Steepest descent, homology, Lefschetz thimbles and numerical multidimensional MB integrals

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Mellin-Barnes integrals

- "Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895),
  "The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$\textit{mathematics} \longrightarrow \frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$

$$\textit{physics} \longrightarrow \frac{1}{(p^2-m^2)^{a}} = \frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z)\frac{(m^2)^z}{(p^2)^{a+z}}$$

It is recursive $\Longrightarrow$ multidimensional complex integrals.

$$\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int\limits_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-s}{M_Z^2}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s\ \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}$$

Overlaped integrals

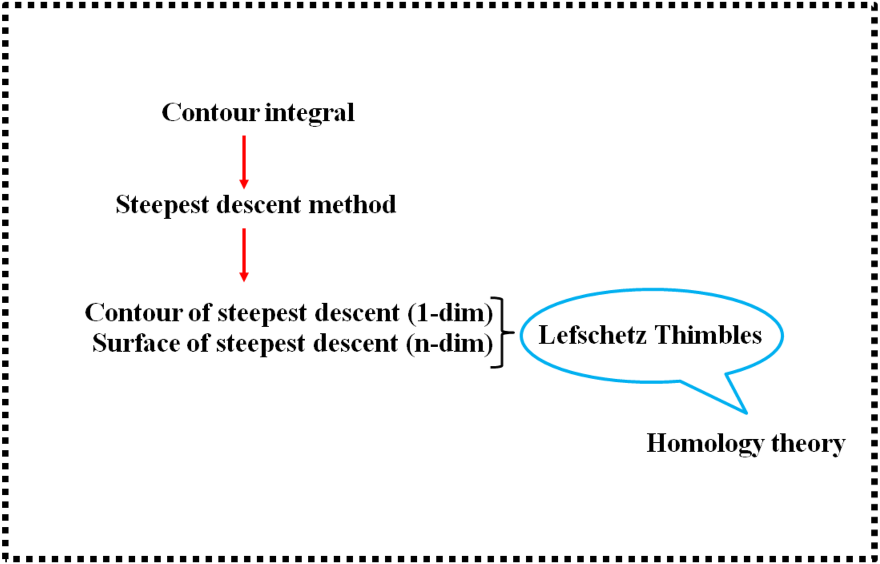
Goal

Find a contour of integration that makes calculation of highly oscillating MB integrals as easy as possible.

↓

Contour of steepest descent.

Overview


Contour integral
Steepest descent method
Contour of steepest descent (1-dim)
Surface of steepest descent (n-dim)
Lefschetz Thimbles
Homology theory

Steepest descent and Lefschetz Thimbles in physics

- Evolution of parton distributions in perturbative QCD [D. Kosower, 1997]
- Infinite-dimensional path integral in QFT [E. Witten, 2011]
- Sign problem in QCD and Condensed Matter [Cristoforetti, Di Renzo, Scorzato,..., Alexandru, Basar, Bedaque,..,Tanizaki, Nishimura, Verbaarschot...]
- Evaluation of Mellin-Barnes integrals [Gluza, Jeliński, Kosower, 2016, Sidorov, Lashkevich, Solovtsova, 2017]

Laplace's method

Laplace's Integral

$$I(k) = \int_a^b g(x)e^{kh(x)},$$

with $g(x), h(x)$ real continuous functions.

The major contribution to this integral comes from the neighborhoods of points where $h(x)$ attains its greatest value.

Steepest descent method

- General integral

$$F(k) = \int_{\mathcal{C}} dz g(z) e^{kh(z)}$$

$$g(z), h(z) - \text{analytic functions}$$

$$h(z) = Re(h(x+iy)) + iIm(h(x+iy)) \equiv R(x,y) + iI(x,y)$$

- Contour deformation to a steepest descent contour

$$\mathcal{C} \longrightarrow \mathcal{C}_*$$

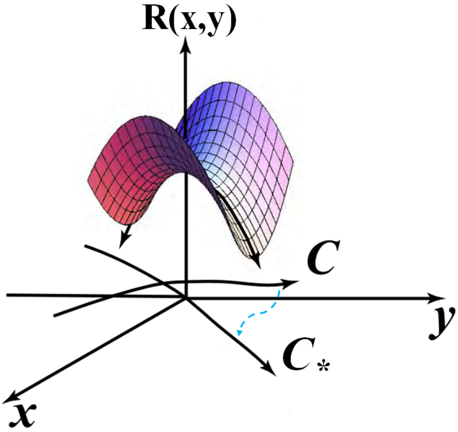
such that

$$I(x,y) = I(x_0, y_0) = const.$$

- Thus $F(k)$ becomes Laplace's Integral along $\mathcal{C}_*$

Steepest descent method

Which $I(x_0, y_0)$ should we use?$\rightarrow \nabla R(x, y) = 0 \Rightarrow$ saddle points of $R(x, y)$



[N. Bleistein, 2012]

