How to calculate/organize higher QED corrections in a way friendly to soft photon resumation?

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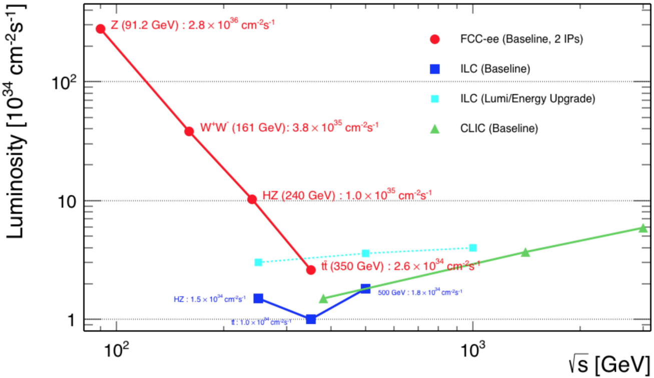
FCCee as Z-factory: 1.5×10^{12} Z's/year! $10^5 \times$ LEP

Luminosities and centre-of mass energies



LEP record at the Z
$2.3 \cdot 10^{31}$ cm^{-2}s^{-1}

LEP2 record
$\approx 10^{32}$ cm^{-2}s^{-1}

From summary talk of A. Blondel, FCCee week, Berlin 2017

QED (lamost) always listed as a challenge!

observable	Physics	Present precision		FCC-ee stat Syst Precision	FCC-ee key	Challenge
M_Z MeV/c2	Input	91187.5 $\pm$2.1	Z Line shape scan	0.005 MeV <$\pm$0.1 MeV	E_cal	QED corrections
Γ_Z MeV/c2	$\Delta\rho$ (T) (no $\Delta\alpha$!)	2495.2 $\pm$2.3	Z Line shape scan	0.008 MeV <$\pm$0.1 MeV	E_cal	QED corrections
$R_l \equiv \frac{\Gamma_h}{\Gamma_l}$	α_s, δ_b	20.767 (25)	Z Peak	0.0001 (2-20)	Statistics	QED corrections
N_ν	Unitarity of PMNS, sterile ν's	2.984 $\pm$0.008	Z Peak Z+γ(161 GeV)	0.00008 (40) 0.001	->lumi meast Statistics	QED corrections to Bhabha scat.
R_b	δ_b	0.21629 (66)	Z Peak	0.000003 (20-60)	Statistics, small IP	Hem. correlations
A_{LR}	$\Delta\rho$, ε_3, $\Delta\alpha$ (T, S)	$\sin^2\theta_W^{eff}$ 0.23098(26)	Z peak, Long. polarized	$\sin^2\theta_W^{eff}$ $\pm$0.000006	4 bunch scheme	Design experiment
A_{FB}^{lept}	$\Delta\rho$, ε_3, $\Delta\alpha$ (T, S)	$\sin^2\theta_W^{eff}$ 0.23099(53)		$\sin^2\theta_W^{eff}$ $\pm$0.000006	E_cal & Statistics	
M_W MeV/c2	$\Delta\rho$, ε_3, ε_2, $\Delta\alpha$ (T, S, U)	80385 $\pm$ 15	Threshold (161 GeV)	0.3 MeV <0.5 MeV	E_cal & Statistics	QED corections
m_{top} MeV/c2	Input	173200 $\pm$ 900	Threshold scan	~10 MeV	E_cal & Statistics	Theory limit at 50 MeV?

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INTRODUCTION

- In most of experimental measurements considered the uncertainties due to QED corrections are quoted as a major/dominant ones.
- It is necessary to re-discuss how efficiently these trivial but large QED effects can be controlled and/or eliminated more precisely by factor 10-100 than in LEP.
- Already at LEP, without resumming infrared (IR) big Sudakov double logs due to soft photons, it was not possible to get QED corrections under control with a sufficient precision.
- There was no single example in the LEP era data analysis, where the so-called "fixed-order" QED calculation was precise enough!
- In the IR-resumed calculation "fixed-order" calculation is a "raw material" from which IR-finite objects (IRFIN) are extracted.
- In LEP era most of QED IRFINs were extracted from $\mathcal{O}(\alpha^1)$ "fixed-order" calculation (except of $\mathcal{O}(\alpha^2)$ implementation in KKMC).
- At FCCee $\mathcal{O}(\alpha^2)$ will be the baseline and soft resummation mandatory.

