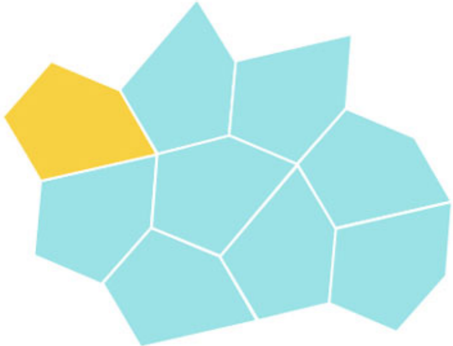
Bootstrapping pentagon functions



Johannes M. Henn

based on

[arXiv:1712.09610 [hep-th]] with D. Chicherin and V. Mitev

Talk at CERN, Jan 13, 2017

Cluster of Excellence
PRISMA
Precision Physics,
Fundamental Interactions
and Structure of Matter

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

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Multi-particle collisions as the next frontier

picture: Quanta Magazine

- at high energies, many particles produced
- challenge to evaluate the virtual corrections
- Les Houches 2017 wishlist, e.g.

  $pp \to 3 \text{ jets}$ $pp \to H + 2 \text{ jets}$ $pp \to V + 2 \text{ jets}$

- challenge: 5-particle processes at NNLO

Bootstrap approach

$\vec{x}$ kinematic dependence

$D = 4 - 2\epsilon$ dimension

$$\mathcal{A}(\vec{x}, \epsilon) = \sum_{i,j,k} c_{ijk} \frac{1}{\epsilon^i} r_j(\vec{x})\, f_k(\vec{x}) + \mathcal{O}(\epsilon)$$

- Laurent expansion in ϵ

- rational/algebraic normalization factors

controlled by leading singularities, generalized cuts

- special functions

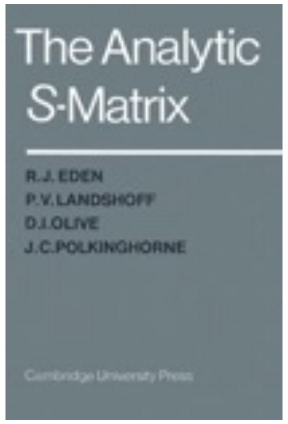
ansatz (educated guess, information from Feynman integrals)

- unknowns: finite number of coefficients

fix from physical input, e.g. soft and Regge limit, discontinuities

Bootstrap (pre)history

- 1960's: determine S-matrix from analytic properties



The Analytic S-Matrix

R.J. EDEN
P.V. LANDSHOFF
D.I. OLIVE
J.C. POLKINGHORNE

Cambridge University Press

- 1994: 'One loop n point gauge theory amplitudes, unitarity and collinear limits' [Bern, Dixon, Dunbar, Kosower]

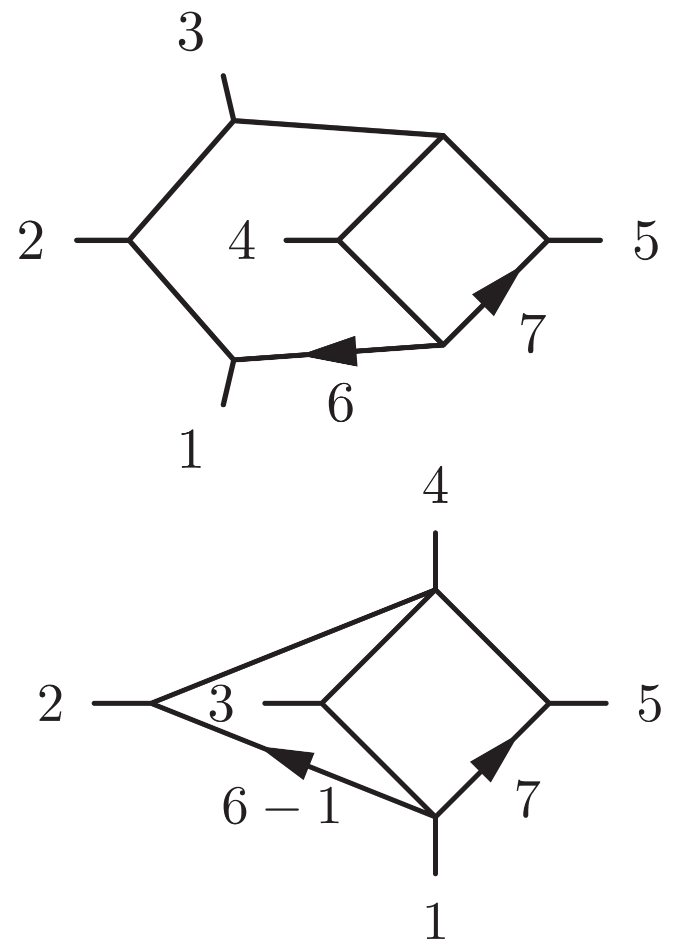
- 2011: bootstrap in planar maximally supersymmetric Yang-Mills theory [Dixon, Drummond, JMH]

