Exploring the function space of Feynman integrals

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I.: **Periodic functions and periods**

II.: **Review of differential equations and multiple polylogarithms**

III.: **Elliptic generalisations**

Part I

Periodic functions and periods

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A period ω of the function f is a constant such that for all z:

$$f(z+\omega) = f(z)$$

The set of all periods of f forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\},$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods ω_1 and ω_2.

Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z \quad = \quad \ln(x)\,.$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z \quad = \quad \int\limits_{x}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Periods as integrals over algebraic functions

In both examples the periods can be expressed as integrals involving only algebraic functions.

- Period of the exponential function:

$$2\pi i \quad = \quad 2i \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that g_2 and g_3 are two given algebraic numbers. Then

$$\omega_1 = 2\int_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad \omega_2 = 2\int_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

where t_1, t_2 and t_3 are the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$.

Numerical periods

Kontsevich and Zagier suggested the following generalisation:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace "rational" with "algebraic".
- The set of all periods is countable.
- Example: $\ln 2$ is a numerical period.

$$\ln 2 \quad = \quad \int_1^2 \frac{dt}{t}.$$

Part II

Review of differential equations and multiple polylogarithms

Differential equations

Let t be an external invariant (e.g. $t = (p_i + p_j)^2$) or an internal mass. Let $I_i \in \{I_1, ..., I_N\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial t} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial t} I_i = \sum_{j=1}^{N} a_{ij} I_j$$

(Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99)

Differential equations

More generally:

$\vec{I} = (I_1, ..., I_N)$, set of master integrals,

$\vec{x} = (x_1, ..., x_n)$, set of kinematic variables the master integrals depend on.

We obtain a system of differential equations of Fuchsian type

$$d\vec{I} = A\vec{I},$$

where A is a matrix-valued one-form

$$A = \sum_{i=1}^{n} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A = 0.$$

Multiple polylogarithms

Definition based on nested sums:

$$\text{Li}_{m_1,m_2,\dots,m_k}(x_1,x_2,\dots,x_k) = \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \dots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on iterated integrals:

$$G(z_1,\dots,z_k;y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \dots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$$

Conversion:

$$\text{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = (-1)^k G_{m_1,\dots,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \dots, \frac{1}{x_1 \dots x_k}; 1\right)$$

Short hand notation:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = G(\underbrace{0,\dots,0}_{m_1-1}, z_1, \dots, z_{k-1}, \underbrace{0\dots,0}_{m_k-1}, z_k; y)$$

The ε-form of the differential equation

If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$d\vec{J} = A'\vec{J}, \qquad A' = UAU^{-1} - U\,dU^{-1}$$

Suppose one finds a transformation matrix U, such that

$$A' \quad = \quad \varepsilon \sum_j C_j \, d\ln p_j(\vec{x}),$$

where

- ε appears only as prefactor,
- C_j are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn '13

Transformation to the ε-form

We may

- perform a rational / algebraic transformation on the kinematic variables

$$(x_1, ..., x_n) \quad \to \quad (x'_1, ..., x'_n),$$

  often done to absorb square roots.

- change the basis of the master integrals

$$\vec{I} \quad \to \quad U\vec{I},$$

  where U is rational in the kinematic variables

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

Numerical evaluations of multiple polylogarithms

Multiple polylogarithms have branch cuts.

Numerical evaluation of multiple polylogarithms $\text{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k)$ as a function of k complex variables x_1, x_2, ..., x_k:

- Use truncated sum representation within its region of convergence.
- Use integral representation to map arguments into this region.
- Acceleration techniques to speed up the computation.

Implementation in GiNaC, using arbitrary precision arithmetic in C++.

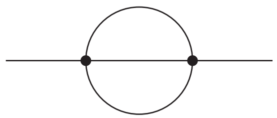
J. Vollinga, S.W. '04

Part III

Elliptic generalisations

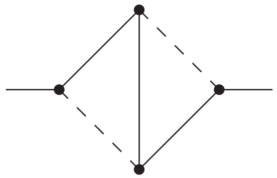
Single scale integrals beyond multiple polylogarithms

Starting from two-loops, there are integrals which cannot be expressed in terms of multiple polylogarithms.



Simplest example: Two-loop sunrise integral with equal masses.

Slightly more complicated: Two-loop kite integral.



Both integrals depend on a single scale t/m^2.

Change variable from t/m^2 to the nome q or the parameter τ with $q = e^{i\pi\tau}$.

