MB-suite 1: AMBRE news: non-planar 3-loop vertices

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Based on collaboration with:

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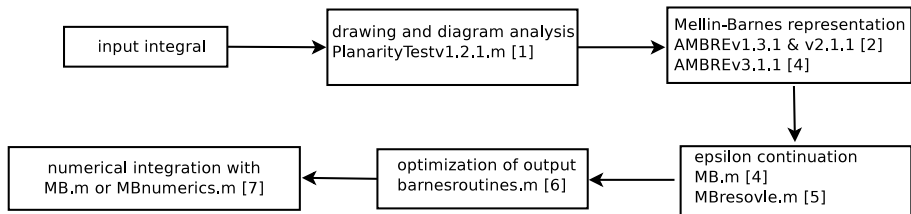
Mini-workshop on multiloop/multiscale methods and techniques
in the context of precise Z-boson studies

CERN, 12 January 2018

Outline

Calculation of Feynman integrals via MB method

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┌─────────────────┐      ┌──────────────────────────┐      ┌──────────────────────────────┐
│  input integral │ ───▶ │ drawing and diagram analysis│ ──▶ │ Mellin-Barnes representation   │
│                 │      │ PlanarityTestv1.2.1.m [1]  │      │ AMBREv1.3.1 & v2.1.1 [2]       │
└─────────────────┘      └──────────────────────────┘      │ AMBREv3.1.1 [4]                │
                                                            └──────────────────────────────┘
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┌────────────────────────┐      ┌──────────────────────────┐      ┌──────────────────────────┐
│ numerical integration with│ ◀─ │ optimization of output    │ ◀─ │ epsilon continuation      │
│ MB.m or MBnumerics.m [7]  │    │ barnesroutines.m [6]      │    │ MB.m [4]                  │
└────────────────────────┘      └──────────────────────────┘    │ MBresovle.m [5]           │
                                                                 └──────────────────────────┘
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[1] I. Dubovyk, K. Bielas: '13
http://us.edu.pl/~gluza/ambre/planarity/

[2] J Gluza, K. Kajda, I. Dubovyk: '11
http://prac.us.edu.pl/~gluza/ambre/

[3] I. Dubovyk: '15

[4] M. Czakon: '06
https://www.hepforge.org/downloads/mbtools

[5] A. Smirnov: '09

[6] D. Kosower: '09

[7] J. Usovitsch: '15 see next talk

Feynman parameters representation

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod\limits_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1}\delta(1 - \sum_{i=1}^{N} x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

<div align="center" style="color:blue">Some remarks</div>

Change of variables in Symanzik polynomials U and F is effective as:

- They are homogeneous in the Feynman parameters, U is of degree L, F is of degree $L+1$
- U is linear in each Feynman parameter. If all internal masses are zero, then also F is linear in each Feynman parameter
- In expanded form each monomial of U has coefficient $+1$

Construction of graph polynomials

- algebraic

$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2, \quad U = det[M], \quad F_0 = QM^TQ$$

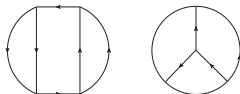with $M(x_i) - (L \times L)$ matrix and $Q(x_i, p_k) - L$-vector, L – number of loops.

- graphical:
  spanning trees and the spanning 2-forests, C.Bogner, S.Weinzierl: '10

- graphical:
  chain diagrams, Kinoshita: '74

- ....

Construction of Mellin-Barnes representation

"Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895),
"The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

General Mellin-Barnes relation:

$$\frac{1}{(A_1 + \ldots + A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i} \, A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \, \Gamma(\lambda + z_1 + \ldots + z_{n-1})$$

Integration over Feynman parameters:

$$\int_0^1 \prod_{i=1}^N dx_i \, x_i^{n_i-1} \, \delta(1 - x_1 - \ldots - x_N) = \frac{\Gamma(n_1) \ldots \Gamma(n_N)}{\Gamma(n_1 + \ldots + n_N)}$$