many further developments [Almelid, Bartels, Bargheer, Caron-Huot, Del Duca, Dixon, Druc, Drummond, Duhr, Dulat, Gardi, Harrington, JMH, von Hippel, Marzucca, McLeod, Paulos, Pennington, Parker, Papathanasiou, Scherlis, Schomerus, Sprenger, Spradlin, Trnka, Verbeek, Volovich]

- 2017: first application to multi-loop QCD integrals, non-planar [Chicherin, JMH, Mitev]

Bootstrapping pentagon functions

[arXiv:1712.09610 [hep-th]] with D. Chicherin and V. Mitev



Dima

Vladimir

Many thanks to my fantastic collaborators!

Strange Numbers Found in Particle Collisions

An unexpected connection has emerged between the results of physics experiments and an important, seemingly unrelated set of numbers in pure mathematics.

By Kevin Hartnett

Quanta magazine

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Xiaolin Zeng for Quanta Magazine

Strange numbers, e.g. $\zeta_2 = \sum_{k \geq 1} \frac{1}{k^2} = \frac{\pi^2}{6}$ …coming from

interesting special functions, e.g. $\mathrm{Li}_2(x) = -\int_0^x \log(1-y)\frac{dy}{y}$, $\mathrm{Li}_2(1) = \zeta_2$

$$\mathcal{A}(\vec{x}, \epsilon) = \sum_{i,j,k} c_{ijk} \frac{1}{\epsilon^i} r_j(\vec{x})\, f_k(\vec{x}) + \mathcal{O}(\epsilon)$$

Multiple polylogarithms, iterated integrals, symbols, and all that…

- many Feynman integrals evaluate to multiple polylogarithms; conveniently described by ‘symbols’

[Goncharov, Spradlin, Volovich, Vergu, 2010]

- those special functions are best thought of as solutions to differential equations in canonical form

[JMH, 2013]

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k \underset{\swarrow}{A_k} \log \underset{\searrow}{\alpha_k}(\vec{x})\right]\vec{f}(\vec{x};\epsilon)$$

constant matrices letters (alphabet)

Iterated integrals

- Logarithm and dilogarithm are first examples of iterated integrals with special ``d-log`` integration kernels

$$\frac{dt}{t} = d\log t \qquad \frac{-dt}{1-t} = d\log(1-t) \qquad \frac{dt}{1+t} = d\log(1+t)$$

- these are called harmonic polylogarithms (HPL) [Remiddi, Vermaseren]

e.g. $H_{1,-1}(x) = \int_0^x \frac{dx_1}{1-x_1} \int_0^{x_1} \frac{dx_2}{1+x_2}$

- Natural generalization: multiple polylogarithms

[also: hyperlogarithms; Goncharov polylogarithms]

allow kernels $w = d\log(t-a)$

$$G_{a_1,\dots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\dots,a_n}(t)$$

- Chen iterated integrals

$$\int_C \omega_1 \omega_2 \dots \omega_n \qquad C : [0,1] \longrightarrow M$$

(space of kinematical variables)

Alphabet: set of differential forms $\omega_i = d\log \alpha_i$

The kinematic invariants

For five points we have

- $s_{ij} = 2p_i \cdot p_j = \langle ij \rangle [ji]$ of which five are independent:

$$v_1 = s_{12} \qquad v_2 = s_{23} \quad \cdots \quad v_5 = s_{51}$$

- $\sqrt{\Delta}$

where $\Delta = \det(2p_i \cdot p_j)$

The alphabet

The planar case $\mathbb{A}_P$

- $W_1 = v_1 = 2p_1 \cdot p_2$ and 4 cyclic
- $W_6 = v_3 + v_4 = 2p_4 \cdot (p_3 + p_5)$ and 4 cyclic
- $W_{11} = v_1 - v_4 = 2p_3 \cdot (p_4 + p_5)$ and 4 cyclic
- $W_{16} = v_1 + v_2 - v_4 = -2p_1 \cdot p_3$ and 4 cyclic

