A mini review of methods of evaluating Feynman integrals

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- Integration by parts

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Evaluating a family of Feynman integrals corresponding to a given graph which are also functions of integer powers of propagators (indices)

$$F_\Gamma(q_1, \ldots, q_n; d; a_1, \ldots, a_L)$$
$$= \int \ldots \int I(q_1, \ldots, q_n; k_1, \ldots, k_h; a_1, \ldots, a_L) \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \mathrm{d}^d k_h$$

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where $I(q_1,\ldots,a_L) = \prod_{i=1}^{L} \frac{1}{(m_i^2-p_i^2)^{a_i}}$ and momenta of the lines p_i are linearly expressed in terms of the loop momenta k_i and external momenta q_j.

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The whole problem of evaluation→

- constructing a reduction procedure
- evaluating master integrals

Public codes to solve IBP relations

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Kira [P. Maierhoefer, J. Usovitsch, P. Uwer]

Methods to evaluate master integrals

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- Feynman (alpha) parameters

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[F. Brown, E. Panzer, O. Schnetz, A. von Manteuffel,
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- Sector decompositions (numerical evaluation)

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[T. Binoth & G. Heinrich; C. Bogner & S. Weinzierl;
A.V. Smirnov & M.N. Tentyukov; J. Carter & G. Heinrich,
S. Borowka, G. Heinrich' et al.'13-17]. Talk by S. Borowka

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Public computer codes:
`SecDec`, `sector_decomposition`, `FIESTA`

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Talks by
E. Dubovyk, J. Usovitsch, M. Prausa, W. Flieger, R. Boels

- Differential equations

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Henn: use canonical bases.
The rhs is proportional to ε and singularities are Fuchsian.

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DE in talks by O. Gituliar, J. Henn, C. Papadopoulos, S. Weinzierl, P. Marquard, VS

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Talk by H. Ita

- FDR: direct calculation of multiloop integrals in $d = 4$.
  Talk by R. Pittau

- Solving differential equations for Feynman integrals by expansions near singular points

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- Three-loop massive form factors: complete light-fermion corrections for the vector current

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- Perspectives

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$$\partial_x \boldsymbol{J} = M\left(x, \epsilon\right) \boldsymbol{J}\,,$$

where $\boldsymbol{J}$ is a column-vector of N primary master integrals, and M is an $N \times N$ matrix with elements which are rational functions of x and $\epsilon = (4 - D)/2$.
Turn to a canonical basis (ε-basis) where DE take the form

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The ε-form is not always possible. The simplest counter example is the two-loop sunset diagram with three equal non-zero masses.

Elliptic generalization of multiple polylogarithms motivated by two-loop examples, where the ε-form is impossible
[L. Adams, C. Bogner, A. Schweitzer & S. Weinzierl'16;
E. Remiddi & L. Tancredi'17; M. Hidding & F. Moriello'17;
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Elliptic functions appear only in two sectors and final results are expressed either in terms of multiple polylogarithms or, for the elliptic sectors, in terms of two and three-fold iterated integrals suitable for numerical evaluation.

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'What is the class of functions which can appear in results for Feynman integrals in situations where ϵ-form is impossible'?

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Some properties of the integrals are more accessible via DE.
Singularities of DE provide a way to examine the branching properties of integrals.
Numerical values of the integrals can be obtained from a numerical solution of DE.

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Using such series is very well known in mathematics.

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[X. Liu, Y.Q. Ma & C.Y. Wang'17]
(solving DE wrt η in propagators $1/(k^2 + i0) \to 1/(k^2 + i\eta)$)

We present (in the case of one variable)

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- An algorithm to solve difference equations for coefficients of the series expansions at a given singular point.
- A matching procedure which enables us to connect series expansions at two neighboring points and thereby obtain a solution of DE at all real values.
- As a proof of concept: a computer code where this algorithm is implemented for a simple example of a family of four-loop Feynman integrals where the ϵ-form is impossible.

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We imply that all the singular points of DE are regular, i.e. we can reduce the DE to a local Fuchsian form at any singular point, i.e. if x_i is a singular point then

$$M(x) = \frac{A_i(x)}{x - x_i}$$

where $A_i(x)$ is regular at $x = x_i$ and $A_i(x_i) \neq 0$.

General solution

$$\boldsymbol{J}(x) = U(x)\,\boldsymbol{C}\,,$$

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where $\boldsymbol{C}$ is a column of constants, and U is an evolution operator

$$U(x) = P\exp\left[\int M(x)\,dx\right].$$

Expanding in a vicinity of each singular point.

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The expansion is

$$U(x) = \sum_{\lambda \in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(n+\lambda, k)\, x^n \ln^k x\,,$$

where S is a finite set of powers of the form $\lambda = r\epsilon$ with integer r, $K_\lambda \geqslant 0$ is an integer number corresponding to the the maximal power of the logarithm.

