Mellin-Barnes meets Method of Brackets

Mario Prausa

Institute of Physics
University of Freiburg

January 13, 2018
CERN

based on arXiv:1706.09852

Mellin-Barnes representation

- evaluation by Mellin-Barnes (MB) representation is a powerful tool for the calculation of Feynman integrals
- after MB representation is found, various packages exist for its computation:
  - `MB` [Czakon (2005)]
  - `MBresolve` [Smirnov,Smirnov (2009)]
  - `MBasymptotic` [Czakon (2006)]
  - `MBsums` [Ochman,Riemann (2015)]
  - `MBnumerics` [Usovitsch] yesterday's talk
  - ...

Mellin-Barnes representation

- MB representation = multi-dimensional contour integral of the form

$$\int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_N}{2\pi \mathrm{i}} \, {x_1}^{A_1(z_1,\cdots,z_N,d)} {x_2}^{A_2(z_1,\cdots,z_N,d)} \cdots \frac{\Gamma(B_1(z_1,\cdots,z_N,d))\Gamma(B_2(z_1,\cdots,z_N,d))\cdots}{\Gamma(C_1(z_1,\cdots,z_N,d))\Gamma(C_2(z_1,\cdots,z_N,d))\cdots}$$

  - red: kinematic variables,
  - blue: linear combinations,
  - $d = 4 - 2\epsilon$: number of space-time dimensions.
- the contour of the z_i-integral is chosen so that it seperates left poles $(\Gamma(\cdots + z_i))$ from right poles $(\Gamma(\cdots - z_i))$ of the integrand.

Mellin-Barnes representation

constructing MB representations:

- *simple cases*: rewrite massive propagators into massless ones using

$$\frac{1}{(A+B)^a} = \frac{1}{\Gamma(a)} \int \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{B^z}{A^{a+z}} \Gamma(a+z)\Gamma(-z)\,.$$

If the resulting Feynman integral can be solve in terms of Γ-functions, we are done.

Mellin-Barnes representation

constructing MB representations:

- *complicated Feynman integrals*:
  - loop-by-loop approach (`AMBRE v2`) [Gluza,Kajda,Riemann (2010)]
  - global approach (`AMBRE v3`) [Dubovyk,Gluza,Kajda,Riemann (2014)]

  (based on Feynman parameterization)

yesterday's talk

Mellin-Barnes representation

In this talk: a new approach for the construction of MB representations.

why bother?

- a representation with a rather low dimensionality is preferrable for the evaluation.
- different methods lead to different MB representations with a different dimensionality.
- for planar integrals the loop-by-loop approach provides very good representations.
- for non-planar integrals the global approach is applicable but leads sometimes to many MB dimensions.

Outline

The (original) Method of Brackets

- Method of Brackets [González,Schmidt (2007)] = a technique to express Feynman integrals directly in terms of *multi-fold sums*.
- based on Ramanujan's master theorem:

$$g(x) = \sum_{n=0}^{\infty} G(n) \frac{(-x)^n}{n!} \Rightarrow \int_0^{\infty} \mathrm{d}x\, x^{\alpha-1} g(x) = \Gamma(\alpha) G(-\alpha)\,.$$

- "The Bracket":

$$\langle \alpha \rangle \equiv \int_0^{\infty} \mathrm{d}x\, x^{\alpha-1}$$

- Property:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} G(n) \langle n+\alpha \rangle = \Gamma(\alpha) G(-\alpha)\,.$$

The (original) Method of Brackets

- generalization to multi-fold sums:

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_K=0}^{\infty}\frac{(-1)^{n_1+\cdots+n_K}}{n_1!\cdots n_K!}G(\vec{n})\,\langle\beta_1+\vec{\alpha}_1\cdot\vec{n}\rangle\cdots\langle\beta_K+\vec{\alpha}_K\cdot\vec{n}\rangle$$

$$=\frac{1}{|\det(A)|}\Gamma(-n_1^*)\cdots\Gamma(-n_K^*)G(\vec{n}^{\,*}),$$

$$A=\begin{pmatrix}\alpha_{11} & \cdots & \alpha_{1K}\\ \vdots & \ddots & \vdots\\ \alpha_{K1} & \cdots & \alpha_{KK}\end{pmatrix},\quad \vec{n}=\begin{pmatrix}n_1\\ \vdots\\ n_K\end{pmatrix},\quad \vec{n}^{\,*}=-A^{-1}\vec{\beta}$$