- $W_{26} = \frac{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 - \sqrt{\Delta}}{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 + \sqrt{\Delta}}$ and 4 cyclic
- $W_{31} = \sqrt{\Delta}$

The alphabet

The non-planar case $\mathbb{A}_{\text{NP}}$

- $W_1 = v_1 = 2p_1 \cdot p_2$ and 4 cyclic
- $W_6 = v_3 + v_4 = 2p_4 \cdot (p_3 + p_5)$ and 4 cyclic
- $W_{11} = v_1 - v_4 = 2p_3 \cdot (p_4 + p_5)$ and 4 cyclic
- $W_{16} = v_1 + v_2 - v_4 = -2p_1 \cdot p_3$ and 4 cyclic
- $W_{21} = v_3 + v_4 - v_1 - v_2 = 2p_3 \cdot (p_1 + p_4)$ and 4 cyclic
- $W_{26} = \frac{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 - \sqrt{\Delta}}{v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 + \sqrt{\Delta}}$ and 4 cyclic
- $W_{31} = \sqrt{\Delta}$

Even and odd

If $p_i \in \mathbb{R}^{1,3} \;\Rightarrow\; \sqrt{\Delta}^* = -\sqrt{\Delta}$

Hence $W_j^* = W_j^{-1}$ and $\log(W_j^*) = -\log(W_j)$ for $j = 26, \ldots, 30$

Symbols of the functions

Once we have the alphabet, we can construct
the symbol of all the functions F

At weight w, the symbol looks like

$$\mathrm{SB}[F] = \sum \underbrace{\alpha^{a_1,\ldots,a_w}}_{\text{constants}} [W_{a_1}, \ldots, W_{a_w}]$$

It's a short form for “iterated integrals”

$$[W_{a_1}, \ldots, W_{a_w}] = \int d\log(W_{a_1}) \cdots \int d\log(W_{a_w})$$

‘symbols’ vs. iterated integrals

- roughly speaking, symbols are iterated integrals, forgetting about integration constants
- upgrade to functions by specifying boundary point
- example:

Definition: boundary point $v_i = -1.$

$$[W_3/W_1, W_{13}/W_1] = [v_3/v_1, 1 - v_3/v_1]$$

$$= [y, 1-y]$$

$$y = v_3/v_1$$

$$= \int_1^{v_3/v_1} d\log(1-y) \int_1^{y} d\log(z)$$

$$= -\mathrm{Li}_2\left(1 - \frac{v_3}{v_1}\right)$$

The transcendental functions

Planar case

All planar pentagon functions F_j are known to 2-loops from

[Gehrmann, Henn, LoPresti, 2015]

[related work: Papadopoulos, Tommasini, Wever, 2015]

Its alphabet $\mathbb{A}_{\text{P}}$ has 26 letters

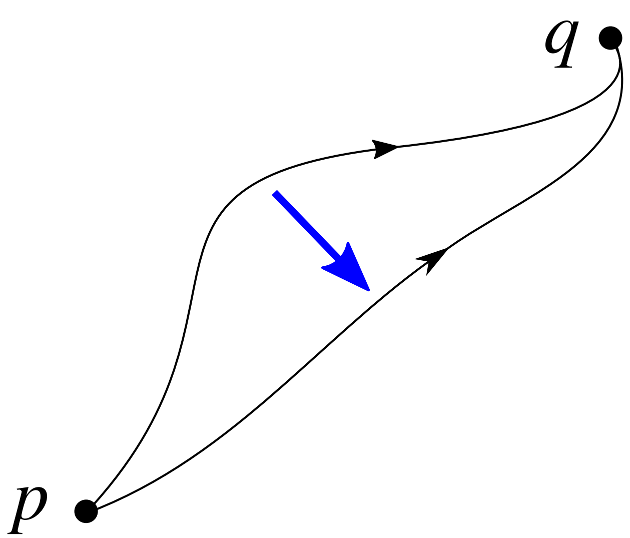
Non-planar case

At the non-planar level, there is a natural generalization of the alphabet by making it permutation invariant

The alphabet $\mathbb{A}_{\text{NP}}$ has 31 letters

[Chicherin, JMH, Mitev, 2017]