Steepest descent method for Mellin-Barnes integrals

- Transformation into an exponential form

$$I(s) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_0} dz F(z,s) \longrightarrow I(s) = \frac{1}{2\pi \mathrm{i}} \int_{c_0 - \mathrm{i}\infty}^{c_0 + \mathrm{i}\infty} dz e^{-f(z,s)}$$

$$f(z) = \ln(F(z,s))$$

- Deform the original integration contour $\mathcal{C}_0$ to a Lefschetz thimble $\mathcal{J}(z_*)$, $f = \mathrm{Re}\, f + i\, \mathrm{Im}\, f$

$$\int_{\mathcal{C}_0} dz\, e^{-f} = \underbrace{e^{-i\,\mathrm{Im}\, f|_{\mathcal{J}(z_*)}}}_{\text{Overal factor}} \underbrace{\int_{\mathcal{J}(z_*)} dz\, e^{-\mathrm{Re}\, f}}_{\text{Damping factor}} + \underbrace{2\pi i \sum_{\mathcal{C}_0 \to \mathcal{J}(z_*)} \mathrm{Res}\, e^{-f}}_{\text{Remnants}}$$

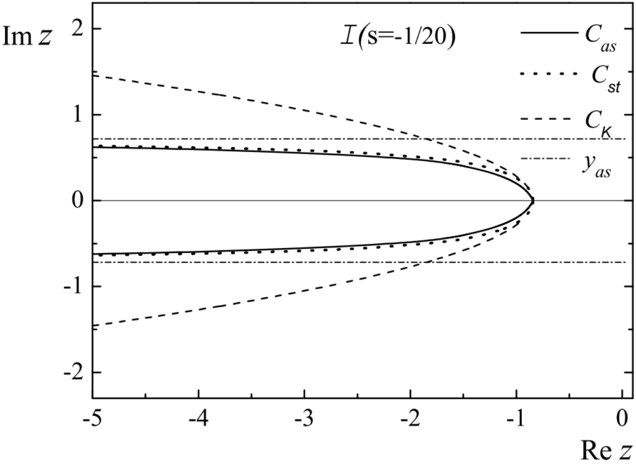
Calculation of the contour of steepest descent

- direct: differential equation

$$\frac{dz^i(t)}{dt} = -\left(\frac{\partial f(z)}{\partial z^i}\right)^*, \quad z(+\infty) = z_*,$$

  Definition: Lefschetz thimble $\mathcal{J}(z_*)$ is a union of curves $t \to z(t) \in \mathbb{C}^n$ which satisfy above equations.
- approximations:
  - [K] Stationary phase approximation [D. Kosower, 1997]
  - Pade approximation [Gluza, Jeliński, Kosower, 2016]
  - [SLS] Asymptotic stationary phase approximation [Sidorov, Lashkevich, Solovtsova, 2017]

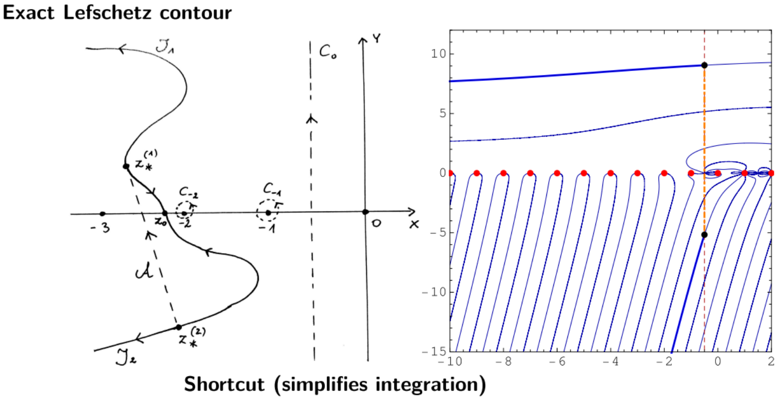
Calculation of the contour of steepest descent



$\mathrm{C_{st}}$ - exact contour
$\mathrm{C_K}$ - [K]
$\mathrm{C_{as}}$ - [SLS]

$$(-s)^{-z}\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)\Gamma(1-z)\Gamma(2+z)}$$

Shortcuts



[Gluza, Jeliński, Kosower, 2016]

Multi-dimensional steepest descent contour

$$I(k) = \int_{\mathcal{S}} e^{-kh(z)} g(z) dz_1 ... dz_n,$$

where $f, g : \mathbb{C}^n \to \mathbb{C}$ analytic functions of $z = (z_1, ..., z_n)$.

$Im[h(z)] = const.$ not sufficient $\to$ (2n-1)-dim variety!
We need n-dim integration surface.

