Direct calculation of multiloop integrals in d$= 4$

Roberto Pittau
University of Granada

CERN mini-workshop

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Outline

1. Intro on
   Four Dimensional Regularization / Renormalization
2. Results
3. Conclusions

The main aim

- Embedding the UV subtraction in the **definition** of the loop integration

$\Longrightarrow$

- Renormalized Green's functions are **directly** computed (no CTs in $\mathcal{L}$) in four dimensions

Properties to be kept by loop integrals

- **SHIFT INVARIANCE**, needed for **Routing Invariance**

$$\int_R d^4q_1 \cdots d^4q_\ell \, J(q_1, \cdots, q_\ell) \stackrel{?}{=} \int_R d^4q_1 \cdots d^4q_\ell \, J(q_1+p_1, \cdots, q_\ell+p_\ell)$$

$\Uparrow$

Some UV regulator

Properties to be kept by loop integrals

$$\int_R d^4q_1 \cdots d^4q_\ell \frac{\not{D}_i}{D_0 \cdots \not{D}_i \cdots D_k} \stackrel{?}{=} \int_R d^4q_1 \cdots d^4q_\ell \frac{1}{D_0 \cdots D_k}$$

$\Uparrow$

Some UV regulator

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$$\int_R d^4q_1 \cdots d^4q_\ell \frac{\not{D}_i}{D_0 \cdots \not{D}_i \cdots D_k} \stackrel{?}{=} \int_R d^4q_1 \cdots d^4q_\ell \frac{1}{D_0 \cdots D_k}$$

⇑

Some UV regulator

- I dub this **NUMDEN** cancellation, which is essential to ensure gauge cancellations ⇒ **Gauge Invariance**

Properties to be kept by loop integrals

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$\Uparrow$

Some UV regulator

- I dub this **NUMDEN** cancellation, which is essential to ensure gauge cancellations $\Rightarrow$ **Gauge Invariance**
- Is this enough?

Properties to be kept by loop integrals

$$\int_R d^4q_1 \cdots d^4q_\ell \frac{\not{D}_i}{D_0 \cdots \not{D}_i \cdots D_k} \stackrel{?}{=} \int_R d^4q_1 \cdots d^4q_\ell \frac{1}{D_0 \cdots D_k}$$

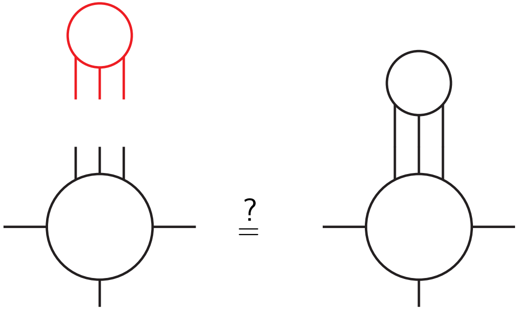
$\Uparrow$

Some UV regulator

- I dub this **NUMDEN** cancellation, which is essential to ensure gauge cancellations $\Rightarrow$ **Gauge Invariance**
- Is this enough?

NO!

Properties to be kept by loop integrals



- I dub this **SUBINTEGRATION** consistency, which is essential to ensure **Unitarity:**

$$T - T^{\dagger} = i\, T^{\dagger} T$$

Dimensional regularization (DReg)

- DReg keeps **SHIFT INVARIANCE** and **NUMDEN** cancellations, but introduces order-by-order CTs in $\mathcal{L}$ to preserve **SUBINTEGRATION** consistency

- In DReg **it is not enough** to drop $\frac{1}{\epsilon^\ell}$ poles in the loop integrals to define a decent renormalization scheme beyond 1-loop!

$$\int d^n q_1 d^n q_2 \frac{1}{(q_1^2 - M^2)^2} \frac{1}{(q_2^2 - M^2)^2}\Bigg|_{\frac{1}{\epsilon}=0} \neq \left(\int d^n q \frac{1}{(q^2 - M^2)^2}\Bigg|_{\frac{1}{\epsilon}=0} \right)^2$$

FDR

- Let a UV divergent integrand be $J(q) = \dfrac{1}{q^2(q+p)^2}$

$$\int [d^4q]\, J(q) \;\equiv\; \lim_{\mu\to 0}\int_R \left(J(q) - \frac{1}{\bar{q}^4} \right) \;\equiv\; \int [d^4q]\, \frac{1}{\bar{q}^2 \bar{D}_p}$$

