# Prospects for the Numerical Unitarity Approach @ Multi Loops

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# Scope

Questions:

Prediction for 3-loop EW/QCD Z-boson observables?

Predictions from 2-loop, 3-loop unitarity methods for multi-loop/scale effects beyond NNLO?

Available amplitudes from unitarity methods:

1-loop QCD/EW

N=4 SYM multi-loop

2-loop QCD

=> Indicators & extrapolate

# Indicator: 1-loop QCD

Multi-leg amplitudes as cuts of multi-loop amplitudes:

- However: simpler integrals, less inclusive => full NLO predictions better indicator

Multi-leg NLO predictions:

- 7-point: e.g. 5jets, tt3jets [NJet, OpenLoops, Recola] — cut of 3-point 3-loop
- 8-point: e.g. V5jets, Wbb3jets [BlackHat; Goetz, Reuschle, Schwan, Weinzierl] — cut of 4-point 3-loop

Explicit helicity amplitudes: 5-point: V*+4-partons [Bern, Dixon, Kosower], partial results 6-partons [collected by Dunbar]

=> generic 3/4-point 3-loop amplitudes achievable

# Indicator: N=4 SYM

N=4 SYM is toy model for QCD-like theories:

- Conformal symmetries [Drummond, Henn, Korchemsky, Sokatchev], color-kinematic duality [Bern, Carrasco, Johansson]
- Simpler/absent integration-by-parts reduction

Example amplitudes:

- 4 gluon 5-loop non-planar integrand [Bern, Carrasco, Dixon, Johansson, Roiban]
- 4 gluon 1/2/3-loop non-planar [Bern, Rozowsky, Yan; Henn, Mistlberger]

=> 4-point multi-loop amplitudes achievable

=> Many ideas to exploit for QCD/EW theory

# Indicator: 2-loop unitarity

Status unitarity methods @ 2-loop QCD:

- analytic: 4-partons [Bern, Dixon, Kosower]

  all-plus helicity 5/6/7-gluons [contributions by groups: Badger, Frellesvig, Mogull, Peraro; Gehrmann, Henn, Lo Presti; Dunbar, Perkins] — N=4 SYM-like

- floating-point: 4-gluons planar [Abreu et al.]

- exact arithmetics: 4/5-gluons planar [Badger et al., Abreu et al.; integrals from Frellesvig, Papadopoulos, Wever]

=> first steps and much more possible, indicators see below

# Unitarity variants

Master equation:

$$A(p_i) = \int [d^{(nD)}\ell]\ \tilde{A}(\ell, p_i) = \sum_{\text{integral basis}} c_j(p_i) \int [d^{(nD)}\ell]\ \frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N}$$

Cut equations:

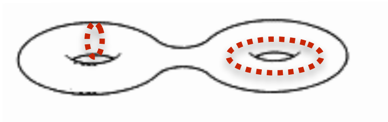
$$\int_{\mathcal{C}} [\text{dLIPS}]\ \tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i)$$

$$= \sum_{\text{integrals with cuts}} c_j(p_i) \int_{\mathcal{C}} [\text{dLIPS}]\ \frac{m_j(\ell, p_i)}{(\text{uncut propagator terms})}$$

Remarks:

- Tree-diagram input
- Passarino-Veltman reduction from cutting
- Integral reduction from integration (challenging)
- Analytic work to disentangle hierarchy of cuts

Ongoing amplitudes field [Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; Veneziano; Virasoro, Shapiro; …Bern, Dixon, Dunbar, Kosover; Arkani-Hammed, Cachazo, … amplitudes community]

$\mathcal{C}$ : 

Duality of master integrals & contours [Kosower, Larsen 11; Caron-Huot, Larsen 12; Georgoudis, Zhang 15; Sogaard, Zhang 14; HI 15; Harley, Moriello, Schabinger; Bosma, Sogaard, Zhang; Primo, Tancredi 17]

# Unitarity — integrands

Master equation for integrand basis:

$$\tilde{A}(\ell, p_i) = \sum_{\text{j in integrand basis}} c_j(p_i) \frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N}$$

Classification of integrands [Badger, Frellesvig, Zhang; Mastrolia, Mirabella, Ossola, Peraro]

Algebraic generalised cut equations:

$$\tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i) = \sum_{\text{j in large integrands}} c_j(p_i)\, m_j(\ell, p_i) + \text{previously computed topologies}$$

Remarks:

