Numerical evaluation of Mellin-Barnes integrals in Minkowskian regions
Workshop at Cern

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Outline

Motivation

- In the Z-boson resonance physics the Feynman integrals contain up to four dimensionless parameters:

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(M_Z^2 + i\varepsilon)}{M_Z^2} \right\} \tag{1}$$

- Many of them contain ultraviolet and infrared singularities, even though the divergences cancel in the final result
- In general, it is not possible to compute all integrals analytically with available methods and tools, but instead one has to resort to numerical integration strategies

Some numerical Methods

Sector decomposition

FIESTA 3 [A.V.Smirnov, 2014], SecDec 3 [Borowka, et. al., 2015] and pySecDec [Borowka, et. al., 2017]

Mellin-Barnes integral approach

- With AMBRE 2 [Gluza, et. al., 2011] (AMBRE 3 [Dubovyk, et. al., 2015]) we derive Mellin-Barnes representations for planar (non-planar) topologies. One may use PlanarityTest [Bielas, et. al, 2013] for automatic identification.
- Expansion in terms of $\epsilon = (4 - D)/2$ is done with MB [Czakon, 2006], MBresolve [A. Smirnov, V. Smirnov, 2009], barnesroutines (D. Kosower).
- For the numerical treatment of massive Mellin-Barnes integrals with Minkowskian regions, the package MBnumerics is being developed since 2015.

Construction of Mellin-Barnes integrals with AMBRE

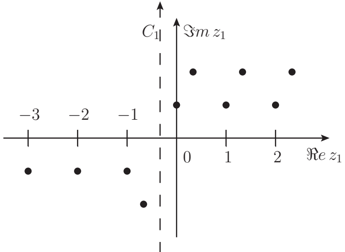
Mellin-Barnes integral master formula

$$\frac{1}{(A+B)^{\nu}} = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{A^z B^{-z-\nu}\Gamma(-z)\Gamma(\nu+z)}{\Gamma(\nu)}, \quad |\arg A - \arg B| < \pi \qquad (2)$$

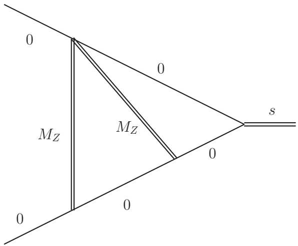
Integral representation of the Euler's Beta-function

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re e\, x > 0,\ \Re e\, y > 0 \qquad (3)$$

Satisfy all relations with a straight contour in the complex plane $z_i = x_i + it_i$, where x_i is a constant generated with MB.m or MBresolve.m $t_i \in [-\infty, +\infty]$



Example



$$I_{\mathsf{0h0w14r}} = \int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \quad \frac{k_1 p_1}{k_1^2((k_1-k_2)^2-M_Z^2)k_2^2((k_2+p_1)^2-M_Z^2)(k_1+p_1+p_2)^2}$$

$$= \int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} \frac{\mathrm{d}z_2}{2\pi i} \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-\epsilon-z_1)\Gamma(2\epsilon+z_1+1)\left(-\frac{M_Z^2}{s}\right)^{z_1}}{2\Gamma(1-z_2)\Gamma(-3\epsilon-z_1+2)\Gamma(-2\epsilon-2z_1-z_2)}$$

$$\times (-s)^{-2\epsilon}\,\Gamma(-2\epsilon-z_1-z_2)^2\Gamma(-\epsilon-z_1-z_2)\Gamma(\epsilon+z_1+z_2+1)$$

Expansion in ϵ

$$-\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\frac{\mathrm{d}z_1}{2\pi i}\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\frac{\mathrm{d}z_2}{2\pi i}\,\frac{\Gamma(-z_1)^2\Gamma(z_1+1)\Gamma(-z_2)\Gamma(z_2+1)\left(-\frac{M_Z^2}{s}\right)^{z_1}\Gamma(-z_1-z_2)^3}{2\Gamma(2-z_1)\Gamma(1-z_2)\Gamma(-2z_1-z_2)}$$
$$\times\Gamma(z_1+z_2+1)+\mathcal{O}(\epsilon)$$