Integrability of the symbols



The function F that

$$\mathrm{SB}[F] = \sum \alpha^{a_1,\ldots,a_w}\ [W_{a_1},\ldots,W_{a_w}]$$

represents should be invariant under infinitesimal deformations of the integration contour

“The rotation is zero”

This imposes

$$\sum \alpha^{a_1,\ldots,a_w}[\cdots \widehat{W}_{a_i},\ \widehat{W}_{a_{i+1}},\cdots]\left\{\frac{\partial \log W_{a_i}}{\partial v_r}\frac{\partial \log W_{a_{i+1}}}{\partial v_s} - (r \leftrightarrow s)\right\} = 0$$

for all $r < s = 1,\ldots,5$ and $i = 1,\ldots,w-1$

The first and second entry conditions

First entry condition

- Planar case: s_{12} and 4 cyclic permutation: $\{W_i\}_{i=1}^{5}$
- Non-planar case: s_{12} and 9 permutation: $\{W_i\}_{i=1}^{5} \cup \{W_j\}_{i=16}^{20}$

Second entry condition

- Experimental fact in the planar case that some pairs $[W_i, W_j, \cdots]$ do not appear in the Feynman integrals
- In the non-planar case we close the pairs under S_5 permutations

How many functions do we have?

- Weight 0: one function, the constant
- Weight 1: 10 functions due to the first entry condition

$$\log(s_{ij}) \quad \text{with symbols} \quad [W_1], \ldots, [W_5], [W_{16}], \ldots, [W_{20}]$$

- Weight 2: 79 functions written as products of logs and dilogarithms in the W_i letters
- Weight 3: 616 functions
- Weight 4: 4927 functions

Mellin-Barnes integral representation

Main identity

$$\frac{1}{(X+Y)^a} = \frac{1}{\Gamma(a)} \int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(a+z) X^z Y^{-a-z} \qquad (\star)$$

Procedure and Properties

- Start with Feynman parametrization of I and use $(\star)$

  $$I = \int [dx] \frac{\cdots}{F^{\#}} \qquad F = x_1 x_3 s_{12} + \cdots$$

  $N+1$ terms in Feynman polynomial $\Rightarrow$ N-fold MB integral

- The MB representation is not unique!

- In the non-planar case beware of $(-1)^z$
  Use global Feynman parametrization

Advantages of the MB representation

Derivatives and limits are easy

Using packages such as `MB`, `MBasymptotics`, `MBsum`, ...
one can easily compute limits

The limits reduce the number of integrations

Discontinuities are easy

$$\mathrm{Disc}_x f(x)_{x=-y} = \frac{1}{2\pi i}\left[f(ye^{-i\pi}) - f(ye^{i\pi})\right]\,,\;\; y > 0\,.$$

$$\Rightarrow\;\mathrm{Disc}_x\left[\int dz\, x^z g(z)\right]_{x=-y} = -\int dz\, y^z \frac{g(z)}{\Gamma(-z)\Gamma(1+z)}\,.$$