Optimal MB representations

- Feynman parameters + MB
  - iteratively to each subloop – loop-by-loop (LA) approach
    (`AMBREv1.3.1` & `AMBREv2.1.1`)
  - in one step to the complete U and F polynomials – global (GA) approach
    (`AMBREv3.1.1`)
  - combination of the above methods – Hybrid approach (under development -
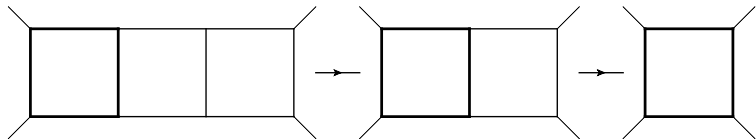   AMBREv4)
    (see also R. Boels, arXiv:1711.08449)

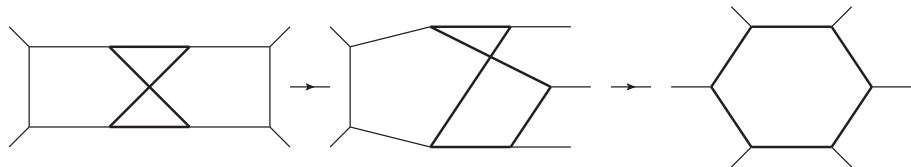 Examples, description, links to basic tools and literature:
  http://prac.us.edu.pl/~gluza/ambre/

- Method of Brackets (see talk by M. Prausa)
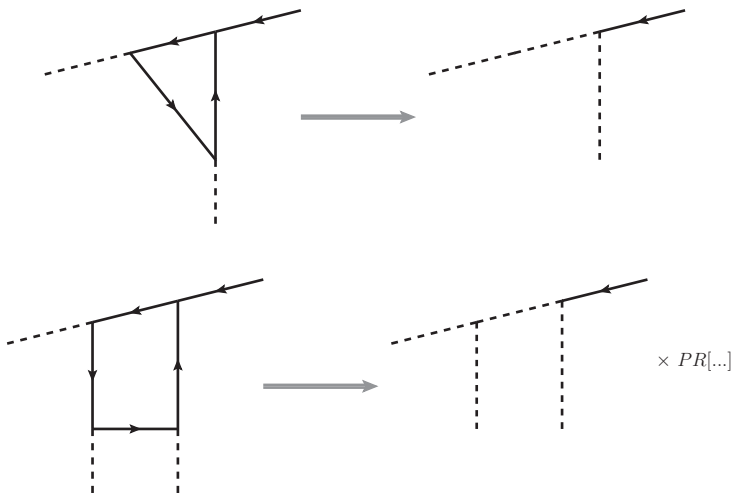
Limitations of LA approach I

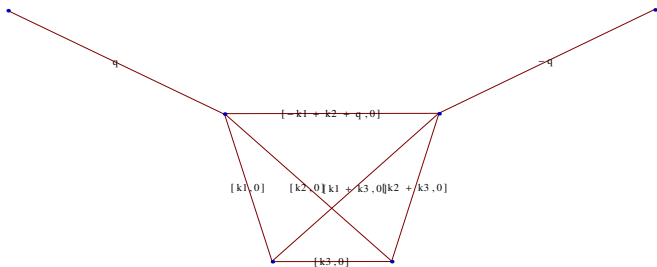Planar case:

Non-planar case:

Limitations of LA approach II

$\times PR[...]$

Limitations of GA approach

U polynomial for non-planar 3-loop box (64 terms)

```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
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x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
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x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
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AMBRE HOW TO I

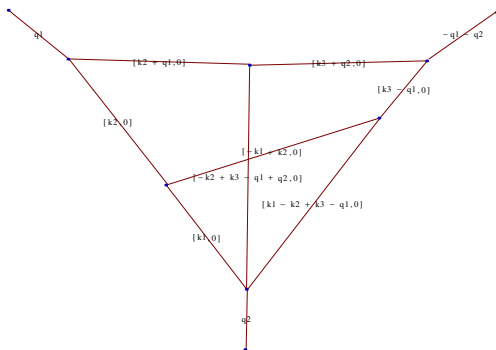- LA is basicaly for planar diagrams and GA – for non-planar.

- For LA momentum flow really matters.
- 1st and 2nd Barnes lemmas are key ingredients.

$$-sx_1x_2 - sx_1x_4 + \ldots = -sx_1(x_2 + x_4) + \ldots \leftrightarrow \text{1st Barnes lemma}$$
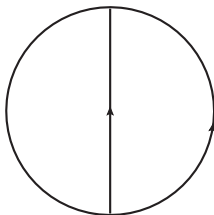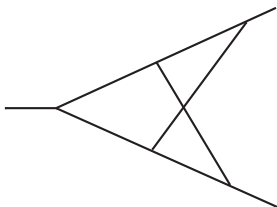
AMBRE HOW TO II

- LA works also for non-planar diagrams:

  - dimensionality must be checked after expansion in ϵ
  - pros: AMBREv2.1.1 - 6-dim vs AMBREv3.1.1 - 13-dim
  - cons: always "minkowskian" - representation contains $(-s)^z s^z$
- GA works for both planar and non-planar diagrams
- please contact us in case of problems

GA 2-loop example (non-planar vertex)

$$\iint d^d k_1 d^d k_2 \frac{1}{[k_1^2]^{n_1} [(p_1 - k_1)^2]^{n_2} [(p_1 - k_1 - k_2)^2]^{n_3}}$$

$$\frac{1}{[(p_2 + k_1 + k_2)^2]^{n_4} [(p_2 + k_2)^2]^{n_5} [k_2^2]^{n_6}}$$

$$U = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_3 x_4 + x_1 x_5 + x_2 x_5 + x_4 x_5 + x_2 x_6 + x_3 x_6 + x_4 x_6 + x_5 x_6$$

$$F = U \sum_{i=1}^{6} m_i^2 x_i - s x_1 x_4 x_5 - s x_1 x_2 x_6 - s x_1 x_3 x_6 - s x_2 x_3 x_6 - s x_1 x_4 x_6 - s x_1 x_5 x_6$$

Variables transformation

$$\{\vec{x}\}_i : x_k \rightarrow v_i \xi_{ik}$$

i denotes a subset of feynman parameters associated to propagators with different combinations of loop momenta

$$
\begin{array}{rcll}
m^2 = \sum x_i D_i & = & x_1 (p_1 - k_1 - k_2)^2 & x_1 \rightarrow v_1 \xi_{11} \\
& + & x_2 (p_2 + k_1 + k_2)^2 & x_2 \rightarrow v_1 \xi_{12} \\
& + & x_3 (k_1)^2 & x_3 \rightarrow v_2 \xi_{21} \\
& + & x_4 (p_1 - k_1)^2 & x_4 \rightarrow v_2 \xi_{22} \\
& + & x_5 (p_2 + k_2)^2 & x_5 \rightarrow v_3 \xi_{31} \\
& + & x_6 (k_2)^2 & x_6 \rightarrow v_3 \xi_{32}
\end{array}
$$

$$\delta \left(1 - \sum_{i=1}^{6} x_i \right) \Rightarrow \delta(1 - v_1 - v_2 - v_3)\delta(1 - \xi_{11} - \xi_{12})\delta(1 - \xi_{21} - \xi_{22})\delta(1 - \xi_{31} - \xi_{32})$$

Jacobian:

$$J = v_1^{N_{\xi_1} - 1} v_2^{N_{\xi_2} - 1} v_3^{N_{\xi_3} - 1} = v_1 v_2 v_3$$

- Using $\prod_i \delta \left(1 - \sum_k \xi_{ik}\right)$ we can simplify U and F

$$U = v_1 v_2 + v_1 v_3 + v_2 v_3 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1 v_2 v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1 v_2 v_3$$
$$-s\xi_{31}\xi_{32}v_1 v_3^2 - s\xi_{31}\xi_{32}v_2 v_3^2$$

Chang–Wu theorem:
delta function in the feyman parameters representation can be replaced by

$$\delta \left(\sum_{i \in \Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1, \ldots, L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.