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where S is a finite set of powers of the form $\lambda = r\epsilon$ with integer r, $K_{\lambda} \geqslant 0$ is an integer number corresponding to the the maximal power of the logarithm.
The goal is to determine S, K_{λ}, and the matrix coefficients $C(n+\lambda, k)$.

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In particular, the 'elliptic' cases, as a rule, can algorithmically be reduced to a global normalized Fuchsian form using, e.g., the algorithm of Lee [R.N. Lee'14].

Multiply both sides by the common denominator $xQ(x)$, where

$$Q(x) = \prod_{k=1}^{s} (x - x_k) = \sum_{m=0}^{s} q_m x^m .$$

with $q_0 \neq 0$.

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with $q_0 \neq 0$.
Define the polynomial matrix $B(x, \alpha)$ and its coefficients $B_m(\alpha)$ by

$$B(x, \alpha) = Q(x)(xM(x) - \alpha) = \sum_{m=0}^{s} B_m(\alpha) x^m \, .$$

with $B_0(\alpha) = q_0(A_0 - \alpha)$.

Then the DE lead to the following recurrence relations

$$-\,\mathrm{BJF}(B_0(\lambda+n),-q_0,K_\lambda)C\,(\lambda+n,0..K_\lambda)$$
$$=\sum_{m=1}^{s}\mathrm{BJF}(B_m\,(\lambda+n-m)\,,-q_m,K_\lambda)C\,(\lambda+n-m,0..K_\lambda)\;.$$

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$$
\begin{aligned}
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built from blocks $C(\alpha, k)$,

$$\mathrm{BJF}(A,B,K) = \underbrace{\begin{bmatrix} A & B & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & B \\ 0 & 0 & 0 & A \end{bmatrix}}_{K+1}$$

The matrix $-\mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda)$ on the lhs of the difference equation is invertible for $\lambda \in S$ and $n > 0$ because

$$\det \mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda) = (\det B_0(\lambda+n))^{K_\lambda+1}$$
$$= q_0^{(K_\lambda+1)n} \left[\det(A_0 - \lambda - n)\right]^{K_\lambda+1}$$

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$$C\left(\lambda+n, 0..K_\lambda\right) = \sum_{m=1}^{s} T(\lambda, n, m) C\left(\lambda+n-m, 0..K_\lambda\right) \,,$$

with

$$\begin{aligned}T(\lambda, n, m) = -&\left[\mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda)\right]^{-1} \\ &\times \mathrm{BJF}(B_m\left(\lambda+n-m\right), -q_m, K_\lambda) \,.\end{aligned}$$

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We determine S, i.e. the set of distinct eigenvalues of A_0, and K_λ, i.e. the highest power of the logarithm, and the leading coefficients $C(\lambda, k)$, representing

$$x^{A_0} = \sum_{\lambda\in S} x^\lambda \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(\lambda, k) \ln^k x\,.$$

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Suppose that the next singular point closest to the origin is $x = 1$.
We can construct the evolution operator also in an expansion near this point. Let it be $\tilde{U}(x)$. Due to the freedom in definition of the evolution operator, we have

$$U(x) = \tilde{U}(x)\, L\,.$$

where L is a constant matrix.

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then for each $0 \leqslant k \leqslant s$ we make the (Moebius) transformation

$$y_k(x) = \frac{ax+b}{cx+d}$$

which maps the points $x_{k-1}, \ x_k, \ x_{k+1}$ to $\mp 1, \ 0, \ \pm 1$, respectively.

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Explicitly,

$$y_k(x) = \pm \frac{(x - x_k)\,(x_{k+1} - x_{k-1})}{(x - x_{k+1})(x_{k-1} - x_k) + (x - x_{k-1})(x_{k+1} - x_k)}$$

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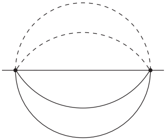
The boundary conditions are included at one of the points, e.g. $x = 0$ and then series expansions at other points can be obtained by matching, step by step, pairs of expansions at neighboring points.

Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines



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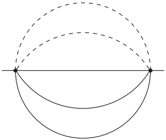

$$F_{a_1,\ldots,a_{14}} =$$

$$\int \ldots \int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4 \; (k_1\cdot p)^{a_6}(k_2\cdot p)^{a_7}(k_3\cdot p)^{a_8}(k_4\cdot p)^{a_9}}{(-k_1^2)^{a_1}(-k_2^2)^{a_2}(m^2-k_3^2)^{a_3}(m^2-k_4^2)^{a_4}(m^2-(\sum k_i+p)^2)^{a_5}} \times (k_1\cdot k_2)^{a_{10}}(k_1\cdot k_3)^{a_{11}}(k_1\cdot k_4)^{a_{12}}(k_2\cdot k_3)^{a_{13}}(k_2\cdot k_4)^{a_{14}} \; ,$$

with $x = p^2/m^2$.

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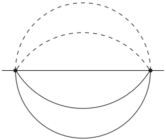

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There are four master integrals in this family.