Minkowskian regions: $s > 0$
Euclidean regions: $s < 0$
With MBnumerics we integrate numerically the coefficients in the ϵ expansion

Asymptotic behavior in Minkowskian regions

$$\lim_{|z|\to\infty} \frac{\Gamma(z)}{z^{z-1/2}e^{-z}} = \sqrt{2\pi}, \quad |\arg z| < \pi \tag{4}$$

Thus the following approximation is valid:

$$\Gamma(z) \underset{|z|\to\infty}{\approx} z^{z-1/2}e^{-z}\sqrt{2\pi}, \quad |\arg z| < \pi \tag{5}$$

For our example: $z_1 = -\frac{1}{3} + it_1$ and $z_2 = -\frac{1}{3} + it_2$
$t_1 \to -t$ and $t_2 \to t$

$$\mathcal{I}_{\mathsf{0h0w14r}} \underset{t\to\infty}{\approx} t^{-2+2x_1+2x_2} \tag{6}$$

everywhere else exponential damping factor.

Linear transformation

$$z_2 \to z_2 - z_1 \tag{7}$$

$$I_{\text{0h0w14}} = -\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \int\limits_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} \frac{\mathrm{d}z_2}{2\pi i} \frac{(-\frac{M_Z^2}{s})^{z_1}\Gamma(-z_1)^2\Gamma(1+z_1)\Gamma(z_1-z_2)\Gamma(-z_2)^3}{2\Gamma(2-z_1)\Gamma(-z_1-z_2)\Gamma(1+z_1-z_2)} \times\Gamma(1+z_2)\Gamma(1-z_1+z_2) \tag{8}$$

$z_1 = -\frac{1}{3} + it_1$ and $z_2 = -\frac{2}{3} + it_2$
$t_1 \to -t$ and $t_2 \to 0$

$$\mathcal{I}_{\text{0h0w14}} \underset{t\to\infty}{\approx} t^{-2+2x_2} \tag{9}$$

Numerical integration is more stable

Mappings of integration regions to finite intervals

- Logarithmic mapping $t = \ln[\frac{d}{1-d}]$ applied to a polynomial function:

$$\frac{1}{t^a} = \frac{1}{\ln[\frac{d}{1-d}]^a}, \text{ Jacobian: } \frac{1}{d(1-d)} \tag{10}$$

$$\lim_{d\to 0, d\to 1} \frac{1}{d(1-d)} \frac{1}{\ln(\frac{d}{1-d})^a} = \frac{1}{0}, \ \forall\, a. \tag{11}$$

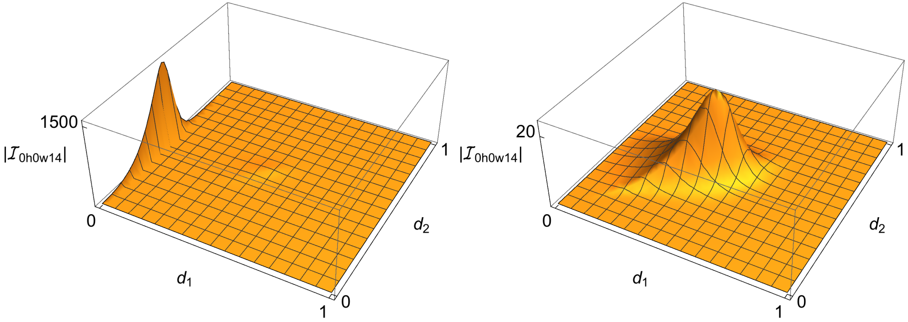
- Cotangent mapping $t = \frac{1}{\tan(-\pi d)}$ applied to the polynomial function:

$$\frac{1}{t^a} = \tan(-\pi d)^a, \text{ Jacobian: } \frac{\pi}{\sin(\pi d)^2} \tag{12}$$

