varepsilon$ of the photon emitted in the horizontal plane on x_β [Sands].

Assuming that the emission of photon is described as a continuous loss of energy (no random fluctuations in the energy of the photon emitted), we have

$$\delta\varepsilon = -P dt$$

both P and dt depend on the betatron coordinate of the electron

$$dt = \frac{dl}{c} = \frac{1}{c}\left(1 + \frac{x_\beta}{\rho}\right) ds$$

And, since $P \propto B^2$, to the first order in x_β

$$P(x_\beta) = P + 2\frac{P}{B}\frac{dB}{dx} x_\beta$$

Transverse plane: horizontal oscillations (V)

The energy change reads

$$\delta\varepsilon = -Pdt = -\frac{P}{c}\left[1+\left(\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right)x_\beta\right]\delta s$$

Substituting in

$$A\delta A = -(D\,x_\beta + \beta^2 D'\,x_\beta'\,)\frac{\delta\varepsilon}{E_0}$$

We get

$$A\delta A = \frac{2P_0}{U_s c}[\gamma x_\beta D + \alpha(x_\beta D' + x'_\beta\, D) + \beta x'_\beta\, D']\left(1 - 2k\rho\, x_\beta + \frac{x_\beta}{\rho}\right)\delta s$$

The change in the Courant-Snyder invariant depends on the position and angle x_β and x'_β for this particular electron. Let us consider now all the electrons in the phase space travelling on the ellipse, and therefore having all the same invariant A

Transverse plane: horizontal oscillations (VI)

For each different x_β and x'_β the change in the invariant will be different. However averaging over the electron phases, assuming a uniform distribution along the ellipse, we have

$$\frac{<\delta A>_{e^-}}{A} = \frac{1}{2cE_0} P_0 D\left(\frac{1}{\rho} - 2k\rho\right)\delta s \qquad k\rho = \frac{1}{B_0}\frac{dB}{dx}$$

The average invariant can now increase or decrease depending on the sign of the previous term, i.e. depending on the lattice.

Let us consider all the photons emitted in one turn. The total energy lost is

$$U_0 = \sum_C \varepsilon$$

Summing the contributions $\delta\varepsilon$ in one turn, we find that in one turn:

$$\frac{<<\delta A>_{e^-}>_{ph}}{A} = \frac{1}{2cE_0}\oint P_0 D\left(\frac{1}{\rho} - 2k\rho\right)\delta s = \frac{U_0}{E_0}\frac{\wp}{2}$$

Transverse plane: horizontal oscillations (VII)

The change in the horizontal average invariant due to the emission of a photon

$$\frac{1}{A}\frac{dA}{dt} = \frac{U_0 \wp}{2E_0 T_0}$$

$\wp$>0 gives an anti-damping term

As in the vertical plane we must add the contribution due to the RF that will replenish all the energy lost.

Adding the RF contribution (as before assuming D_x = 0 at the RF cavities)

$$-\frac{1}{A}\frac{dA}{dt} = \frac{U_0}{2E_0 T_0}(1-\wp) = \frac{1}{\tau_x}$$

The average horizontal invariant decreases (or increases) exponentially with a damping time τ_z .$\tau_z \approx$ half of longitudinal damping time always dependent on $1/\gamma^{3.}$

This remains true for more general distribution of electron in phase space with invariant A (e.g Gaussian)

Transverse plane: horizontal oscillations (VIII)

As in the vertical plane, the horizontal betatron oscillations are damped in presence of synchrotron radiation

$$x(t) = x_0 e^{-t/\tau_x} \sin\left(\omega_\beta t + \varphi\right)$$

Since the emittance of a bunch of particle is given by the average of the square of the betatron amplitude of the particles in the bunch

$$\varepsilon_x = \frac{< x^2 >}{\beta_x}$$

the emittance decays with a time constant which is half the radiation damping time

$$\varepsilon_x(t) = \varepsilon_x(0) \exp\left(-\frac{2t}{\tau_x} \right)$$

Damping partition numbers (I)

The results on the radiation damping times can be summarized as

$$\frac{1}{\tau_i} = \frac{J_i U_0}{2E_0 T_0} \qquad J_x = 1 - \wp; \qquad J_z = 1; \qquad J_\varepsilon = 2 + \wp;$$

The J_i are called damping partition numbers, because the sum of the damping rates is constant for any $\wp$ (any lattice)

$$J_x + J_z + J_\varepsilon = 4$$

(Robinson theorem)

Damping in all planes requires $-2 < \wp < 1$

Fixed U_0 and E_0 one can only trasfer damping from one plane to another

Adjustment of damping rates

Modification of all damping rates:

Increase losses U_0

Adding damping wigglers to increase U_0 is done in damping rings to decrease the emittance

Repartition of damping rates on different planes:

Robinson wigglers: increase longitudinal damping time by decreasing the horizontal damping (reducing dU/dE)

Change RF: change the orbit in quadrupoles which changes $\wp$ and reduces τ_x

Robinson wiggler at CERN

Example: damping rings

Damping rings are used in linear colliders to reduce the emittance of the colliding electron and positron beams:

The emittance produced by the injectors is too high (especially for positrons beams).