(master formula)

Modifications to obtain MB representations

- similar to Ramanujan's master theorem, we have the Mellin inversion theorem

$$f(x) = \int \frac{\mathrm{d}z}{2\pi \mathrm{i}} x^z F(z) \Rightarrow \int_0^\infty \mathrm{d}x\, x^{\alpha-1} f(x) = F(-\alpha)\,.$$

- for multi-dimensional MB integrals (with the bracket notation):

$$\int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_K}{2\pi \mathrm{i}} F(\vec{z})\, \langle \beta_1 + \vec{\alpha}_1 \cdot \vec{z} \rangle \cdots \langle \beta_K + \vec{\alpha}_K \cdot \vec{z} \rangle = \frac{1}{|\det A|} F(-A^{-1}\vec{\beta})$$

(our master formula)

Method of Brackets for MB

- starting point: the Schwinger parameterization

$$\frac{1}{\Gamma(a_1)\cdots\Gamma(a_N)}\int_0^\infty \mathrm{d}x_1\; x_1^{a_1-1}\cdots\int_0^\infty \mathrm{d}x_N\; x_N^{a_N-1}\;\frac{e^{-F/U-\sum_i x_i m_i^2}}{U^{d/2}},$$

  U and F: Symanzik polynomials.
- idea: expand the Schwinger parameterized integral in terms of MB integrals and brackets so that the master formula can be applied
- only 4 simple rules required

Method of Brackets for MB - Rule A

- split exponential function into factors

$$e^{-\sum_i A_i} = \prod_i e^{-A_i}$$

  so that A_i consists only of a monomial or a monomial divided by U.
- apply Cahen-Mellin formula to the factors

$$e^{-A_i} = \int \frac{\mathrm{d}z_i}{2\pi\mathrm{i}} A_i^{z_i} \Gamma(-z_i)$$

- arrange the powers so that every base appears only a single time, e.g.

$$U^{-d/2} \left(\frac{x_1 x_3}{U}\right)^{z_1} \left(\frac{x_1 x_4}{U}\right)^{z_2} = U^{-d/2-z_1-z_2} x_1^{z_1+z_2} x_3^{z_1} x_4^{z_2}$$

Method of Brackets for MB - Rule B

- powers of multinomials are treated by

$$(A_1 + \cdots + A_J)^\alpha = \frac{1}{\Gamma(-\alpha)} \int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_J}{2\pi \mathrm{i}} \langle z_1 + \cdots + z_J - \alpha \rangle \times A_1^{z_1} \cdots A_J^{z_J} \, \Gamma(-z_1) \cdots \Gamma(-z_J)$$

- arrange the powers so that every base appears only a single time

Method of Brackets for MB - Rule C

- Schwinger parameter integrals are rewritten as brackets

$$\int_0^\infty \mathrm{d}x_i \; x_i^{L(a_1,\cdots;z_2,\cdots)-1} = \langle L(a_1,\cdots;z_1,\cdots)\rangle \,.$$

Method of Brackets for MB - Rule D

- after rules A, B and C: "presolution" of the form

$$\int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_J}{2\pi \mathrm{i}} \langle \beta_1 + \vec{\alpha}_1 \cdot \vec{z} \rangle \cdots \langle \beta_K + \vec{\alpha}_K \cdot \vec{z} \rangle \, f(\vec{z}) \,,$$

- K out of the J MB integrals can be solved using the brackets (i.e. Mellin inversion theorem) $\Rightarrow \binom{J}{K}$ possibilities
- w.l.o.g.
  z_i to be solved $\quad \vec{z}_1 = (z_1, \cdots, z_K)^T$
  z_i to remain $\quad \vec{z}_2 = (z_{K+1}, \cdots, z_J)^T$

MB representation:

$$\int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_J}{2\pi \mathrm{i}} \langle \beta_1 + \vec{\alpha}_1 \cdot \vec{z}_1 + \vec{\gamma}_1 \cdot \vec{z}_2 \rangle \cdots \langle \beta_K + \vec{\alpha}_K \cdot \vec{z}_1 + \vec{\gamma}_K \cdot \vec{z}_2 \rangle f(\vec{z}_1, \vec{z}_2)$$
$$= \frac{1}{|\det A|} \int \frac{\mathrm{d}z_{K+1}}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_J}{2\pi \mathrm{i}} f(-A^{-1}\vec{\beta} - A^{-1} C \, \vec{z}_2, \vec{z}_2)$$

Optimization

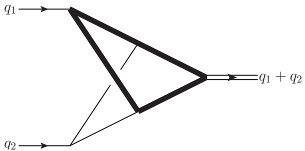
- naïve application of rules

$$\Rightarrow \dim = \#(\text{terms in } U) + \#(\text{terms in } F) + \#(\text{massive lines}) - \#\text{propagators} - 1$$

Huge!