$\Uparrow$ Subtraction term **encoded** in definition

$\Uparrow$ FDR integral **finite** in 4 dim

$$\bar{q}^2 \equiv q^2 - \mu^2$$

$$\bar{D}_p \equiv (q+p)^2 - \mu^2$$

$\mu^2 \equiv$ regulates IR behavior induced by subtraction term

FDR

- **SHIFT INVARIANCE** is preserved, e.g.

$$\int [d^4q]\, \frac{1}{\bar{q}^2\,\bar{D}_p} = \int [d^4q]\, \frac{1}{\bar{q}^2\,\bar{D}_{-p}}$$

since both sides share the **same** subtraction term

- Easy to prove in general when using $R=$ DReg

Extra Integrals

- Does *"naive"* **NUMDEN** cancellation work?

$$\int [d^4q] \frac{q^2}{\bar{q}^4 \bar{D}_p} \stackrel{?}{=} \int [d^4q] \frac{\not{q}^2}{\not{q}^2 \bar{q}^2 \bar{D}_p} = \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_p}$$

$\Uparrow$ subtracts $q^2/\bar{q}^6$ $\qquad$ $\Uparrow$ subtracts $1/\bar{q}^4$

- **NO**, because different subtractions contribute despite $\mu \to 0$

$$\left(\int [d^4q] \frac{\mu^2}{\bar{q}^4 \bar{D}_p} \equiv \right) \int_R d^4q \frac{\bar{q}^2 - q^2}{\bar{q}^6} = -\mu^2 \int d^4q \frac{1}{\bar{q}^6} = \frac{i\pi^2}{2}$$

- The correct **NUMDEN** cancellation is:

$$\int [d^4q] \frac{\bar{q}^2 + \mu^2}{\bar{q}^4 \bar{D}_p} = \int [d^4q] \frac{\not{\bar{q}}^2}{\not{\bar{q}}^2 \bar{q}^2 \bar{D}_p} + \underbrace{\int [d^4q] \frac{\mu^2}{\bar{q}^4 \bar{D}_p}}_{\text{Extra Integral!}}$$

Global prescription (GP)

- To keep gauge cancellations

$$\boxed{q^2 \to \bar{q}^2 \text{ in denominators} \quad \Leftrightarrow \quad q^2 \to \bar{q}^2 \text{ in numerators}}$$

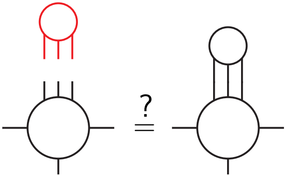
when q^2 originates from Feynman rules (**not** from reduction!)

- Apart from that

algebraic manipulations on FDR integrals are legal

such as tensor decomposition and IBP to reduce them to MI

Unitarity

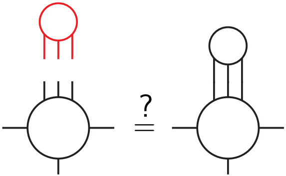


- In FDR

$$\int [d^4q_1][d^4q_2] \frac{1}{(\bar{q}_1^2 - M^2)^2} \frac{1}{(\bar{q}_2^2 - M^2)^2} = \left(\int [d^4q] \frac{1}{(\bar{q}^2 - M^2)^2} \right)^2$$

- That is promising, **but** ...

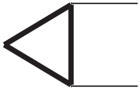
Extra Extra Integrals (EEIs)



- One needs **GP** at the level of the subamplitude  (Sub-Prescription, **SP**) and also **GP** at the level of the full amplitude on the right (**full GP**)
- **SP** and **full GP** clash with each other
- It is possible to correct for this mismatch and ensure **SUBINTEGRATION** consistency by adding **EEIs** derived by **solely** analyzing the loop diagrams on the right

FDR treatment of IR infinities

- Adding μ^2 to propagators regulates **virtual** IR divergences



$$= \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} \equiv \lim_{\mu \to 0} \int d^4q \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2}$$

giving rise to logs of μ

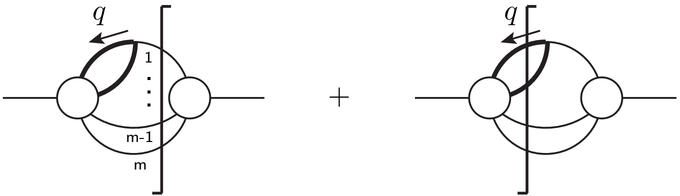
- **Real** matched via *cutting rules* $\boxed{\frac{i}{\bar{q}^2 + i\varepsilon} \to (2\pi)\, \delta_+(\bar{q}^2)}$ e.g.