Sabry, Broadhurst, Fleischer, Tarasov, Bauberger, Berends, Buza, Böhm, Scharf, Weiglein, Caffo, Czyz, Laporta, Remiddi, Groote, Körner, Pivovarov, Bailey, Borwein, Glasser, Adams, Bogner, Müller-Stach, Schweitzer, S.W, Zayadeh, Bloch, Vanhove, Pozzorini, Gunia, Broedel, Duhr, Dulat, Tancredi, ...

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1 x_2 x_3 t + m^2 \left(x_1 + x_2 + x_3\right)\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right) = 0$$

- From the maximal cut:

$$y^2 - \left(x - \frac{t}{m^2}\right)\left(x - \frac{t-4m^2}{m^2}\right)\left(x^2 + 2x + 1 - 4\frac{t}{m^2}\right) = 0$$

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

Set $\quad \tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{i\pi\tau}.$

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}.$$

Elliptic dilogarithm:

$$\mathrm{E}_{2;0}(x;y;q) = \frac{1}{i} \left[\frac{1}{2} \mathrm{Li}_2(x) - \frac{1}{2} \mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right) \right].$$

Various definitions of elliptic polylogarithms can be found in the literature

Beilinson '94, Levin '97, Wildeshaus '97, Brown, Levin '11, Bloch, Vanhove '13, Adams, Bogner, S.W. '14, Remiddi, Tancredi '17, Broedel, Duhr, Dulat, Tancredi '17

Elliptic generalisations

In order to express the sunrise/kite integral to all orders in ε introduce

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) =$$

$$= \sum_{j_1=1}^{\infty}\cdots\sum_{j_l=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}}\cdots\frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}}\cdots\frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1k_1+\ldots+j_lk_l}}{\prod\limits_{i=1}^{l-1}(j_ik_i+\ldots+j_lk_l)^{o_i}}.$$

Numerical evaluation: G. Passarino ’16

The all-order in ε result (ELi-representation)

Taylor expansion of the sunrise integral around $D = 2 - 2\varepsilon$:

$$S = \frac{\psi_1}{\pi} \sum_{j=0}^{\infty} \varepsilon^j E^{(j)}$$

Each term in the ε-series is of the form

$$E^{(j)} \sim \text{linear combination of } \mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}} \text{ and } \mathrm{Li}_{n_1,...,n_l}$$

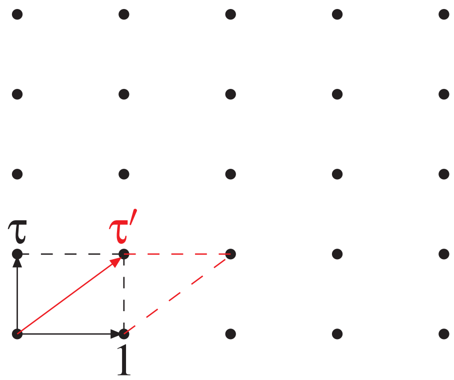
Using dimensional-shift relations this translates to the expansion around $4 - 2\varepsilon$.

$\Rightarrow$ The multiple polylogarithms extended by $\mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}}$ are the class of functions to express the equal mass sunrise graph to all orders in ε.

Adams, Bogner, S.W., '15

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(2, \mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

Congruence subgroups

Apart from $\mathrm{SL}_2(2,\mathbb{Z})$ we may also look at congruence subgroups, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \text{ mod } N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a, d \equiv 1 \text{ mod } N,\ c \equiv 0 \text{ mod } N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a, d \equiv 1 \text{ mod } N,\ b, c \equiv 0 \text{ mod } N \right\}$$

Modular forms for congruence subgroups: Require “nice” transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Dirichlet character

Let N be a positive integer. A function $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character modulo N, if

(i) $\chi(n) = \chi(n+N) \qquad \forall n \in \mathbb{Z}$,

(ii) $\chi(n) = 0$ if $\gcd(n,N) > 1$ and $\chi(n) \neq 0$ if $\gcd(n,N) = 1$,

(iii) $\chi(nm) = \chi(n)\chi(m) \qquad \forall\, n,m \in \mathbb{Z}$.

The conductor of χ is the smallest positive divisor $d|N$ such that there is a character χ' modulo d with

$$\chi(n) = \chi'(n) \qquad \forall\, n \in \mathbb{Z} \qquad \text{with } \gcd(n,N) = 1.$$

Modular forms with character

We may relax the transformation law:

Let N be a positive integer and let χ be a Dirichlet character modulo N. A function $f:\mathbb{H}\to\mathbb{C}$ is a modular form of weight k for $\Gamma_0(N)$ with character χ if

(i) f is holomorphic on $\mathbb{H}$,

(ii) f is holomorphic at the cusps of $\Gamma_1(N)$,

(iii) $f\left(\dfrac{a\tau+b}{c\tau+d}\right)=\chi(d)(c\tau+d)^k f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N)$.