Main message of this talk:

- If you calculate multiloop SM radiative coreections including QED photonic part, DO NOT follow Bloch-Nordcieck!!!
- That is DO NOT ADD REAL soft photons to kill infrared singularities of virtual loops!!!
- Instead, SUBTRACT VIRTUAL part of the-Sudakov Yennie-Frautschi-Suura type from your virtual loop calculations.
- Why? To make life easier for you and others, more of that follows...

General problem and potential solution

- The following concerns mainly virtual corrections.
- Separating $\mathcal{O}(\alpha^1)$ "fixed-order" QED/SM corrections into IR-divergent and IR-finite part is relatively easy using any variant of IR regulator, etc.
- At the $\mathcal{O}(\alpha^2)$, IR-divergent double logs and single collinear logs may account for 99% of the "fixed-order" algebraic and/or numerical results.
- Moreover, this IR/collinear "trash", well known at any perturbative order, obscuring IR-finite part, is completely useless, especially in case of Monte Carlo with the IR-resummation built into it.
- It is of great practical importance to organize SM/QED perturbative calculations such, that IR-subtraction is done as early as possible, before the integrations, at the integrand level, getting rid of this "IR trash" as early as possible.
- In principle, one may also profit from the fact that IR-finite (IR-subtracted) objects consist of many components, each of them is IR-finite.
- Moreover, since subtraction is done before the integration, there is no need of regulating IR divergences at all! (Photon mass or dimensionaly.)

Important misunderstanding concerning real-virtual cancellations

- Among our colleagues involved in the calculations of the “fixed-order” perturbative SM calculation, there is an important misunderstanding concerning cancellations of the QED real and virtual corrections.
- They think that once calculation of the virtual corrections finished, one should ADD soft real photon contributions, to cancel IR divergences.
- In the context of the resummed calculation it is methodological mistake!
- In the IR-resummed calculation (in the MC form) real-virtual cancellations are already done somewhere else, independently, up to $\mathcal{O}(\alpha^\infty)$!
- The correct procedure is to SUBTRACT from virtual fixed order calculations well-known (at any order) “standardized” IR formfactor.
- NB. In LEP era it was impossible to convince our colleagues to do it properly – so typically we had to undo what they did, un-subtract real-soft and subtract virtual-soft formfactor of resummation.
- It would be great to avoid this mess in the future!!!

Unsolved problems

- In the real photon phase space, in the IR-resumed MC one may forget about dimensional or photon-mass regulation in the soft limit!
- IR-subtracted objects used in the MC are just finite.
- Closer examination of any classic $\mathcal{O}(\alpha^1)$ virtual integral shows that in case of the algebraic integration of the IR-subtracted object, it turn out that the introduction of some IR regulator at the intermediate stage is almost unavoidable.
- But may be I am wrong?
- On the other hand, in case of numerical evaluation of the IR-subtracted virtual corrections, this could be feasible and natural?
- The above conjecture should be examined by the experts!
- *NB. What about collinear/mass logs?*
  So far I have assumed that they are regularized by finite masses of fermions.
  Subtraction of collinear/mass logs is, of course, done in the QCD matching scenarios like KrkNLO, POWHEG, MC@NLO...

Let us illustrate the above with some formulas in the following...

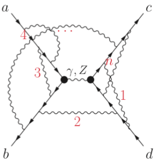
Flagship projects of Krakow group in LEP era 1989-2002

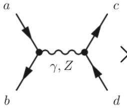
Standard Model calculations for LEP with YFS exponentiation