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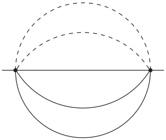

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with $x = p^2/m^2$.
There are four master integrals in this family. We choose

$$\boldsymbol{J_0} = \{F_{1,1,1,1,1,0,\ldots,0},\ F_{1,1,2,1,1,0,\ldots,0},\ F_{1,2,1,1,1,0,\ldots,0},\ F_{1,2,1,1,2,0,\ldots,0}\}\,.$$

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Matching points are $\{-3, 3(3-2\sqrt{2}), 3, 3(3+2\sqrt{2})\}$.

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The code `DESS.m` as well files with input data can be downloaded from
`https://bitbucket.org/feynmanintegrals/dess`.

For example, at $x_0 = 25$, we obtain the following result (shown with a truncation to 10 digits) for the first primary integral:

$$
\begin{aligned}
&-\frac{0.25}{\epsilon^4}+\frac{2.125}{\epsilon^3}-\frac{0.2391337000}{\epsilon^2}-\frac{5.2663306926}{\epsilon} \\
&\quad -185.9464179437+6.5261388472\,\mathrm{i} \\
&\quad -(1825.1476432369-48.9550593728\,\mathrm{i})\epsilon \\
&\quad -(8406.8551978029-176.0638485153\,\mathrm{i})\epsilon^2 \\
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We checked results at sample points (between singular points and matching points) with FIESTA [A.V. Smirnov'16].

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- This code is similar in spirit to the well-known existing codes to evaluate harmonic polylogarithms and multiple polylogarithms, where the problem of evaluation reduces to summing up appropriate series.
- Our public package includes tools for a decomposition of the real axis into domains, a subsequent mapping and an introduction of appropriate new variables.

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complete light-fermion corrections for the vector current
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Planar diagrams. The evaluation of the corresponding planar master integrals
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The quark-photon vertex

$$V^{\mu}(q_1, q_2) = \bar{u}(q_1)\Gamma^{\mu}(q_1, q_2)v(q_2)\,,$$

where the colour indices are suppressed and $\bar{u}(q_1)$ and $v(q_2)$ are the spinors of the quark and anti-quark, respectively. q_1 is incoming and q_2 is outgoing with $q_1^2 = q_2^2 = m^2$.

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Two scalar form factors (electric and magnetic form factors)

$$\Gamma^\mu(q_1, q_2) = Q_q\left[F_1(q^2)\gamma^\mu - \frac{i}{2m}F_2(q^2)\sigma^{\mu\nu}q_\nu\right]\,,$$

where $q = q_1 - q_2$ is the outgoing momentum of the photon and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. Q_q is the charge of the considered quark.

Results in terms of Goncharov polylogarithms of the variable x given by

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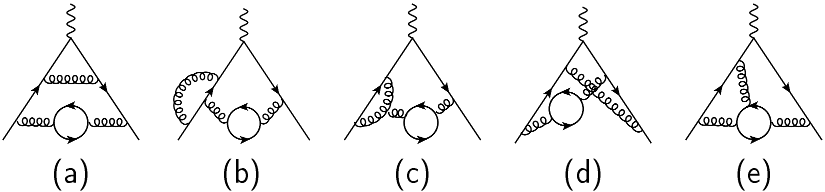
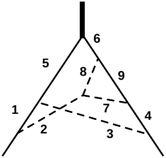
The three-loop QCD corrections to the massive quark-anti-quark-photon form factors F_1 and F_2 involving a closed loop of massless fermions, i.e. proportional to n_l.

Sample three-loop diagrams contributing to F_1 and F_2



Four families of integrals corresponding to these graphs are new, i.e. they were not involved in the large-N_c calculation



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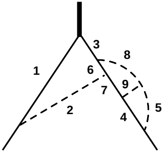

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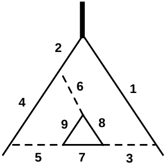

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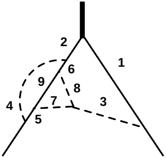

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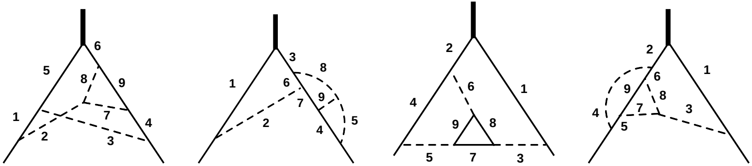

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Agreement of our results with known results in various limits.

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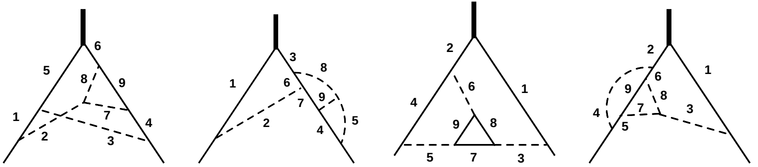

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Agreement of our results with known results in various limits. We have also reproduced the two-loop results for the form factors obtained quite recently
[J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana & C. Schneider]

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to be continued