Summary bootstrap strategy

$\vec{x}$ kinematic dependence

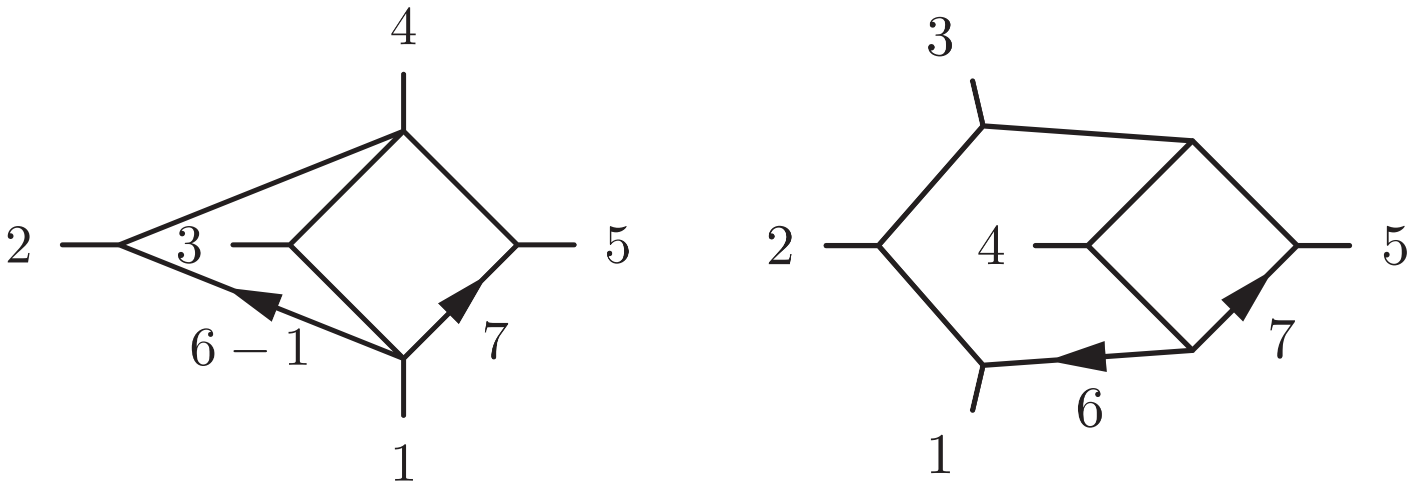
$D = 4 - 2\epsilon$ dimension

$$\mathcal{A}(\vec{x},\epsilon) = \sum_{i,j,k} c_{ijk} \frac{1}{\epsilon^i} r_j(\vec{x})\, f_k(\vec{x}) + \mathcal{O}(\epsilon)$$

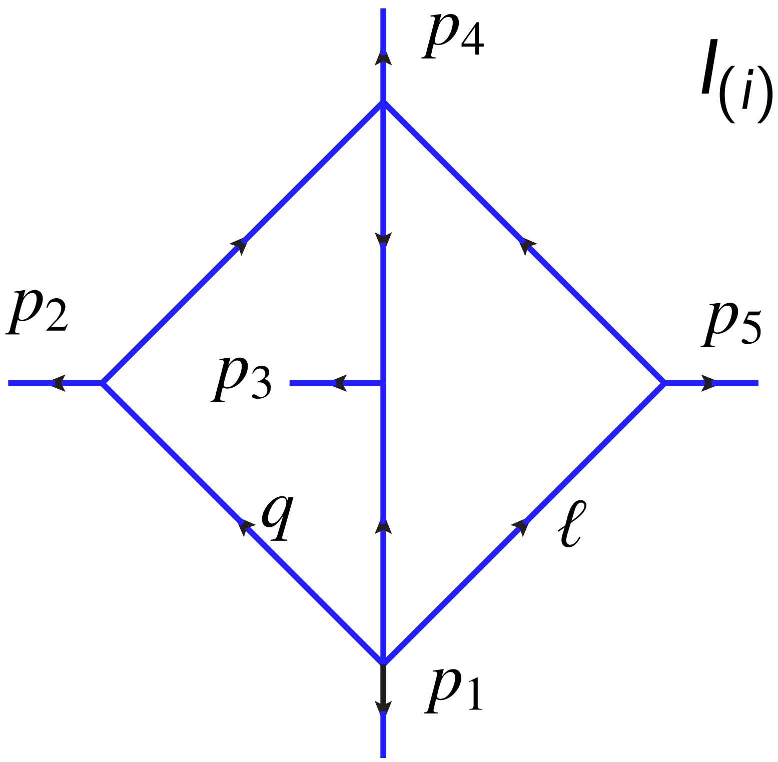
- normalization factors: computed by leading singularities

see e.g. [Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010]

- special functions: pentagon functions
- fix coefficients from Mellin-Barnes limits, discontinuities
- we apply this to integrals of the topologies:



The topology (i)



$$I_{(i)} = \pi^D \frac{\Gamma^3(-\epsilon)}{\Gamma(-3\epsilon)} \int \frac{[dz]}{(2\pi i)^5} \prod_{j=1}^{5} \Gamma(-z_j)$$
$$\times \frac{\Gamma(2+2\epsilon+z_{1,2,3,4,5})\Gamma(1+z_{1,4,5})}{\Gamma(2+z_{1,4,5})}$$
$$\times \frac{\Gamma(-1-2\epsilon-z_{1,2,4})\Gamma(-1-2\epsilon-z_{1,3,5})}{\Gamma(-2\epsilon-z_{1,2,4})\Gamma(-2\epsilon-z_{1,3,5})}$$
$$\times (-s_{15})^{z_1} (-s_{12})^{z_2} (-s_{13})^{z_3} (-s_{25})^{z_4}$$
$$\times (-s_{35})^{z_5} (-s_{23})^{-2-2\epsilon-z_{1,2,3,4,5}}$$

$$z_{i_1,\cdots,i_n} = z_{i_1} + \cdots + z_{i_n} \qquad [dz] = dz_1 \dots dz_5$$