- $e^+e^- \to f\bar{f} + n\gamma$, $f = \tau, \mu, d, u, s, c$, YFS1 (1987-1989) $\mathcal{O}(\alpha^1)_{exp}$ ISR,
  YFS2∈KORALZ (1989-1990), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$ ISR
  YFS3∈KORALZ (1990-1998), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$ ISR+FSR
  KKMC (98-02) $\mathcal{O}(\alpha^2 + h.o.LL)_{exp}$ ISR+FSR+Interf. $d\sigma/\sigma = 0.2\%$
- $e^+e^- \to e^+e^- + n\gamma$ for $\theta < 6^\circ$
  BHLUMI 1.x, (1987-1990), $\mathcal{O}(\alpha^1)_{exp}$
  BHLUMI 2.x, (1990-1996), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$ $d\sigma/\sigma = 0.07\%$
- $e^+e^- \to e^+e^- + n\gamma$ for $\theta > 6^\circ$
  BHWIDE (1994-1998), $\mathcal{O}(\alpha^1 + h.o.LL)_{exp}$
- $e^+e^- \to W^+W^- + n\gamma, W^\pm \to f\bar{f}$
  KORALW (1994-2001)
- $e^+e^- \to W^+W^- + n\gamma, W^\pm \to f\bar{f}$
  YFS3WW (1995-2001), YFS expon. + Leading Pole Approx. $d\sigma/\sigma = 0.4\%$

YFS soft photon resummation scheme

Factorization of virtual IR by Yennie-Frautschi-Suura (1961):

$$\sum_{n=0}^{\infty}$$ 

$$= e^{\alpha B_4}$$ 

$$\times (1 + \Delta_{\rm finite})$$

where $B_4(p_a, ..., p_d) = \int \frac{d^4k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \left| J_I(k) - J_F(k) \right|^2,$

$$J_I = eQ_e\big(\hat{J}_a(k) - \hat{J}_b(k)\big), \quad J_F = eQ_f\big(\hat{J}_c(k) - \hat{J}_d(k)\big), \quad \hat{J}_f^\mu(k) \equiv \frac{2p_f^\mu + k^\mu}{k^2 + 2kp_f + i\epsilon}$$

B_4 **is UV finite because of** k^2 **in the denominator. It is also gauge-invariant.**

The above should be understood, "order by order", as follows:

$\mathcal{O}(\alpha^1)$: $\mathcal{M}^{(1)} = 1 + \alpha B_4 + \alpha\Delta^{(1)}$, where $\Delta^{(1)}$ is IR-finite.

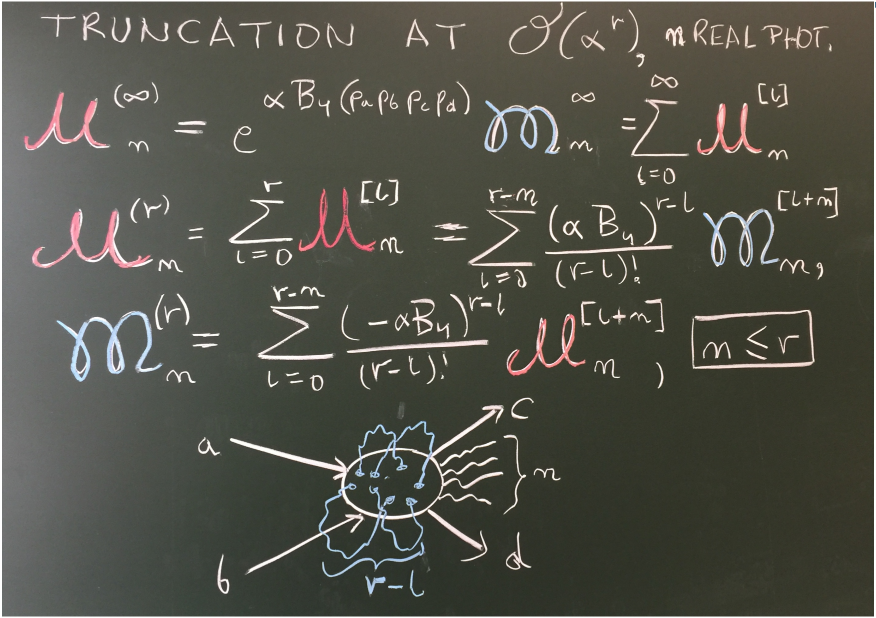
$\mathcal{O}(\alpha^2)$: $\mathcal{M}^{(2)} = 1 + \alpha B_4 + \alpha\Delta^{(1)} + \boxed{\tfrac{1}{2}\alpha^2 B_4^2} + \boxed{\alpha^2 B_4 \Delta^{(1)}} + \alpha^2\Delta^{(2)}$

$$= \{\exp(\alpha B_4)(1 + \alpha\Delta^{(1)} + \alpha^2\Delta^{(2)})\}\big|_{\mathcal{O}(\alpha^2)}$$

UNIVERSALITY: At $\mathcal{O}(\alpha^2)$ terms $\boxed{\alpha^2 \ldots}$ are NOT NEW!