- optimization:
  - identify recursively *common subexpressions* in U, F and $\sum_i m_i^2 x_i$
  - replace common subexpressions by new variables
  - treat new variables like Schwinger parameters in Rule A and B
  - after Rule B, reinsert common subexpressions one-by-one and apply Rule B again
- if a polynomial with N terms appears J times
  $\Rightarrow$ dimensionality is reduced by $(J-1)(N-1)$

Example



——— massless , **———** mass m

$$q_1^2 = q_2^2 = 0\,, \quad (q_1+q_2)^2 = M^2$$

$$\begin{aligned} U &= x_2x_5 + x_1x_2 + x_4x_5 + x_2x_4 + x_2x_6 + x_3x_6 + x_1x_5 \\ &\quad + x_1x_6 + x_3x_5 + x_1x_3 + x_4x_6 + x_3x_4\,, \\ F &= -M^2\left[x_2x_3x_5 + x_1x_3x_6 + x_1x_2x_3 + x_2x_4x_5 + x_2x_3x_4 + x_2x_3x_6\right]\,, \\ \textstyle\sum_i m_i^2 x_i &= m^2\left[x_1+x_2+x_3+x_4\right] \end{aligned}$$

- without optimization: 15 MB dimensions

Example - Optimization

- rearrange U, F and $\sum_i m_i^2 x_i$

$$U = (x_5 + x_6)((x_1 + x_4) + (x_2 + x_3)) + (x2 + x3)(x1 + x4)\,,$$
$$F = -M^2\left[(x_5 + x_6)x_2x_3 + x_1x_3x_6 + (x1 + x4)x_2x_3 + x_2x_4x_5\right]\,,$$
$$\sum_i m_i^2 x_i = m^2((x_1 + x_4) + (x_2 + x_3))$$

- introduce new variables:

$$r_1 = x_1 + x_4\,, \qquad r_2 = x_2 + x_3\,,$$
$$r_3 = r_1 + r_2\,, \qquad r_4 = x_5 + x_6\,,$$

$$U = r_4 r_3 + r_2 r_1\,,$$
$$F = -M^2\left[r_4 x_2x_3 + x_1x_3x_6 + r_1 x_2x_3 + x_2x_4x_5\right]\,,$$
$$\sum_i m_i^2 x_i = m^2 r_3$$

- $\Rightarrow$ 4 MB dimensions

Example - Rule A

- split e-function into factors:

$$e^{-F/U-\sum_i x_i m_i^2} = e^{-M^2 r_4 x_2 x_3/U} e^{-M^2 x_1 x_3 x_6/U} e^{-M^2 r_1 x_2 x_3/U} \times e^{-M^2 x_2 x_4 x_5/U} e^{-m^2 r_3}$$

- apply Cahen-Mellin formula:

$$e^{-F/U-\sum_i x_i m_i^2} = \int \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \cdots \int \frac{\mathrm{d}z_5}{2\pi\mathrm{i}} (M^2)^{z_{1234}} (m^2)^{z_5} \Gamma(-z_1)\cdots\Gamma(-z_5) \times {r_1}^{z_3} {r_3}^{z_5} {r_4}^{z_1} U^{-z_{1234}} x_1^{z_2} x_2^{z_{134}} x_3^{z_{123}} x_4^{z_4} x_5^{z_4} x_6^{z_2}$$

$$\int_0^\infty \mathrm{d}x_1\, x_1^{a_1-1} \cdots \int_0^\infty \mathrm{d}x_6\, x_6^{a_6-1} \int \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \cdots \int \frac{\mathrm{d}z_5}{2\pi\mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5} \times \frac{\Gamma(-z_1)\cdots\Gamma(-z_5)}{\Gamma(a_1)\cdots\Gamma(a_6)} \times {r_1}^{z_3} {r_3}^{z_5} {r_4}^{z_1} U^{-z_{1234}-d/2} x_1^{z_2} x_2^{z_{134}} x_3^{z_{123}} x_4^{z_4} x_5^{z_4} x_6^{z_2}$$