$$\int_{\Phi_2} \Re \left(\int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_1 \bar{D}_2} \right) = \int_{\bar{\Phi}_3} \frac{1}{\bar{s}_{13} \bar{s}_{23}} \qquad \begin{cases} \bar{s}_{ij} = (\bar{p}_i + \bar{p}_j)^2 \\ \bar{p}^2_{i,j} = \mu^2 \end{cases}$$

- Logs of μ can be rewritten as counterterms integrated over a μ-**massive** phase-space $\bar{\Phi}_3$

Unobserved particles

- m-body virtual and $(m+1)$-body real IR divergences compensate each other



- In both cases the divergent splitting is regulated by μ-**massive unobserved particles:**

Results

DReg vs FDR @NLO

- A one-to-one correspondence exists between DReg and FDR

$$\Gamma(1-\epsilon)\,\pi^{\epsilon}\int \frac{d^n q}{\mu_R^{-2\epsilon}}\,\Big(\cdots\Big)\Bigg|_{\mu_R=\mu \text{ and } \frac{1}{\epsilon^i}=0} = \int [d^4 q]\,\Big(\cdots\Big)$$

for both UV and IR divergent loop integrals

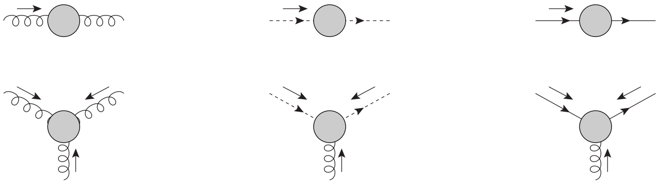
- Analogously for the real contribution

$$\left(\frac{\mu_R^2}{s}\right)^{\epsilon}\int_{\phi_3} dx\,dy\,dz\Big(\cdots\Big)\delta(1-x-y-z)(xyz)^{-\epsilon}\Bigg|_{\mu_R=\mu \text{ and } \frac{1}{\epsilon^i}=0}$$

$$=\int_{\bar{\phi}_3} dx\,dy\,dz\Big(\cdots\Big)\delta(1-x-y-z+3\mu^2/s)$$

DReg vs FDR @NNLO (off shell)

- FDR has been proven to renormalize consistently **off-shell** QCD up to 2 loops (B. Page, R.P., JHEP 1511 (2015) 183)



- α_S and m_q shifts necessary to translate FDR $\Leftrightarrow$ $\overline{\text{MS}}$ have been determined

$\mathbf{EEI_b}$s

- Analyzing **EEIs** led to a fix of 2-loop *"naive"* FDH in DReg

$$\boxed{G^{(2-\text{loop})}_{\text{bare, DReg}}\big|_{n_s=4} \to G^{(2-\text{loop})}_{\text{bare, DReg}}\big|_{n_s=4} + \sum\nolimits_{\text{Diag}} \mathbf{EEI_b}\big|_{n_s=4}}$$

where $n_s = \gamma_\mu \gamma^\mu = g_{\mu\nu} g^{\mu\nu}$

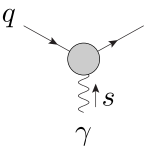
- **EEI$_\mathbf{b}$s** are obtained from FDR **EEIs** by *"dropping"* the subtraction term, e.g.

$$\begin{aligned} \mathbf{EEI} &= \text{Const} \int [d^4q] \frac{1}{\bar{q}^2 \bar{D}_p} \quad \Rightarrow \\ \mathbf{EEI_b}\text{s} &= \text{Const} \int d^nq \frac{1}{q^2 D_p} \end{aligned}$$

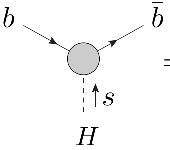
- **EEI$_\mathbf{b}$s** reproduce the effect of the FDH/DRed evanescent operators in DReg, at least **off-shell**