The space of modular forms

- The modular forms for a given congruence subgroup form a vectorspace.
- This vectorspace is finite dimensional.
- It decomposes into a subspace of cusp forms and the Eisenstein subspace.
- We have

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(N,\chi)$$

  and similar for the subspace of cusp forms and the Eisenstein subspace.

- Basis of Eisenstein subspace $\mathcal{E}_k(N,\chi)$ given in terms of generalised Eisenstein series.

Iterated integrals of modular forms

Iterated integrals of modular forms:

$$I(f_1, f_2, ..., f_n; q) = (2\pi i)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1(\tau_1) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2(\tau_2) \dots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n(\tau_n)$$

Notation:

$$I\left(\{f\}^k; q\right) = I\left(\underbrace{f, f, ..., f}_{k}; q\right)$$

An integral over a modular form is in general not a modular form.
Analogy: An integral over a rational function is in general not a rational function.

The all-order in ε result (iterated integrals)

$$
\begin{aligned}
S \quad = \quad & \frac{\psi_1}{\pi} e^{-\varepsilon I(f_2;q)+2\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta_n\varepsilon^n} \\
& \left\{ \left[\sum_{j=0}^{\infty} \left(\varepsilon^{2j} I\left(\{1,f_4\}^j ; q \right) - \frac{1}{2}\varepsilon^{2j+1} I\left(\{1,f_4\}^j , 1 ; q \right) \right) \right] \sum_{k=0}^{\infty} \varepsilon^k B^{(k)} \right. \\
& \left. + \sum_{j=0}^{\infty} \varepsilon^j \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} I\left(\{1,f_4\}^k , 1, f_3, \{f_2\}^{j-2k} ; q \right) \right\}
\end{aligned}
$$

Uniform weight: At order ε^j one has exactly $(j+2)$ integrations.

Alphabet given by modular forms 1, f_2, f_3, f_4.

Adams, S.W., '17

The letters

Example: The modular form f_3 is given by

$$
\begin{aligned}
f_3 &= -\frac{1}{24}\left(\frac{\psi_1}{\pi}\right)^3 \frac{t\left(t-m^2\right)\left(t-9m^2\right)}{m^6} \\
&= \frac{3}{i}\left[\mathrm{ELi}_{0;-2}\left(r_3;-1;-q\right) - \mathrm{ELi}_{0;-2}\left(r_3^{-1};-1;-q\right)\right] \\
&= 3\sqrt{3}\frac{\eta\left(2\tau_2\right)^{11}\eta\left(6\tau_2\right)^{7}}{\eta\left(\tau_2\right)^{5}\eta\left(4\tau_2\right)^{5}\eta\left(3\tau_2\right)\eta\left(12\tau_2\right)} \\
&= 3\sqrt{3}\left[E_3\left(\tau_2;\chi_1,\chi_0\right) + 2E_3\left(2\tau_2;\chi_1,\chi_0\right) - 8E_3\left(4\tau_2;\chi_1,\chi_0\right)\right]
\end{aligned}
$$

with $\tau_2 = \tau/2$, $r_3 = \exp(2\pi i/3)$, Dedekind's eta function η, Dirichlet characters $\chi_0 = (\frac{1}{n})$, $\chi_1 = (\frac{-3}{n})$ and Eisenstein series E_3.

The ε-form of the differential equation for the sunrise/kite

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by the (non-algebraic) change of variables from t to τ and by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form for the kite/sunrise family:

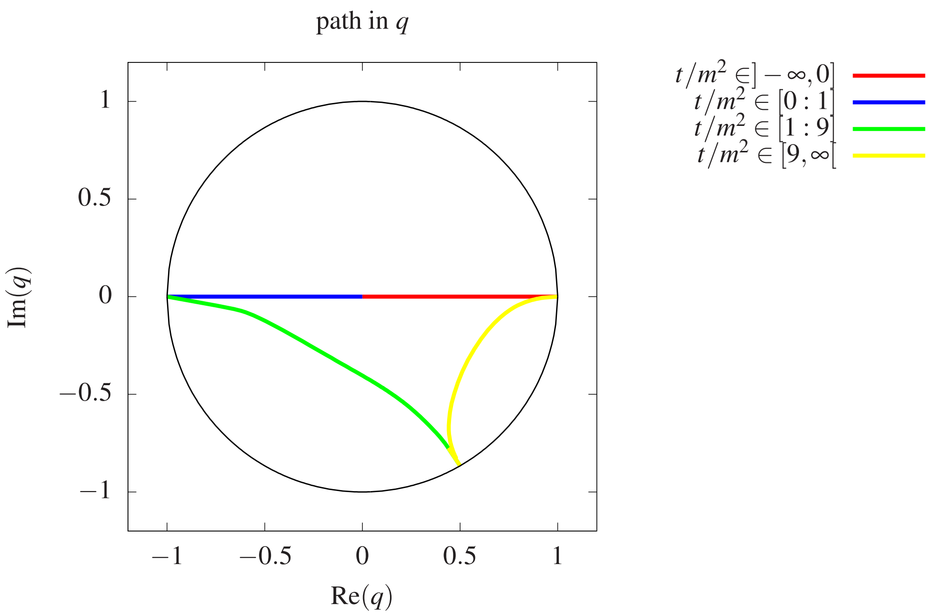
$$\frac{d}{d\tau}\vec{I} \quad = \quad \varepsilon\, A(\tau)\, \vec{I},$$