Example - Rule B

- apply Rule B to power of U:

$$U^{-z_{1234}-d/2} = (r_4 r_3 + r_2 r_1)^{-z_{1234}-d/2}$$
$$= \frac{1}{\Gamma(z_{1234}+d/2)} \int \frac{\mathrm{d}z_6}{2\pi \mathrm{i}} \int \frac{\mathrm{d}z_7}{2\pi \mathrm{i}} \langle z_{123467} + d/2 \rangle$$
$$\times \Gamma(-z_6)\Gamma(-z_7) {r_1}^{z_7} {r_2}^{z_7} {r_3}^{z_6} {r_4}^{z_6}$$

$$\int_0^\infty \mathrm{d}x_1\, x_1^{a_1-1} \cdots \int_0^\infty \mathrm{d}x_6\, x_6^{a_6-1} \int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_7}{2\pi \mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5}$$
$$\times \langle z_{123467} + d/2 \rangle$$
$$\times \frac{\Gamma(-z_1) \cdots \Gamma(-z_7)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2)}$$
$$\times {r_1}^{z_{37}} {r_2}^{z_7} {r_3}^{z_{56}} {r_4}^{z_{16}} x_1^{z_2} x_2^{z_{134}} x_3^{z_{123}} x_4^{z_4} x_5^{z_4} x_6^{z_2}$$

Example - Rule B

- apply Rule B to power of r_4:

$$
\begin{aligned}
{r_4}^{z_{16}} &= (x_5 + x_6)^{z_{16}} \\
&= \frac{1}{\Gamma(-z_{16})} \int \frac{\mathrm{d}z_8}{2\pi \mathrm{i}} \int \frac{\mathrm{d}z_9}{2\pi \mathrm{i}} \langle z_{89} - z_{16} \rangle \\
&\quad \times \Gamma(-z_8)\Gamma(-z_9) x_5^{z_8} x_6^{z_9}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}x_1 \; x_1^{a_1 - 1} \cdots \int_0^\infty \mathrm{d}x_6 \; x_6^{a_6 - 1} \int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_9}{2\pi \mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5} \\
&\quad \times \langle z_{123467} + d/2 \rangle \langle z_{89} - z_{16} \rangle \\
&\quad \times \frac{\Gamma(-z_1) \cdots \Gamma(-z_9)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2) \Gamma(-z_{16})} \\
&\quad \times {r_1}^{z_{37}} {r_2}^{z_7} {r_3}^{z_{56}} x_1^{z_2} x_2^{z_{134}} x_3^{z_{123}} x_4^{z_4} x_5^{z_{48}} x_6^{z_{29}}
\end{aligned}
$$

Example - Rule B

- apply Rule B to power of r_3:

$$
\begin{aligned}
{r_3}^{z_{56}} &= (r_1 + r_2)^{z_{56}} \\
&= \frac{1}{\Gamma(-z_{56})} \int \frac{\mathrm{d}z_a}{2\pi\mathrm{i}} \int \frac{\mathrm{d}z_b}{2\pi\mathrm{i}} \langle z_{ab} - z_{56} \rangle \\
&\quad \times \Gamma(-z_a)\Gamma(-z_b) {r_1}^{z_a} {r_2}^{z_b}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}x_1\, x_1^{a_1 - 1} \cdots \int_0^\infty \mathrm{d}x_6\, x_6^{a_6 - 1} \int \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \cdots \int \frac{\mathrm{d}z_b}{2\pi\mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5} \\
&\quad \times \langle z_{123467} + d/2 \rangle \langle z_{89} - z_{16} \rangle \langle z_{ab} - z_{56} \rangle \\
&\quad \times \frac{\Gamma(-z_1) \cdots \Gamma(-z_b)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2) \Gamma(-z_{16}) \Gamma(-z_{56})} \\
&\quad \times {r_1}^{z_{37a}} {r_2}^{z_{7b}} x_1^{z_2} x_2^{z_{134}} x_3^{z_{123}} x_4^{z_4} x_5^{z_{48}} x_6^{z_{29}}
\end{aligned}
$$