Fully determined by $\mathcal{O}(\alpha^1)$. The same at $\mathcal{O}(\alpha^3)$ etc.

Virtual subtraction



YFS soft photon resummation scheme

YFS Factorisation of real IR singularities proceeds order by order

Consider the case up to $\mathcal{O}(\alpha^2)$ with real photons only:

$\mathcal{O}(\alpha^0)$: $\mathcal{M}^{(0)} = \hat{\beta}_0,$

$\mathcal{O}(\alpha^1)$: $\mathcal{M}^{(1)\mu_1}(k_1) = \hat{\beta}_0 j^{\mu_1}(k_1) + \hat{\beta}_1^{\mu_1}(k_1)$

$\mathcal{O}(\alpha^2)$: $\mathcal{M}^{(2)\mu_1,\mu_2}(k_1,k_2) =$

$$\hat{\beta}_0 j^{\mu_1}(k_1) j^{\mu_2}(k_2) + \hat{\beta}_1^{\mu_1}(k_1) j^{\mu_2}(k_2) + \hat{\beta}_1^{\mu_2}(k_2) j^{\mu_1}(k_1) + \hat{\beta}_2(k_1,k_2)$$

where

$$j_I = eQ_e\big(\hat{j}_a(k) - \hat{j}_b(k)\big),\quad j_F = eQ_f\big(\hat{j}_c(k) - \hat{j}_d(k)\big),\quad \hat{j}_f^{\mu}(k) \equiv \frac{2p_f^{\mu}}{2kp_f}$$

encapsulate all IR (real) divergences, while $\hat{\beta}_i$ are IR-finite.

UNIVERSALITY:

At $\mathcal{O}(\alpha^2)$, $\hat{\beta}_1$ and $\hat{\beta}_0$ and NOT NEW! Determined by $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(\alpha^0)$.

The inductive proof of the above decomposition $\mathcal{O}(\alpha^n) \to \mathcal{O}(\alpha^{n+1})$.

Amplitudes at $\mathcal{O}(\alpha^n)$ for l real and m virtual photons, $n = l + m$, analysed similarly.

Virtual $\exp(\alpha B_4)$ factorises first, decomposition in terms of $j^{\mu}(k)$ done next.

UNIVERSALITY: In $\mathcal{O}(\alpha^{n+1})$ amplitude many components the same as in $\mathcal{O}(\alpha^{n-k})$.

EEX resum. scheme, Monte Carlo implement. of YFS 1961

Classsic EEX/YFS schematically; β's truncated to $\mathcal{O}(\alpha^1)$, example of ISR

$$e^-(p_1,\lambda_1)+e^+(p_2,\lambda_2)\to f(q_1,\lambda_1')+\bar{f}(q_2,\lambda_2')+\gamma(k_1,\sigma_1)+\ldots+\gamma(k_n,\sigma_n)$$

$$\sigma=\sum_{n=0}^{\infty}\int_{m_\gamma} d\Phi_{n+2}\, e^{Y(m_\gamma)} D_n(q_1,q_2,k_1,\ldots,k_n)$$

$$D_0=\bar{\beta}_0$$
$$D_1(k_1)=\bar{\beta}_0\tilde{S}(k_1)+\bar{\beta}_1(k_1)$$
$$D_2(k_1,k_2)=\bar{\beta}_0\tilde{S}(k_1)\tilde{S}(k_2)+\bar{\beta}_1(k_1)\tilde{S}(k_2)+\bar{\beta}_1(k_2)\tilde{S}(k_1)$$
$$D_n(k_1,k_2\ldots k_n)=\bar{\beta}_0\tilde{S}(k_1)\tilde{S}(k_2)\ldots\tilde{S}(k_n)+\bar{\beta}_1(k_1)\tilde{S}(k_2)\tilde{S}(k_3)\ldots\tilde{S}(k_n)$$
$$+\tilde{S}(k_1)\bar{\beta}_1(k_2)\tilde{S}(k_3)\ldots\tilde{S}(k_n)+\ldots+\tilde{S}(k_1)\tilde{S}(k_2)\tilde{S}(k_2)\ldots\bar{\beta}_1(k_n)$$