In presence of synchrotron radiation losses the emittance is damped according to

$$\varepsilon_{fin} = \varepsilon_{eq} + (\varepsilon_{in} - \varepsilon_{eq}) \cdot e^{-2T/\tau_x}$$

The time it takes to reach an acceptable emittance will depend on the transverse damping time

The emittance needs to be reduced by large factors in a short store time T. If the natural damping time is too long, it must be decreased.

This can be achieved by introducing damping wigglers. Note that damping wigglers also generate a smaller equilibrium emittance ε_{eq} (see CAS).

Example: damping rings

Using ILC parameters

ε_i **= 0.01 m** ε_f **= 10 nm** $\varepsilon_f / \varepsilon_i = 10^{-6}$

The natural damping time is T ~ 400 ms while it is required that $T/\tau_x \sim 15$**, i.e. a damping time** $\tau_x \sim 30$ **ms (dictated by the repetition rate of the following chain of accelerators – i.e. a collider usually)**

Damping wigglers reduce the damping time by increasing the energy loss per turn

$$\frac{1}{\tau_i} = \frac{J_i U_0}{2 E_0 T_0}$$

With the ILC damping ring data

E = 5 GeV, ρ **= 106 m,** **C = 6700 m,**

we have

U_0 **= 520 keV/turn** $\tau_x = 2ET_0/U_0 = 430$ **ms**

Example: damping rings

The damping time τ_x has to be reduced by a factor 17 to achieve e.g. 25 ms.

Damping wigglers provide the extra synchrotron radiation energy losses without changing the circumference of the ring.

The energy loss of a wiggler E_w with peak field B and length L and are given by (see lecture on wigglers)

$$E_w = \frac{2}{3}\frac{r_e e^2}{m^3 c^4} E^2 B_w^2 L_w$$

or in practical units the energy loss per electron reads

$$E_w(\mathrm{eV}) = 0.07257\,\frac{E[\mathrm{GeV}]^2 K^2}{\lambda_u[\mathrm{m}]^2} L_w[\mathrm{m}] \qquad K = \frac{e\lambda_u B_w}{2\pi mc}$$

A total wiggler length of 220 m will provide the required damping time.

Radiation integrals

Many important properties of the stored beam in an electron synchrotron are determined by integrals taken along the whole ring:

$$I_1 = \oint \frac{D_x}{\rho} ds \qquad I_2 = \oint \frac{1}{\rho^2} ds \qquad I_3 = \oint \frac{1}{\rho^3} ds$$

$$I_4 = \oint \frac{D_x}{\rho} \left(\frac{1}{\rho^2} - 2k \right) ds \qquad I_5 = \oint \frac{H}{\rho^3} ds \qquad H = \gamma D_x^2 + 2\alpha D_x D'_x + \beta D'^2_x$$

In particular

$$U_0 = \frac{e^2}{6\pi\varepsilon_0} \gamma^4 \oint \frac{1}{\rho^2} ds = \frac{e^2}{6\pi\varepsilon_0} \gamma^4 I_2$$

Energy loss per turn

$$J_x = 1 - \frac{I_4}{I_2} \qquad J_\varepsilon = 2 + \frac{I_4}{I_2}$$

Damping partition numbers

$$\tau_x = \frac{3T_0}{r_0 \gamma^3} \frac{1}{I_2 - I_4} \qquad \tau_z = \frac{3T_0}{r_0 \gamma^3} \frac{1}{I_2} \qquad \tau_\varepsilon = \frac{3T_0}{r_0 \gamma^3} \frac{1}{2I_2 + I_4}$$

Damping times

Summary

Synchrotron radiation losses and RF energy replacement generate a damping of the oscillation in the three planes of motion

The damping times depend on the energy as $1/\gamma^3$ and on the magnetic lattice parameters (stronger for light particles)

The damping times can be modified, but at a fixed energy losses, the sum of the damping partition number is conserved regardless of the lattice type

Radiation damping combined with radiation excitation determine the equilibrium beam distribution and therefore emittance, beam size, energy spread and bunch length.