- Choose now v_3 as Chang-Wu variable $\int_0^\infty dv_3 \int_0^1 dv_1 dv_2 \delta(1 - v_1 - v_2)$

$$U = v_3 + v_1 v_2 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1 v_2 v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1 v_2 v_3$$
$$-s\xi_{31}\xi_{32}v_1 v_3^2$$

- Apply MB relation for F

$$\frac{1}{(A_1 + \ldots + A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1}$$

$$\times \prod_{i=1}^{n-1} A_i^{z_i} \; A_n^{-\lambda - z_1 - \ldots - z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \; \Gamma(\lambda + z_1 + \ldots + z_{n-1})$$
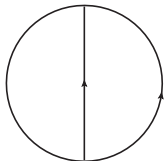
- lintegrate over $v3$ using

$$\int\limits_0^\infty dx \; x^{z_1} (x+y)^{z_2} = \frac{y^{1+z_1+z_2} \Gamma(1+z_1) \Gamma(-1-z_1-z_2)}{\Gamma(-z_2)}$$

- lintegrate over each subset of variables $\{v, \xi_i\}$ separately using

$$\int_0^1 \prod_{i=1}^N dx_i \; x_i^{n_i-1} \; \delta(1 - x_1 - \ldots - x_N) = \frac{\Gamma(n_1) \ldots \Gamma(n_N)}{\Gamma(n_1 + \ldots + n_N)}$$

U polynomial gives no additional MB integration and final dimentionality depends only on length of F $\longrightarrow$ similar to one loop integrals and/or LA approach

$$U = v_1 v_2 + v_1 v_3 + v_2 v_3$$
$$F = -p^2 v_1 v_2 v_3 + U \sum_i v_i m_i^2$$

$$G(X) \sim \int \prod d\xi_{ik} \delta \left(1 - \sum_k \xi_{ik} \right)$$

$$\int d^d k_1 d^d k_2 \frac{1}{[k_1^2 - m_1^2(S, \xi_{ik})]^{n_1} [k_2^2 - m_2^2(S, \xi_{ik})]^{n_2} [(p + k_1 + k_2)^2 - m_3^2(S, \xi_{ik})]^{n_3}}$$
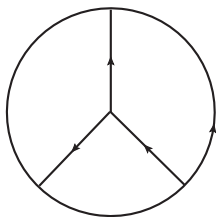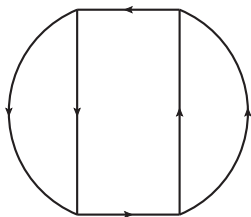
$$p^2 = Q(S, \xi_{ik})$$

In case of massless non-planar vertex from above

$$p^2 = -s(\xi_{12}\xi_{22}\xi_{31} + \xi_{12}\xi_{21}\xi_{32} - \xi_{31}\xi_{32})$$

$$m_1^2 = -s\xi_{31}\xi_{32}, \quad m_2^2 = m_3^2 = 0.$$

3-loop GA

Case I:

$$U = v_1 v_2 v_3 + v_1 v_2 v_4 + v_2 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_1 v_4 v_5 + v_3 v_4 v_5$$

Case II:

$$U = v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_2 v_4 v_5 + v_3 v_4 v_5$$
$$+ v_1 v_2 v_6 + v_2 v_3 v_6 + v_1 v_4 v_6 + v_2 v_4 v_6 + v_3 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6$$

Now in the Chang-Wu theorem we choose 3 variables

$$\int_0^\infty dv_2 dv_3 dv_4 \int_0^1 dv_1 dv_5 dv_6 \delta(1 - v_1 - v_5 - v_6)$$

$U_{CW} = v_2 v_3 + v_2 v_4 + v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_1 v_2 v_6 + v_1 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6$

Factorization trick:

$$U_{CW} = v_2(v_3 + v_4 + v_1 v_5) + v_3(v_4 + v_1 v_5) + v_1 v_6(v_2 + v_5) + v_4 v_6(v_1 + v_5) + v_3 v_5 v_6$$
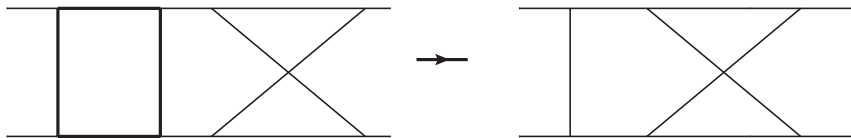
24 possibilities to choose of CW variables and factorize U
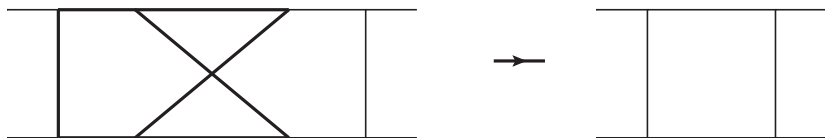
U polynomial gives 4 additional MB integration!

GA usually gives optimal representation if from the beginning Length(U) $\gtrsim$ Length(F)

3-loop mixed approach

Mixed approach starting with planar subloop:

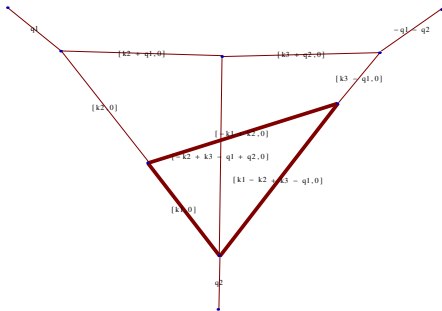Mixed approach starting with non-planar subloop:

3-loop mixed approach: example