$$\lim_{d\to 0, d\to 1} \frac{\pi \tan(-\pi d)^a}{\sin(\pi d)^2} = \begin{cases} \frac{1}{0}, & a < 2, \\ \pi, & a = 2, \\ 0, & a > 2, \end{cases} \tag{13}$$

$$\prod_i \Gamma_i \to \exp\left(\sum_i \log \Gamma_i\right) \tag{14}$$

Example: Mappings of integration intervals to finite intervals



$$\mathcal{I}_{0h0w14} = -\frac{(-\frac{M_Z^2}{s})^{z_1}\Gamma(-z_1)^2\Gamma(1+z_1)\Gamma(z_1-z_2)\Gamma(-z_2)^3}{2\Gamma(2-z_1)\Gamma(-z_1-z_2)\Gamma(1+z_1-z_2)} \times\Gamma(1+z_2)\Gamma(1-z_1+z_2) \quad (15)$$

Shifts may improve asymptotic behavior

$$z_i = x_i + it_i + n_i \tag{16}$$

x_i is a constant generated with MB.m or MBresolve.m
$t_i \in [-\infty, \infty]$

$$n_i \in \mathbb{R} \text{ is a shift}$$

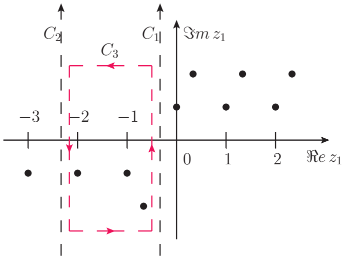
First observation: Shifts may change the asymptotic behavior:

$$\mathcal{I}_{\text{0h0w14r}} \underset{t\to\infty}{\approx} t^{-2+2x_1+2x_2+2n_1+2n_2} \tag{17}$$

Shifts may change order of magnitude

$$z_i = x_i + it_i + n_i \tag{18}$$

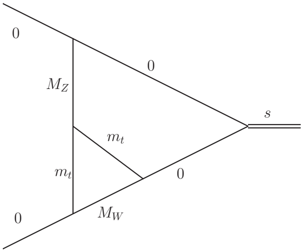
Second observation: Integrals with the shifted contour may have a value of lower order of magnitude than the original integral



Original integral at C_1: $0.392382858885\boldsymbol{7} + 0.745638853661\boldsymbol{3}i$
Shifted integral at C_2 with $n_1 = -2$: $-0.0097496582320\boldsymbol{2}$
3 residues due to C_3: $0.40213251711780\boldsymbol{7} + 0.74563885366131\boldsymbol{8}i$

$$\int_{C_1} \mathcal{I}_{\mathsf{0h0w14}} \mathrm{d}z_1 \mathrm{d}z_2 = \int_{C_2} \mathcal{I}_{\mathsf{0h0w14}} \mathrm{d}z_1 \mathrm{d}z_2 + \int_{C_3} \mathcal{I}_{\mathsf{0h0w14}} \mathrm{d}z_1 \mathrm{d}z_2 \tag{19}$$

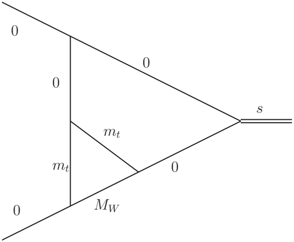
0hxwxtxz96 [MB - 4 dim] [SD - 5 dim]



$$\int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \qquad \frac{\exp(2\epsilon\gamma_E)(k_2 p_2)}{(k_1)^2((k_1-k_2)^2-m_t^2)((k_2)^2-M_W^2)((k_1+p_1)^2-M_Z^2)} \times \frac{1}{((k_2+p_1)^2-m_t^2)(k_1+p_1+p_2)^2}$$