- Tree diagram input

- Passarino-Veltman reduction from cutting

- Integral reduction postponed

- Fitting & solving equations

# Unitarity — surface terms

Integrand ansatz as masters and surface terms:

$$\tilde{A}(\ell, p_i) = \sum_{\text{j in master integrands}} c_j(p_i)\,\frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N} + \sum_{\text{j in surface terms}} \hat{c}_j(p_i)\,\frac{\hat{m}_j(\ell, \_\ )}{\rho^1 \cdots \rho^N}$$

@ one-loop [Ossola Papadopoulos, Pittau 07; Ellis, Giele Kunszt 07; Giele Kunszt, Melnikov 08]

@ two/multi-loop [HI15]

Algebraic cut equations:

$$\tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i) = \sum_{\text{j in large integrands}} c_j(p_i)\, m_j(\ell, p_i) + \text{previously computed topologies}$$

Remarks:

-Tree-diagram input

- Passarino-Veltman reduction from cutting

- Fitting and solving for coefficients

- IBP reduction from equation solving

# Surface Terms

Surface terms as integration-by-parts (IBP) identities.

$$\int \prod_{l=1,L} d^D \ell_l \sum_k \frac{\partial}{\partial \ell_k^\nu} \left[ \frac{u_k^\nu}{\prod_{j \in P_\Gamma} \rho_j} \right] = 0$$

- No need to do Laporta inversion but stopping IBP reduction early.

A) Control propagator powers with special vector fields [Gluza, Kajda, Kosower].

$$\partial_\mu \left( \frac{u^\mu}{\rho^i} \right) = \frac{1}{\rho^i} \partial_\mu u^\mu - \frac{1}{(\rho^i)^2} u^\mu \partial_\mu \rho^i$$

$$\rightarrow \quad u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j$$

- Methods in algebraic geometry (see below)

B) Determine complete and independent set of IBP relations:

- On-shell, numerical for simplification, natural to change power counting [Ita; Zhang, Larsen; Abreu et al.]

# Syzygy-module equations

Formulations of equations:

- Syzygy-module equations [Gluza, Kajda, Kosower; Ita; Abreu et al.]

- Syzygy equations from Baikov polynomial [Zhang, Larsen]

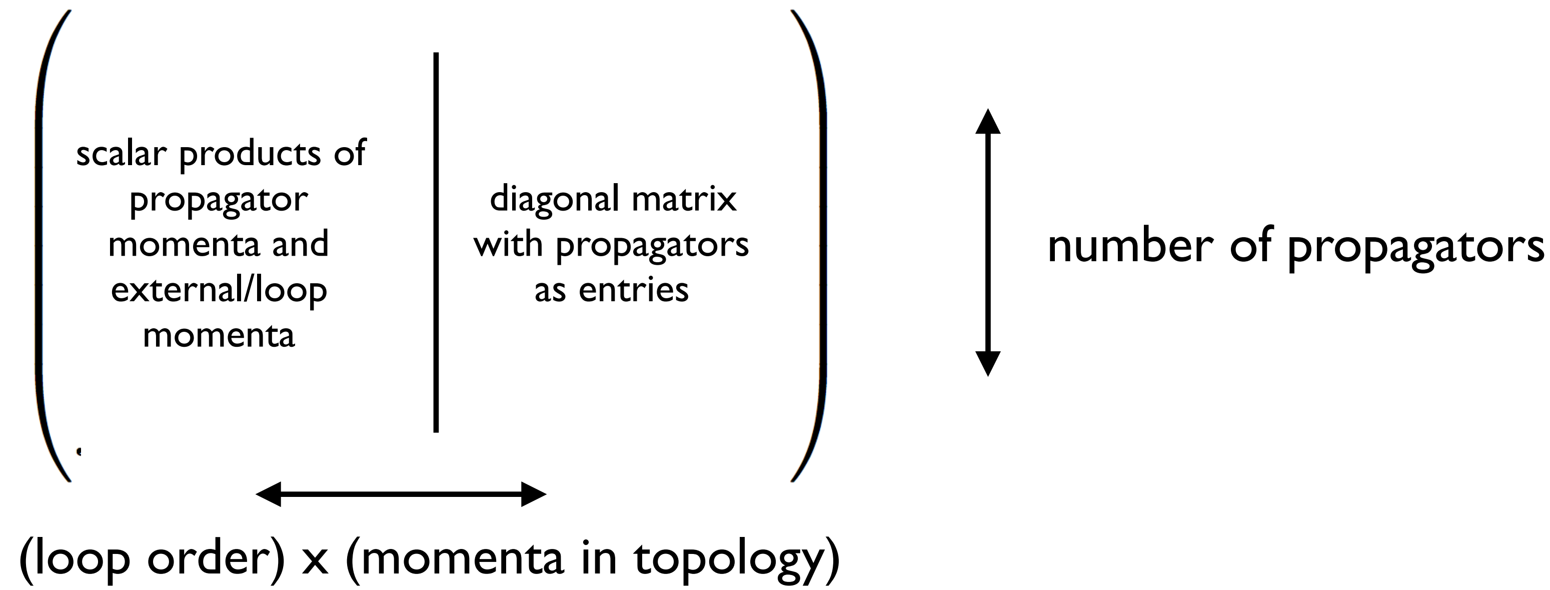
Origin of syzygy module equations (‘compute polynomial kernel of polynomial matrix’):