Real soft factors: $\tilde{S}(k)=\sum_\sigma|\mathfrak{s}_\sigma(k)|^2=|\mathfrak{s}_+|^2(k)+|\mathfrak{s}_-(k)|^2=-\frac{\alpha}{\pi}\left(\frac{q_1}{kq_1}-\frac{q_2}{kq_2}\right)^2$

IR-finite building blocks:

$\bar{\beta}_0=\left(e^{-2\alpha\Re B_4}\sum_\lambda|\mathfrak{M}_\lambda^{\mathrm{Born+Virt.}}|^2\right)\Big|_{\mathcal{O}(\alpha^1)}$, λ = fermion helicities, σ = photon hel.

$$\bar{\beta}_1(k)=\sum_{\lambda\sigma}|\mathfrak{M}_{\lambda\sigma}^{1-\mathrm{PHOT}}|^2-\sum_\sigma|\mathfrak{s}_\sigma(k)|^2\sum_\lambda|\mathfrak{M}_\lambda^{\mathrm{Born}}|^2$$

Everything in terms of $\sum_{spin}|\ldots|^2$! **Distr. <0 possible for hard 2γ.**

CEEX soft photon resum. scheme, generalization of YFS 1961

CEEX schematically, ISR $\mathcal{O}(\alpha^1)$ Example:

$$e^-(p_1,\lambda_1) + e^+(p_2,\lambda_2) \to f(q_1,\lambda'_1) + \bar{f}(q_2,\lambda'_2) + \gamma(k_1,\sigma_1) + \ldots + \gamma(k_n,\sigma_n)$$

$$\sigma = \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2} \sum_{\lambda,\sigma_1,\ldots,\sigma_n} |e^{\alpha B(m_\gamma)} \mathcal{M}^{\lambda}_{n,\sigma_1,\ldots,\sigma_n}(k_1,\ldots,k_n)|^2$$

$\mathcal{M}^{\lambda}_0 = \hat{\beta}^{\lambda}_0$, λ=fermion helicities,

$$\mathcal{M}^{\lambda}_{1,\sigma_1}(k_1) = \hat{\beta}^{\lambda}_0 \mathfrak{s}_{\sigma_1}(k_1) + \hat{\beta}^{\lambda}_{1,\sigma_1}(k_1)$$

$$\mathcal{M}^{\lambda}_{2,\sigma_1,\sigma_2}(k_1,k_2) = \hat{\beta}^{\lambda}_0 \mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2) + \hat{\beta}^{\lambda}_{1,\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2) + \hat{\beta}^{\lambda}_{1,\sigma_2}(k_2)\mathfrak{s}_{\sigma_1}(k_1)$$

$$\mathcal{M}^{\lambda}_{n,\sigma_1,\ldots\sigma_n}(k_1,k_2,\ldots k_n) = \hat{\beta}^{\lambda}_0 \mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n) + \hat{\beta}^{\lambda}_{1,\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n)$$
$$+ \mathfrak{s}_{\sigma_1}(k_1)\hat{\beta}^{\lambda}_{1,\sigma_2}(k_2)\ldots\mathfrak{s}_{\sigma_n}(k_n) + \ldots + \mathfrak{s}_{\sigma_1}(k_1)\mathfrak{s}_{\sigma_2}(k_2)\ldots\hat{\beta}^{\lambda}_{1,\sigma_2}(k_2)$$

IR-finite building blocks:

$$\hat{\beta}^{\lambda}_0 = \left(e^{-\alpha B_4} \mathcal{M}^{\text{Born+Virt.}}_{\lambda}\right)\Big|_{\mathcal{O}(\alpha^1)},$$

$$\hat{\beta}^{\lambda}_{1,\sigma}(k) = \mathcal{M}^{\lambda}_{1,\sigma}(k) - \hat{\beta}^{\lambda}_0 \mathfrak{s}_{\sigma}(k)$$

Everything in terms of $\mathcal{M}$-spin-amplitudes!