```
PR[k1, 0, n1] PR[-k1 + k2, 0, n2] PR[k2, 0, n3] PR[k2 + q1, 0, n4] PR[k3 + q2, 0, n5]
PR[k3 - q1, 0, n6] PR[-k2 + k3 - q1 + q2, 0, n7] PR[k1 - k2 + k3 - q1, 0, n8]

--iteration nr: 1 with momentum: k1

   F polynomial during this iteration
-PR[k2,0] X[1] X[2]-PR[k2-k3+q1,0] X[1] X[3]-PR[k3-q1,0] X[2] X[3]
```



```
PR[k2, 0, nz3] PR[k3 - q1, 0, nz6] PR[k2 + q1, 0, n4] PR[k3 + q2, 0,n5] PR[-k2 + k3 - q1 + q2, 0, n7] PR[k2 - k3 + q1, 0, z]
```
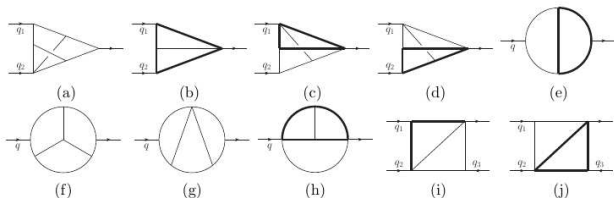
Comparison with method of brackets

Figure 5.1: Example two- and three-loop diagrams. Bold (thin) lines represent massive (massless) propagators.

diagram	Method of Brackets	AMBRE	planarity	AMBREv4*
fig.5.1(a)	7	13	NP	4
fig.5.1(b)	1	2	P	1
fig.5.1(c)	7	9	NP	5
fig.5.1(d)	7	8	NP	8
fig.5.1(e)	5	3	P	3
fig.5.1(f)	9	4	P	4
fig.5.1(g)	7	4	P	4
fig.5.1(h)	5	4	P	4
fig.5.1(i)	2	2	P	2
fig.5.1(j)	2	2	P	2

materials are take from arXiv:1706.09852: M. Prausa, *"Mellin-Barnes meets Method of Brackets"*.

Conclusions and Outlook

- Dimensionality of MB representations strongly depends on topology, number of legs and loops, internal and external masses.
- `AMBRE` software is based on two different approaches:
  LA – general planar and some non-planar diagrams
  GA – 2-loop planar and non-planar, 3-loop non-planar diagrams with massless external legs
- new `AMBREv4` which combines all advantages of methods above is underway

  ```
  MBrepr[{numerator}, {propagators}, {{k1, k3, k2}}];
                                  {k1, {k2, k3}}
                          {{k1, k2}, {k3, k4}}
  ```

- MB-suite is ready to 3-loop challenge
- next step – 4-loop
- MB approach to Feynman integrals reached Minkowskian region next talk
- and still much must be done on tools and methods to get it beyond the present status