Example - Rule B

- apply Rule B to power of r_2:

$$
\begin{aligned}
r_2{}^{z_{7b}} &= (x_2 + x_3)^{z_{7b}} \\
&= \frac{1}{\Gamma(-z_{7b})} \int \frac{\mathrm{d}z_c}{2\pi \mathrm{i}} \int \frac{\mathrm{d}z_d}{2\pi \mathrm{i}} \langle z_{cd} - z_{7b} \rangle \\
&\quad \times \Gamma(-z_c)\Gamma(-z_d) x_2^{z_c} x_3^{z_d}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}x_1\, x_1^{a_1-1} \cdots \int_0^\infty \mathrm{d}x_6\, x_6^{a_6-1} \int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_d}{2\pi \mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5} \\
&\quad \times \langle z_{123467} + d/2 \rangle \langle z_{89} - z_{16} \rangle \langle z_{ab} - z_{56} \rangle \langle z_{cd} - z_{7b} \rangle \\
&\quad \times \frac{\Gamma(-z_1) \cdots \Gamma(-z_d)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2) \Gamma(-z_{16}) \Gamma(-z_{56}) \Gamma(-z_{7b})} \\
&\quad \times r_1{}^{z_{37a}} x_1^{z_2} x_2^{z_{134c}} x_3^{z_{123d}} x_4^{z_4} x_5^{z_{48}} x_6^{z_{29}}
\end{aligned}
$$

Example - Rule B

- apply Rule B to power of r_1:

$$
\begin{aligned}
{r_1}^{z_{37a}} &= (x_1 + x_4)^{z_{37a}} \\
&= \frac{1}{\Gamma(-z_{37a})} \int \frac{\mathrm{d}z_e}{2\pi \mathrm{i}} \int \frac{\mathrm{d}z_f}{2\pi \mathrm{i}} \langle z_{ef} - z_{37a} \rangle \\
&\quad \times \Gamma(-z_e)\Gamma(-z_f) x_1^{z_e} x_4^{z_f}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}x_1\, x_1^{a_1 - 1} \cdots \int_0^\infty \mathrm{d}x_6\, x_6^{a_6 - 1} \int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_f}{2\pi \mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5} \\
&\times \langle z_{123467} + d/2 \rangle \langle z_{89} - z_{16} \rangle \langle z_{ab} - z_{56} \rangle \langle z_{cd} - z_{7b} \rangle \langle z_{ef} - z_{37a} \rangle \\
&\times \frac{\Gamma(-z_1) \cdots \Gamma(-z_f)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2) \Gamma(-z_{16}) \Gamma(-z_{56}) \Gamma(-z_{7b}) \Gamma(-z_{37a})} \\
&\times x_1^{z_{2e}} x_2^{z_{134c}} x_3^{z_{123d}} x_4^{z_{4f}} x_5^{z_{48}} x_6^{z_{29}}
\end{aligned}
$$

Example - Rule C

- replace all Schwinger integrals with brackets, e.g.:

$$\int_0^\infty \mathrm{d}x_1 \; x_1^{a_1+z_{2e}-1} = \langle a_1 + z_{2e} \rangle$$

$$\int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_f}{2\pi \mathrm{i}} (-M^2)^{z_{1234}} (m^2)^{z_5} \\ \times \langle z_{123467} + d/2 \rangle \langle z_{89} - z_{16} \rangle \langle z_{ab} - z_{56} \rangle \langle z_{cd} - z_{7b} \rangle \langle z_{ef} - z_{37a} \rangle \\ \times \langle a_1 + z_{2e} \rangle \langle a_2 + z_{134c} \rangle \langle a_3 + z_{123d} \rangle \langle a_4 + z_{4f} \rangle \langle a_5 + z_{48} \rangle \langle a_6 + z_{29} \rangle \\ \times \frac{\Gamma(-z_1) \cdots \Gamma(-z_f)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2) \Gamma(-z_{16}) \Gamma(-z_{56}) \Gamma(-z_{7b}) \Gamma(-z_{37a})}$$

- 15 MB integrals, 11 brackets $\Rightarrow$ 4 MB integrals will remain

Example - Rule D

- $\binom{15}{11} = 1365$ possible choices of which integrations should remain
- "only" 497 lead to a non-singular matrix A
- we choose z_2, z_4, z_6, z_d to remain