Distr. ≥ 0 by construction!

Discussion 1

Flexibility in the choice of the IR regulator in the CEEX/YFS

The IR cancellations do occur independently in two places:

[a] between the exponential formfactor and the real-photon $\int PhaseSpace$

[b] between the various term *inside* the well defined IR-finite β-functions.

Hence, a freedom to choose different IR regulators (a) and (b).

Case (a) of IR cancellations: YFS formfactor vs. real γ integrals

$$\sigma = \sum_{n=0}^{\infty} \int_{m_\gamma} d\Phi_{n+2} \sum_{\lambda,\sigma_1,\dots,\sigma_n} \left| e^{\alpha B(m_\gamma)} \mathcal{M}^{\lambda}_{n,\sigma_1,\dots,\sigma_n}(k_1,\dots,k_n) \right|^2$$

Here, for $\int d^3k$ going into $D \neq 4$ dimensions makes little sense.

One may choose finite photon mass or any other convenient “photon energy cut” method which works at $D = 4$.

Discussion 2

Case (b) of IR cancellations: in construction and evaluation of βs

2 kinds of cancellations: real-real and virtual-virtual (no real-virt.)

- The real-real ones occur for the integrand of the real-emission phase space before $\int d^3k$ integration, even better, they occur for the spin amplitudes, before taking square!
  Example: $\hat{\beta}^{\lambda}_{1,\sigma}(k) = \mathcal{M}^{\lambda}_{1,\sigma}(k) - \hat{\beta}^{\lambda}_{0}\mathfrak{s}_{\sigma}(k)$.
  No need for any IR regulation, it works entirely numerically.
- In virtual components of the β's, the numerical virtual-virtual IR cancellations can be executed before $\int d^Dk$ integration.
  $\hat{\beta}^{\lambda}_{0} = \left(e^{-\alpha B}\mathcal{M}^{\text{Born+Virt.}}_{\lambda}\right)\Big|_{\mathcal{O}(\alpha^1)} = \mathcal{M}^{\text{Born+Virt.}}_{\lambda} - \alpha B\;\mathcal{M}^{\text{Born}}_{\lambda}$.
  The IR-regulator may be even unnecessary!
  For D-regularization the cancellation of term $1/\epsilon$ of IR origin always done after $\int d^Dk$ integration.

Discussion 2b

Traditional real photon regulator.

Optionally, the traditional IR-cut $k^0 > \varepsilon\sqrt{s}/2$ on real γ's can be introduced.

Phase space integral $m_\gamma < k^0 < \varepsilon \frac{\sqrt{s}}{2}$ done analiticaly (rigorously) $\Rightarrow e^{2\alpha \tilde{B}_4}$ factor,

where: $\tilde{B}_4(p_a,...,p_d) = Q_e^2 \tilde{B}_2(p_a,p_b) + Q_f^2 \tilde{B}_2(p_c,p_d)$

$+ Q_e Q_f \tilde{B}_2(p_a,p_c) + Q_e Q_f \tilde{B}_2(p_b,p_d) - Q_e Q_f \tilde{B}_2(p_a,p_d) - Q_e Q_f \tilde{B}_2(p_b,p_c)$,

$$\tilde{B}_2(p,q) \equiv \int_{k^0<\varepsilon\sqrt{s}/2} \frac{d^3k}{k^0} \frac{(-1)}{8\pi^2} \left(\frac{p}{kp} - \frac{q}{kq}\right)^2.$$

New Master Formula:

$$\sigma^{(r)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{k_i^0>\varepsilon\sqrt{s}/2} d\tau_n(p_a+p_b;p_c,p_d,k_1,\dots,k_n)\, e^{2\alpha\Re B_4 + 2\alpha\tilde{B}_4} \sum_{\sigma_i,\lambda,\bar\lambda} \sum_{i,j,l,m=0}^{3}$$
$$\hat{\varepsilon}_a^i \hat{\varepsilon}_b^j \sigma^i_{\lambda_a\bar\lambda_a} \sigma^j_{\lambda_b\bar\lambda_b} \mathfrak{M}_n^{(r)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \left[\mathfrak{M}_n^{(r)}\begin{pmatrix} p & k_1 & k_2 & \dots & k_n \\ \bar\lambda & \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix}\right]^\star \sigma^l_{\bar\lambda_c\lambda_c} \sigma^m_{\bar\lambda_d\lambda_d} \hat{h}_c^l \hat{h}_c^m$$