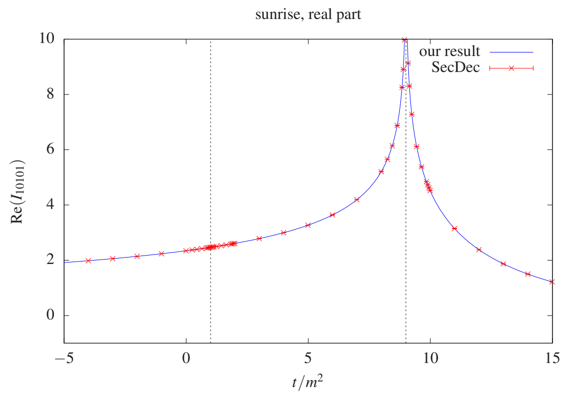
where $A(\tau)$ is an ε-independent 8×8-matrix whose entries are modular forms.

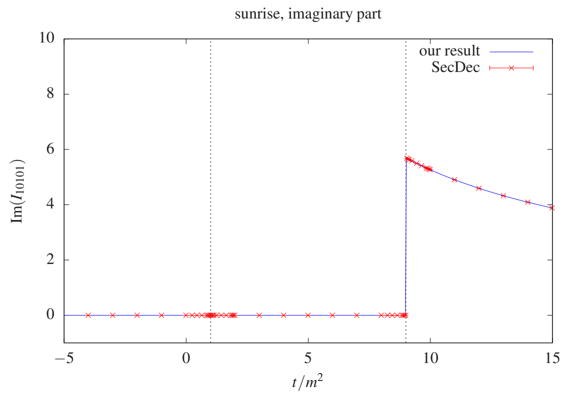
Analytic continuation and numerical evaluations of the kite and sunrise integral

Complete elliptic integrals efficiently computed from arithmetic-geometric mean.

q-series converges for all $t \in \mathbb{R} \backslash \{m^2, 9m^2, \infty\}$.







No need to distinguish the cases $t < 0$, $0 < t < m^2$, $m^2 < t < 9m^2$, $9m^2 < t$!

Bogner, Schweitzer, S.W., '17

Summary numerical evaluation

Given t and m, compute the periods ψ_1 and ψ_2 through arithmetic-geometric mean.

Set $\tau = \frac{\psi_2}{\psi_1}$, $q = e^{i\pi\tau}$.

Evaluate the truncated series

$$
\begin{aligned}
S \quad = \quad & 3\sqrt{3}\frac{\psi_1}{\pi}\left\{\frac{1}{\sqrt{3}}\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) + q + \frac{5}{4}q^2 + q^3 + \frac{11}{16}q^4 + \frac{24}{25}q^5 + \frac{5}{4}q^6 + \frac{50}{49}q^7 + \frac{53}{64}q^8 \right. \\
& + q^9 + \frac{6}{5}q^{10} + \frac{120}{121}q^{11} + \frac{11}{16}q^{12} + \frac{170}{169}q^{13} + \frac{125}{98}q^{14} + \frac{24}{25}q^{15} + \frac{203}{256}q^{16} + \frac{288}{289}q^{17} \\
& \left. + \frac{5}{4}q^{18} + \frac{362}{361}q^{19} \right\} + O\left(q^{20}\right).
\end{aligned}
$$

Conclusions

- Differential equations are a powerful tool to compute Feynman integrals.
- If a system can be transformed to an ε-form, a solution in terms of multiple polylogarithm is easily obtained.
- There are system, where within rational transformations at order ε^0 two coupled equations remain.

  Kite/sunrise family:

  - Sum representation in terms of ELi-functions.
  - Iterated integral representation involving modular forms
  - Analytic continuation / numerical evaluation easy.