Example - Rule D

write the "presolution" in the form

$$\int \frac{\mathrm{d}z_1}{2\pi \mathrm{i}} \cdots \int \frac{\mathrm{d}z_f}{2\pi \mathrm{i}} \langle \beta_1 + \vec{\alpha}_1 \cdot \vec{z}_1 + \vec{\gamma}_1 \cdot \vec{z}_2 \rangle \cdots \langle \beta_{11} + \vec{\alpha}_{11} \cdot \vec{z}_1 + \vec{\gamma}_{11} \cdot \vec{z}_2 \rangle f(\vec{z}_1, \vec{z}_2),$$

$$\vec{z}_1 = (z_1, z_3, z_5, z_7, z_8, z_9, z_a, z_b, z_c, z_e, z_f)^T, \quad \vec{z}_2 = (z_2, z_4, z_6, z_d)^T,$$

$$A = \begin{pmatrix} \alpha_1^T \\ \vdots \\ \alpha_{11}^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \gamma_1^T \\ \vdots \\ \gamma_{11}^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\vec{\beta} = (d/2, 0, 0, 0, 0, a_1, a_2, a_3, a_4, a_5, a_6)^T,$$

$$f(\vec{z}_1, \vec{z}_2) = \frac{(-M^2)^{z_{1234}} (m^2)^{z_5} \Gamma(-z_1) \cdots \Gamma(-z_f)}{\Gamma(a_1) \cdots \Gamma(a_6) \Gamma(z_{1234} + d/2) \Gamma(-z_{16}) \Gamma(-z_{56}) \Gamma(-z_{7b}) \Gamma(-z_{37a})}$$

Example - Rule D

MB representation is given by

$$\frac{1}{|\det A|}\int\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}\int\frac{\mathrm{d}z_4}{2\pi\mathrm{i}}\int\frac{\mathrm{d}z_6}{2\pi\mathrm{i}}\int\frac{\mathrm{d}z_d}{2\pi\mathrm{i}} f(-A^{-1}\vec{\beta}-A^{-1}C\,\vec{z}_2,\vec{z}_2)$$

$$= (m^2)^{-a_{12456}+d}(-M^2)^{-a_3}\int\frac{\mathrm{d}z_2}{2\pi\mathrm{i}}\int\frac{\mathrm{d}z_4}{2\pi\mathrm{i}}\int\frac{\mathrm{d}z_6}{2\pi\mathrm{i}}\int\frac{\mathrm{d}z_d}{2\pi\mathrm{i}}\left(-\frac{m^2}{M^2}\right)^{-z_4+z_d}$$

$$\times\frac{\Gamma(-z_2)\Gamma(-z_4)\Gamma(-z_6)\Gamma(-z_d)\Gamma(a_1+z_2)\Gamma(a_4+z_4)\Gamma(a_5+z_4)\Gamma(a_6+z_2)}{\Gamma(a_1)\cdots\Gamma(a_6)}$$

$$\times\frac{\Gamma(a_{1456}-d/2+z_{24})\Gamma(a_{12456}-d+z_4-z_d)\Gamma(a_3-a_{56}-z_{46}+z_d)}{\Gamma(a_{14}+z_{24})\Gamma(a_{56}+z_{24})}$$

$$\times\frac{\Gamma(a_2-a_3-z_{2d}+z_4)\Gamma(a_2-d/2-z_{26d})}{\Gamma(a_2-a_3-z_2+z_4-2z_d)\Gamma(-a_3+d/2+z_4-z_d)}$$

$$\times\frac{\Gamma(-a_3+d/2+z_{46}-z_d)\Gamma(a_{56}+z_{246})}{\Gamma(a_{12456}-d+z_4-z_{6d})}$$

Open Questions

- In some rare cases (e.g. some one-scale 4-loop integrals) Rule D fails (A is always singular), why?
- Is there a more efficient algorithm to identify *common subexpressions* than the one in the paper?
- Can we "tune" the optimization procedure in a way that Barnes' lemmas are applicable in the end?
- The method produces not one but many different MB representations. Are some of them in some sense better than others?

Summary

- a novel method for construction of Mellin-Barnes representations for Feynman integrals
  - reformulation of the "Method of Brackets"
  - based on the Cahen-Mellin formula and Mellin's inversion theorem
- crucial point: optimization procedure (i.e. find common subexpressions in the Symanzik polynomials)
- good results for many non-trivial Feynman integrals with not too many scales (even non-planar ones)