On-shell QCD @2-loops

- B. Page and I started computing 2-loop IR divergent vertices



$= \quad V_\gamma^{(2)}$



$= \quad V_H^{(2)}$

$$\mathsf{EEI}_{\mathbf{b}}(V_\gamma^{(2)}) = \frac{\alpha_s^2}{16\pi^2}\, C_F \left(\frac{2N_c + n_f}{3} + \frac{1}{N_c} \right) \left(\frac{1}{\epsilon} + 1 + \ln \frac{\mu_{\mathsf{R}}^2}{-s - i\varepsilon} \right) V_\gamma^{(0)}$$

$$\mathsf{EEI}_{\mathbf{b}}(V_H^{(2)}) = \frac{\alpha_s^2}{16\pi^2}\, 2\, C_F \left(\frac{2N_c + n_f}{3} + \frac{1}{N_c} \right) \left(\frac{1}{\epsilon} + 2 + \ln \frac{\mu_{\mathsf{R}}^2}{-s - i\varepsilon} \right) V_H^{(0)}$$

- Work ongoing with A. Signer and C. Gnendiger:

EEI$_{\mathbf{b}}$s *compensate the wrong UV (sub)renormalization of the evanescent couplings when using "naive" FDH*

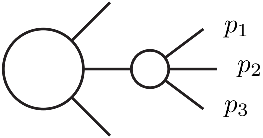
Local subtraction of IR divergences @NLO

- Disintegrating virtual logs

$$\sigma_{\rm NLO} = \int_{\Phi_2} \left(|M|^2_{\rm Born} + \underbrace{|M|^2_{\rm Virt}}_{\text{devoid of logs of } \mu} \right) F_J^{(2)}(p_1,p_2)$$

$$+ \underbrace{\int_{\Phi_3}}_{\mu \to 0 \text{ in here!}} \left(|M|^2_{\rm Real}\, F_J^{(3)}(p_1,p_2,p_3) - |M|^2_{\rm CT}\, F_J^{(2)}(\underbrace{\hat{p}_1,\hat{p}_2}_{\text{mapped kinematics}}) \right)$$

- *"Tripole"* arrangements when more final state particles



$e^+e^- \to \gamma^* \to q\bar{q}$ @NLO

- The local counterterm reads

$$|M|^2_{\mathsf{CT}} = \frac{16\pi\alpha_s}{s} C_F |M|^2_{\mathsf{Born}}(\hat{p}_1, \hat{p}_2)\left(\frac{s^2}{s_{13}s_{23}} - \frac{s}{s_{13}} - \frac{s}{s_{23}} + \frac{s_{13}}{2s_{23}} + \frac{s_{23}}{2s_{13}} - \frac{17}{2}\right)$$

- The mapping is: $\hat{p}_1^{\alpha} = \kappa \Lambda^{\alpha}_{\ \beta}\, p_1^{\beta}\left(1 + \frac{s_{23}}{s_{12}}\right)$, $\hat{p}_2^{\alpha} = \kappa \Lambda^{\alpha}_{\ \beta}\, p_2^{\beta}\left(1 + \frac{s_{13}}{s_{12}}\right)$

  where $\kappa = \sqrt{\frac{s s_{12}}{(s_{12}+s_{13})(s_{12}+s_{23})}}$ and $\Lambda^{\alpha}_{\ \beta}$ brings $\hat{p}_1 + \hat{p}_2 = (\sqrt{s}, 0, 0, 0)$

- The correct limiting behavior is obtained for both $s_{13} \to 0$ and $s_{23} \to 0$ $\Rightarrow$ *"tripole"*

- Sanity checks
  - Inclusive $\sigma_{\text{NLO}} = \sigma_0 \left(1 + C_{\text{F}} \frac{3}{4} \frac{\alpha_s}{\pi}\right)$ reproduced by montecarlo
  - $\sigma^{\text{cut}}_{\text{NLO}}$ (when available analytically) reproduced by montecarlo

- Comparisons with MadGraph5_aMC@NLO + FastJet
  (thanks to M. Moretti)

n_j	σ_{MG5} [pb]	σ_{FDR} [pb]
≥ 1	0.1729(2) $\times 10^6$	0.1730(1) $\times 10^6$
≥ 2	0.1268(3) $\times 10^6$	0.1265(2) $\times 10^6$
≥ 3	0.2335(7) $\times 10^4$	0.2333(5) $\times 10^4$