Ansatz

$$I_{(i)} = \frac{1}{\sqrt{\Delta}} \left[\frac{\mathcal{P}_2}{\epsilon^2} + \frac{\mathcal{P}_3}{\epsilon} + \mathcal{P}_4 + \mathcal{O}(\epsilon) \right]$$

where each $\mathcal{P}_n$ is an MB integral

Weight	2	3	4
$\#$ of odd symbols for the topology (i)	9	180	2730
with first entries $s_{12}, s_{34}, \ldots$	1	13	143
$\mathcal{S}_2 \times \mathcal{S}_3$ symmetry	1	4	21
second entry condition	1	3	12

Weight 2

$$\begin{aligned}\mathrm{SB}[\mathcal{P}_2] =& c_2\Big(- [W_1, W_{30}] - [W_3, W_{26}] \\ &+ [W_4, W_{26}] + [W_4, W_{30}] + [W_5, W_{26}] \\ &+ [W_5, W_{30}] - [W_{16}, W_{26}] - [W_{17}, W_{30}]\Big)\end{aligned}$$

Discontinuities

$$\mathrm{Disc}_{v_1\sim 0}\,\mathrm{SB}[\mathcal{P}_2] = -c_2[W_{30}]\Big|_{v_1\sim 0} = c_2\Big([v_2 - v_4] + [v_3] - [v_4] - [v_5]\Big)$$

$$\begin{aligned}\mathrm{Disc}_{v_5\sim 0}\mathrm{Disc}_{v_1\sim 0}\,\frac{\mathcal{P}_2}{\sqrt{\Delta}} &\overset{\mathrm{MB}}{=} \frac{-3}{v_3(v_2 - v_4)} \\ &\overset{!}{=} \mathrm{Disc}_{v_5\sim 0}\mathrm{Disc}_{v_1\sim 0}\,\frac{\mathrm{SB}[\mathcal{P}_2]}{\sqrt{\Delta}} = -\frac{c_2}{\sqrt{\Delta}}\Big|_{\substack{v_5\sim 0\\ v_1\sim 0}}.\end{aligned}$$

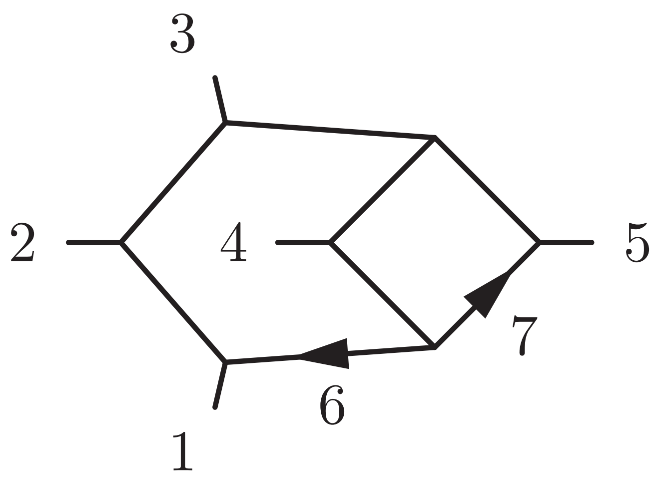
Since $\sqrt{\Delta}\Big|_{\substack{v_5\sim 0\\ v_1\sim 0}} = v_3(v_2 - v_4) \;\Rightarrow\; c_2 = 3$

Results

- For SB[$\mathcal{P}_2$] we find

$$\mathcal{P}_2 = 6\Big[\mathrm{Li}_2(W_{26}) + \mathrm{Li}_2(W_{30}) - \mathrm{Li}_2(W_{26}W_{30}) - \frac{1}{2}\log W_{26}\log W_{30} - \frac{\pi^2}{6}\Big]$$

- We obtained the symbols of $\mathcal{P}_3$ and $\mathcal{P}_4$
- We obtained the symbols of $I_{(c)}$



Conclusions

- first application of bootstrap method to non-planar multi-leg integrals

- proposed space of pentagon functions for scattering amplitudes

- conjectural second-entry condition - relation to Steinmann relations?

- possible applications to amplitudes where 'integrand' is available, e.g. all-plus amplitude, supergravity

Pentagons are full of surprises…!

The (Math) Problem With Pentagons

Triangles fit effortlessly together, as do squares. When it comes to pentagons, what gives?

By Patrick Honner