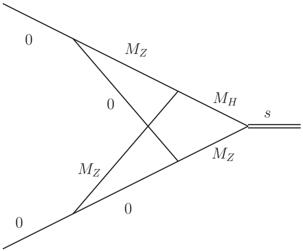
MB 1	(-0.03489556***6***6	-0.10101869***9***8 i) + $\mathcal{O}(\epsilon)$
MB 2	(-0.03489553***3***4	-0.10101866***6***5 i) + $\mathcal{O}(\epsilon)$
SD - 540 Mio	(-0.034895529***3***	-0.101018681***3*** i) + $\mathcal{O}(\epsilon)$
SD - 90 Mio	(-0.03489542***2***9	-0.10101882***2***6 i) + $\mathcal{O}(\epsilon)$
SD - 15 Mio	(-0.0348905***5***7	-0.1010253***3***0 i) + $\mathcal{O}(\epsilon)$

soft7 ϵ^0:[MB - 3 dim] [SD - 5 dim], ϵ^{-1}:[MB - 2 dim] [SD - 4 dim], ϵ^{-2}:[MB - 1 dim] [SD - 3 dim]



MB	$0.06026648655769\boldsymbol{9}\,\epsilon^{-2}$	
SD - 90 Mio	$0.0602664865\boldsymbol{5}\,\epsilon^{-2}$	
MB	$(-0.031512489\boldsymbol{0}3$	$+0.189332751\boldsymbol{4}2i)\,\epsilon^{-1}$
SD - 90 Mio	$(-0.03151248\boldsymbol{1}6$	$+0.18933271\boldsymbol{6}96i)\,\epsilon^{-1}$
MB 1	$(-0.2282318675\boldsymbol{1}1$	$-0.0882479456\boldsymbol{9}1i) + \mathcal{O}(\epsilon)$
MB 2	$(-0.2282318675\boldsymbol{5}1$	$-0.0882479457\boldsymbol{3}9i) + \mathcal{O}(\epsilon)$
SD - 90 Mio	$(-0.228226\boldsymbol{5}3$	$-0.088245\boldsymbol{9}6i) + \mathcal{O}(\epsilon)$
SD - 15 Mio	$(-0.2281\boldsymbol{6}2$	$-0.0882\boldsymbol{0}9i) + \mathcal{O}(\epsilon)$

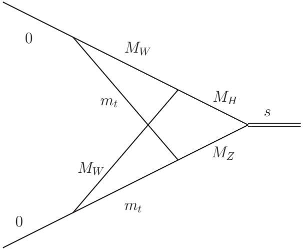
xh0w16 [MB - 6 dim] [SD - 5 dim]



$$\int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \quad \frac{\exp(2\epsilon\gamma_E)(k_1 p_1)^2}{(k_1^2-M_H^2)((k_1-k_2)^2-M_Z^2)(k_2^2-M_Z^2)(k_1-k_2+p_1)^2} \times \frac{1}{(k_2+p_2)^2((k_1+p_1+p_2)^2-M_Z^2)}$$

MB	$(0.04361 \pm 0.00003) + \mathcal{O}(\epsilon)$
SD - 15 Mio	$(0.04361665537\boldsymbol{5}) + \mathcal{O}(\epsilon)$

xhxwxz63 [MB - 8 dim] [MB - 5 dim]



$$\int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \quad \frac{\exp(2\epsilon\gamma_E)(k_1 p_1)^2}{(k_1^2 - M_Z^2)((k_1-k_2)^2 - m_t^2)(k_2^2 - m_t^2)((k_1-k_2+p_1)^2 - M_W^2)}$$
$$\times \frac{1}{((k_2+p_2)^2 - M_W^2)((k_1+p_1+p_2)^2 - M_H^2)}$$

MB	(0.0029 ± 0.0008)
SD - 15 Mio	$(0.003133833\mathbf{6}\boldsymbol{3}) + \mathcal{O}(\epsilon)$

Conclusions and Outlook

- The main challenge so far in the electroweak Z-boson resonance physics was the calculation of massive two-loop vertex Feynamn diagrams
- No reduction of integrals to masters
- New automatized tools AMBRE 3 and MBnumerics for the evaluation of the Mellin-Barnes integrals in Minkowskian kinematics were developed and used together with sector decomposition programs SecDec 3 and Fiesta 3