$\sqrt{s} = 1\,\text{GeV}$, $p_{\text{T}} > 0.2$, $|\eta| < 4$, $R = 0.7$

n_j	σ_{MG5} [pb]	σ_{FDR} [pb]
≥ 1	0.8875(4) $\times 10^5$	0.8878(5) $\times 10^5$
≥ 2	0.778(2) $\times 10^5$	0.7755(7) $\times 10^5$
≥ 3	0.1415(2) $\times 10^5$	0.1412(2) $\times 10^5$

$\sqrt{s} = 1.2\,\text{GeV}$, $p_{\text{T}} > 0.2$, $|\eta| < 4$, $R = 0.7$

Slicing-like treatment of IR singularities

- $$\boxed{\bar{\Phi}_{m+1} \stackrel{\text{mapping}}{\longrightarrow} \Phi_{m+1}}$$

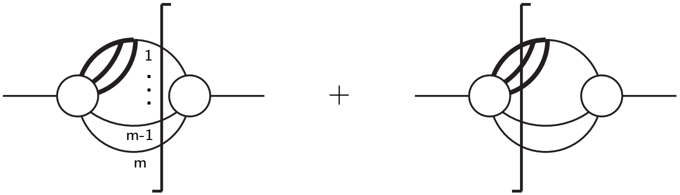
$$\sigma^{\mathrm{R}}_{\mathrm{NLO}} = \lim_{\mu \to 0} \int_{\bar{\Phi}_{m+1}} \underbrace{d\sigma^{\mathrm{R}}_{\mathrm{NLO}}(\Phi_{m+1})}_{\text{gauge invariant!}} \prod_{i<j} \frac{s_{ij}}{\bar{s}_{ij}}$$

based on $d\sigma^{\mathrm{R}}_{\mathrm{NLO}}(\Phi_{m+1}) \sim \frac{1}{s_{ij}}$ if $s_{ij} \to 0$

- $$\boxed{e^+e^- \to \gamma^* \to jets}$$

	Local	$\mu/s = 10^{-4}$	$\mu/s = 10^{-3}$
Total	33899(10)	34219(196)	33789(116)
$n_j \geq 2$	26855(35)	27104(163)	26789(132)
$n_j \geq 1$	33409(26)	33352(121)	33507(80)

- **NNLO ansatz** cancellation of double unresolved singularities



- $$\boxed{\sigma = \sigma_{\text{LO}} + \sigma_{\text{NLO}} + \sigma_{\text{NNLO}}}$$

$$\sigma_{\text{LO}} = \int_{\Phi_m} d\sigma^{\text{B}}_{\text{LO}}(\Phi_m)$$

$$\sigma_{\text{NLO}} = \int_{\Phi_m} d\sigma^{\text{V}}_{\text{NLO}}(\Phi_m) + \lim_{\mu\to 0}\int_{\bar\Phi_{m+1}} d\sigma^{\text{R}}_{\text{NLO}}(\Phi_{m+1}) \prod_{i<j}\frac{s_{ij}}{\bar s_{ij}}$$

$$\sigma_{\text{NNLO}} = \int_{\Phi_m} d\sigma^{\text{VV}}_{\text{NNLO}}(\Phi_m) + \lim_{\mu\to 0}\int_{\bar\Phi_{m+1}} d\sigma^{\text{VR}}_{\text{NNLO}}(\Phi_{m+1}) \prod_{i<j}\frac{s_{ij}}{\bar s_{ij}}$$

$$+ \boxed{\lim_{\mu\to 0}\int_{\bar\Phi_{m+2}} d\sigma^{\text{RR}}_{\text{NNLO}}(\Phi_{m+2}) \prod_{i<j}\frac{s_{ij}}{\bar s_{ij}} \prod_{i<j<k}\left(\frac{s_{ijk}}{\bar s_{ijk}}\right)}$$

Conclusions

1. **FDR** is turning to a competitive tool to compute RC
   - UV subtraction incorporated at the integrand level in the definition of loop integration $\Rightarrow$ 4-dim FDR integrals
   - No CTs introduced in $\mathcal{L}$
2. IR regularization *à la* FDR well understood @NLO for FSR
   - 2-jet cross section with local IR subtraction worked out (more to come)
3. Going on-shell @NNLO is feasible in **FDR**
   - A fix to *"naive"* FDH avoiding evanescent couplings is available for realistic observables
4. To do list
   - ISR @NLO (should be trivial) and IR @NNLO
   - Complete FDR calculation of $V^{(2)}_{\gamma,H}$ (in progress)
   - Studying FDR integration as a new mathematical object

Thanks!