Additional conditions are necessary

Multi-dimensional steepest descent contour

- Approaches
  - differential equation (non-uniquely defined [F. Ursell, 1980])

$$\frac{dz^i(t)}{dt} = -\left(\frac{\partial f(z)}{\partial z^i}\right)^*, \quad z(+\infty) = z_*,$$

  - 2-dim real domain D: in term of flow [D. Kaminski, 1994]

$$(\dot{x}, \dot{y}, \dot{u}, \dot{v}) = -\nabla Re[h(x, y, u, v)], \quad z = x + iy, \ w = u + iv$$
$$\Omega \equiv \textit{closure of} \ \cup_{t \geq 0} \{(x(t), y(t), u(t), z(t)) : (x(0), y(0), u(0), z(0)) \in D\}$$

  - algebraic topology: homology theory [Pham, 1983]

$$\mathcal{S} = \sum_{\alpha} \sum_{j=1}^{\mu_\alpha} N_{\alpha_j}(\mathcal{S}) \mathcal{S}_{\alpha_j} \in H_n^\Psi(\mathbb{C}^n)$$

Summary

- Contour of steepest descent $\longrightarrow$ $Im[h(z)] = const.$
- One dimensional case $\longrightarrow$ trivial.
- n-dimensional surface of steepest descent $\longrightarrow$ Lefschetz thimbles.
- Non-trivial calculation

$$\frac{dz^i(t)}{dt} = -\left(\frac{\partial f(z)}{\partial z^i}\right)^*.$$

- Possible solution $\longrightarrow$ **homology theory**.

Calculation of a multi-dimensional contour of steepest descent is a highly nontrivial problem involving complex analysis of several variables, algebraic geometry and algebraic topology.

Will the effort be worth in the case of MB?

Backup slides

Homology theory

A p-chain is a cycle z_p if $\partial z_p = 0$.
A p-chain is a boundary b_p if there exists a (p+1)-dimensional chain c_{p+1} such that $\partial c_{p+1} = b_p$.
A collection of all p-cycles on K forms p-dimensional cycle group $Z_p(K)$.
Similarly a set of all p-boundaries on K forms a p-dimensional boundary group $B_p(K)$.

$$B_p(K) \subset Z_p(K)$$

A p-dimensional homology group $H_p(K)$ of a complex K is defined to by the quotient group

$$H_p(K) = Z_p(K)/B_p(K),$$

Thus two cycles are homologous if they differ by a boundary.

Example [Delabaere, Howls, 2002]

$$I_{\mathcal{S}}(k) = \int_{\mathcal{S}_0} e^{-kh(z)} g(z) dz_1 \wedge \ldots \wedge dz_n$$

Let us consider the following functions

$$h(z) = z_1 + 2z_2 + 3z_3 + z_1 z_2 + z_2 z_4 + z_3 z_4 + z_2^2 + z_4^2$$
$$g(z) = 1$$

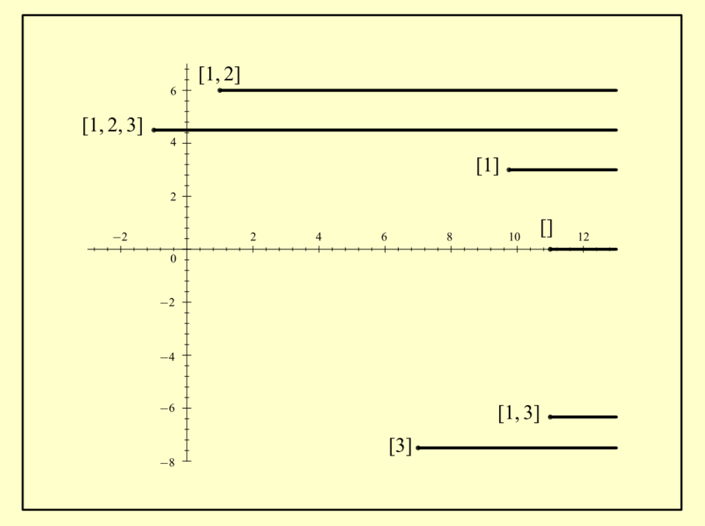
The boundary $S_1 \cup S_2 \cup S_3$ where

$$S_1 : z_1 = 2i,$$
$$S_2 : z_2 = 1 + 2i,$$
$$S_3 : z_3 = 2 - 3i.$$

Now we have to find all isolated critical points of $h(z)$ (six critical points).

Example

A basis of six steepest descent contours generating the space $H_4^{\Psi}(\mathbb{C}^4, S)$.



Example

We can split integration over an initial surface to finite sume of integrals over a set of Lefschetz thimbles

$$\mathcal{S} = \sum_{\alpha} \sum_{j=1}^{\mu_\alpha} N_{\alpha_j}(\mathcal{S}) \mathcal{S}_{\alpha_j} \to I_{\mathcal{S}}(k) = \sum_{\alpha} \sum_{j=1}^{\mu_\alpha} N_{\alpha_j}(\mathcal{S}) \mathcal{S}_{\alpha_j} I_{\mathcal{S}_{\alpha_j}}(k)$$

We can calculate each integral separately

$$I_{\mathcal{S}_0}(k) = \int_{\mathcal{S}_0} e^{-kh(z)} dz_1 \wedge \ldots \wedge dz_4 = \frac{4\pi^2}{k^2} e^{=11k}$$

$$\vdots$$

References

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