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j$$

$$u_k^\nu \frac{\partial}{\partial \ell_k^\nu} = \left( u_{ka}^{\text{loop}} \ell_a^\nu + u_{kc}^{\text{ext}} p_c^\nu \right) \frac{\partial}{\partial \ell_k^\nu}$$

$$\left( u_{ka}^{\text{loop}} \ell_a^\nu + u_{kc}^{\text{ext}} p_c^\nu \right) \frac{\partial}{\partial \ell_k^\nu} \begin{pmatrix} \rho_{j(1)} \\ \rho_{j(2)} \\ \vdots \\ \rho_{j(|\Gamma|)} \end{pmatrix} - \begin{pmatrix} f_{j(1)} \rho_{j(1)} \\ f_{j(2)} \rho_{j(2)} \\ \vdots \\ f_{j(|\Gamma|)} \rho_{j(|\Gamma|)} \end{pmatrix} = 0$$

Form of matrix:




Vectors from kernel of polynomial matrix:

- Finite integrals complete solution (Cramer's rule)
- Singular-program (Groebner basis methods; slimgb)

Expectations:

- Indicator: degrees of freedom, rank of matrix
- Planar vs. non-planar
- Currently tested: all planar and non-planar 2-loop integrals for 5-partons ~ 15 vars

=> 4-parton 3-loops similar (ladder diagram has similar count of variables)

# Fitting procedure

1) Sampling to generate equations:

Evaluation of trees / integrand ansatz / subtraction

2) Inversion of linear system in coefficients

3) Fitting of spin dimension Ds

4) Fitting of D dependence

$$\tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i) = \sum_{\text{j in large integrands}} c_j(p_i)\, m_j(\ell, p_i) + \text{previously computed topologies}$$

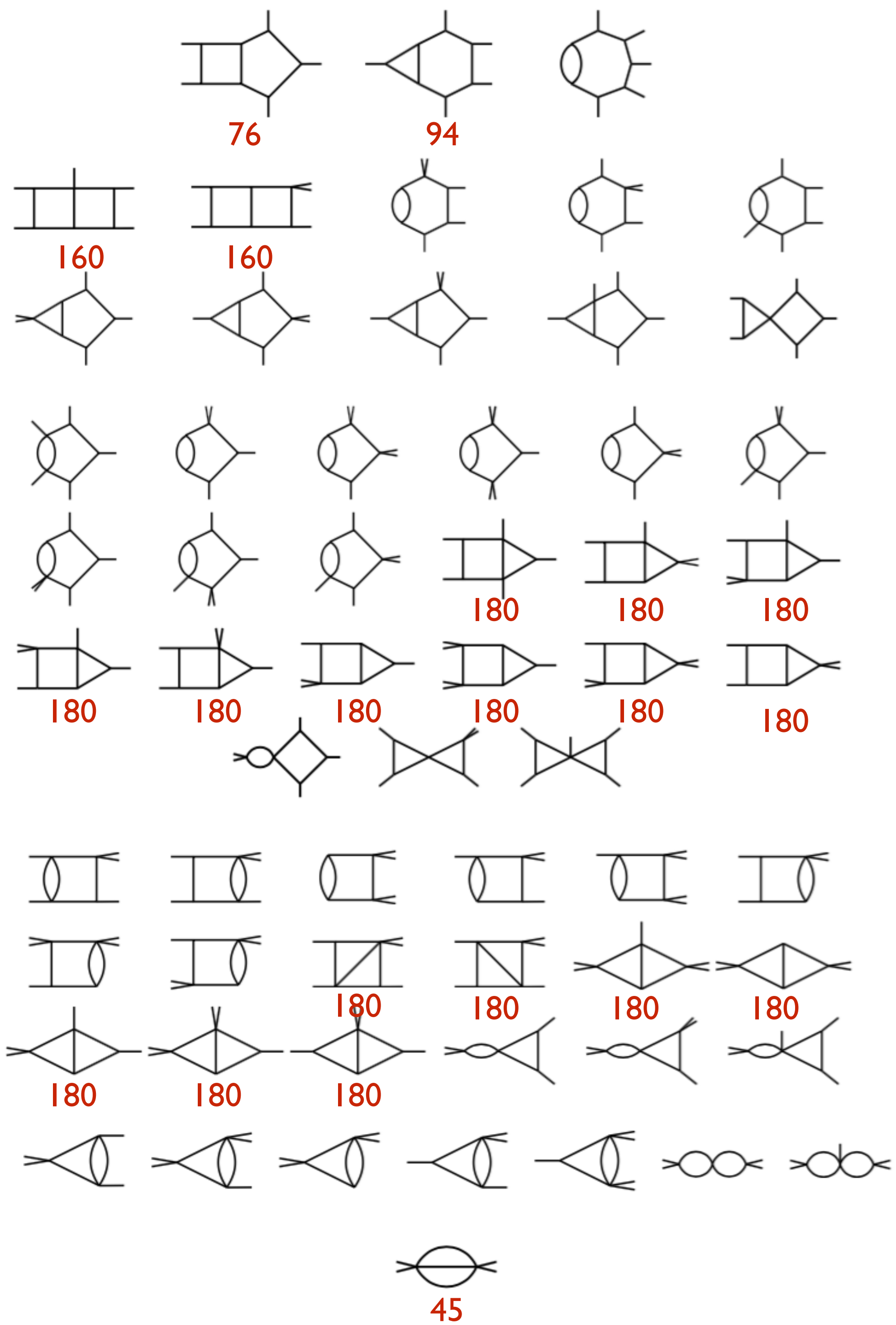
Limiting factors:

Number of numerators & diagrams

Depth of hierarchy => precision loss

# Complexity of fitting

5-gluon hierarchy:




Scaling in loops:

- Number of independent tensor grow by order of magnitude with loop order (estimated from factored topologies).

Planar 4-point 2-loop total timing:

- core-seconds/PS point/helicity

- core-minutes to reconstruct analytic helicity amplitude

=> 4-point 3-loops achievable

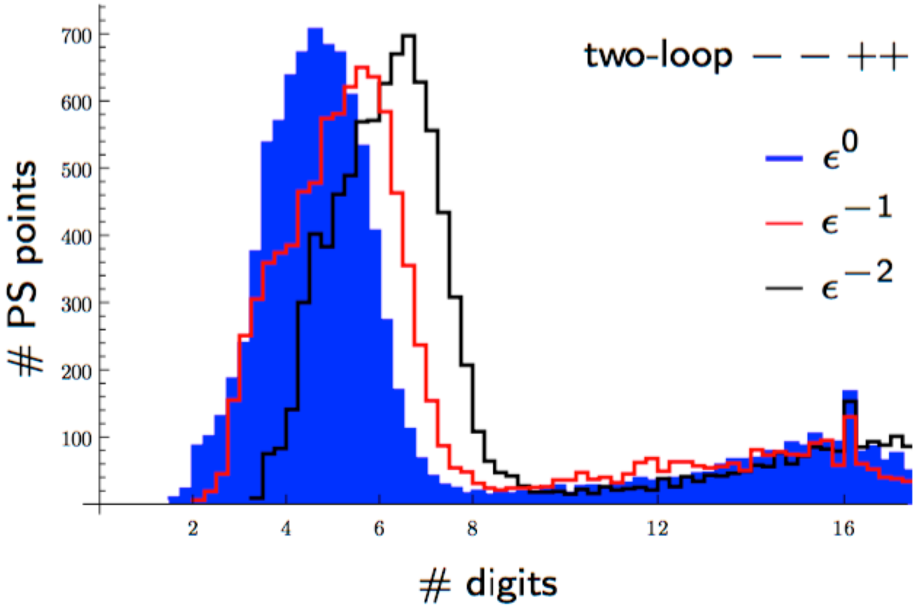
# Numerical strategies

Floating point numerics [one-loop generators; Abreu et al.]:

+ process independence

- stability => improved integral basis for cancellations

Correct digits floating vs analytic
@ planar, 4-gluon, 2-loop




Exact arithmetic [Manteuffel, Schabinger; Peraro; recently: Badger et al.; Abreu et al.]:

+ works already up to two scales: @ 4-gluon 2-loop: minutes/planar helicity amplitude

- many variables {D,Ds, s_ij} => (expected power)^vars evaluations/helicity config.

# Conclusions

Discussed rough estimates from unitarity approach:

- Non-standard in unitarity indicators
- New approach for surface terms / integration-by-parts
- Numerical approach and analytic reconstruction

Future:

- Numerical approach suggests room for adding scales and that 3-loop 4-parton amplitudes are in reach now.

Discussion:

- Bite-size wishlist
- Help from established reduction programs (stop early)
- Numerical evaluation of pure/finite integrals