The resonant part (if present) is multiplied by $e^{2\alpha\delta B_4^\star(p_a,p_b,p_c,p_d;X_\wp)}$

Discussion 2c: YFS formfactor is very simple!

$$Y(\Omega; p_1, ..., p_4) = 2\alpha \tilde{B}_4(p_1, ..., p_4) + 2\alpha \Re B_4(p_1, ..., p_4) \qquad (70)$$

The YFS form-factor B_4 for $e^-(p_a) + e^+(p_b) \to f(p_c) + \bar{f}(p_d) + n\gamma$ reads

$$\alpha B_4(p_a, p_b, p_c, p_d) = \int \frac{d^4k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \left| J_I(k) - J_F(k) \right|^2,$$
$$J_I = eQ_e(\hat{J}_a(k) - \hat{J}_b(k)), \quad J_F = eQ_f(\hat{J}_c(k) - \hat{J}_d(k)), \quad \hat{J}_f^\mu(k) = \frac{2p_f^\mu + k^\mu}{k^2 + 2k \cdot p_f + i\epsilon}. \qquad (50)$$

$$B_4(p_a, p_b, p_c, p_d) = Q_e^2 B_2(p_a, p_b) + Q_f^2 B_2(p_c, p_b)$$
$$+ Q_e Q_f B_2(p_a, p_c) + Q_e Q_f B_2(p_b, p_d) - Q_e Q_f B_2(p_a, p_d) - Q_e Q_f B_2(p_b, p_c), \qquad (51)$$
$$B_2(p_i, p_j) \equiv \int \frac{d^4k}{k^2 - m_\gamma^2 + i\epsilon} \frac{i}{(2\pi)^3} \left(\hat{J}(p_i, k) - \hat{J}(p_j, k) \right)^2. \qquad (52)$$

$$\tilde{B}_4(p_1, ..., p_4) = \int \frac{d^3k_j}{2k_j^0} |\mathfrak{s}(k_j)|^2 \Theta(\Omega, k_j) = Q_e^2 \tilde{B}_2(p_1, p_2) + Q_f^2 \tilde{B}_2(p_3, p_4)$$
$$+ Q_e Q_f \tilde{B}_2(p_1, p_3) + Q_e Q_f \tilde{B}_2(p_2, p_4) - Q_e Q_f \tilde{B}_2(p_1, p_4) - Q_e Q_f \tilde{B}_2(p_2, p_3),$$
$$\tilde{B}_2(p, q) \equiv -\int \frac{d^3k}{2k^0} \Theta(\Omega, k_j) \left(j_p(k) - j_q(k) \right)^2 \equiv \int \frac{d^3k}{2k^0} \Theta(\Omega, k_j) \frac{(-1)}{8\pi^2} \left(\frac{p}{kp} - \frac{q}{kq} \right)^2, \qquad (69)$$

$$Y_e(\Omega_I; p_1, p_2) = \gamma_e \ln \frac{2E_{min}}{\sqrt{2p_1p_2}} + \frac{1}{4}\gamma_e + Q_e^2 \frac{\alpha}{\pi} \left(-\frac{1}{2} + \frac{\pi^2}{3} \right), \quad \gamma = \gamma_e = 2Q_e^2 \frac{\alpha}{\pi} \left(\ln \frac{2p_1p_2}{m_e^2} - 1 \right)$$

Bottom line:

- In the future efforts to calculate multiloop radiative corrections for high precision measurements at FCCe one should plan in advance the proper treatment of the QED part, such that they could be used in the soft photon resummation schemes.
- In particular one should (re)-examine possibility of the cancellation of the IR-divergences before the virtual phase space integration.
- Dont combine real+virtual a la Bloch-Nordsieck!