CUTS OF FEYNMAN INTEGRALS IN BAIKOV REPRESENTATION

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INPP, NCSR "Demokritos"

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Outline

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Space-Time Approach to Quantum Electrodynamics

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Department of Physics, Cornell University, Ithaca, New York
(Received May 9, 1949)

In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.

The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation

D. More Complex Problems

Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-

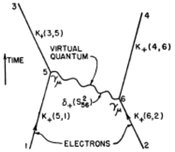
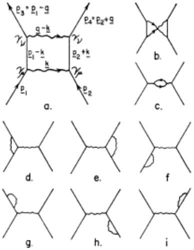


FIG. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.



C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1604**, 078 (2016) [arXiv:1511.09404 [hep-ph]].

T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001 Erratum: [Phys. Rev. Lett. **116** (2016) no.18, 189903] [arXiv:1511.05409 [hep-ph]].

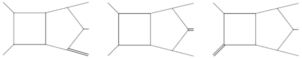


Figure 1. The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

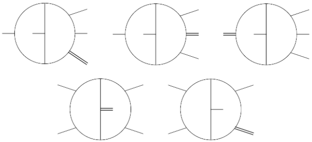


Figure 2. The five non-planar families with one external massive leg.

$$\begin{aligned}
\mathbf{G} = {} & \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
& + \varepsilon^{0}\left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
& + \varepsilon\left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
& + \varepsilon^{2}\left(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-1)}\right. \\
& \left. \quad + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned} \tag{3.6}$$

Best today

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	27.7526	-23.773	-168.117	-175.207±0.004
$P^{(2),[0]}_{--+++}$	12.5	27.7526	-23.773	-168.116	—
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5029	-35.8094	69.661±0.009
$P^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5028	-35.8086	—

TABLE II. The numerical evaluation of $\widehat{A}^{(2),[0]}(1,2,3,4,5)$ using $\{x_1 = -1, x_2 = 79/90, x_3 = 16/61, x_4 = 37/78, x_5 = 83/102\}$ in Eq.(6). The comparison with the universal pole structure, P, is shown. The **+++++** and **-++++** amplitudes vanish to $\mathcal{O}(\epsilon)$ for this $(d_s-2)^0$ component.

S. Badger, C. Brynnum-Hansen, H. B. Hartanto and T. Peraro, "A first look at two-loop five-gluon scattering in QCD," arXiv:1712.02229 [hep-ph].

$\mathcal{A}^{(2)}/\mathcal{A}^{(0)}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1^-, 2^-, 3^+, 4^+, 5^+)$	12.5000000	25.46246919	-1152.843107	-4072.938337	-3637.249567
$(1^-, 2^+, 3^-, 4^+, 5^+)$	12.5000000	25.46246919	-6.121629624	-90.22184215	-115.7836685

TABLE II. Numeric results truncated to 10 significant figures for the two-loop split and alternating MHV amplitudes, normalized to the tree level, at the kinematic point of eq. (IV.1).

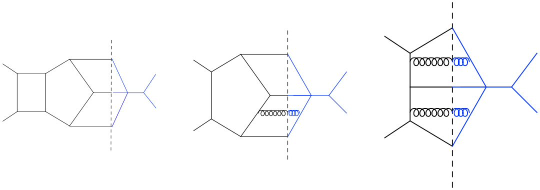
S. Abreu, F. Febres Cordero, H. Ita, B. Page and M. Zeng, "Planar Two-Loop Five-Gluon Amplitudes from Numerical Unitarity," arXiv:1712.03946 [hep-ph].

A REMARK

At the end of the day everything is about numerical applications,
but we need efficiency and reliability
and not only better computing machines

What do we need for an NNLO calculation ?

$$p_1, p_2 \rightarrow p_3, \ldots, p_{m+2}$$



PERTURBATIVE QCD AT NNLO

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NNLO} \quad \to \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) & VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) & RR
\end{aligned}$$

$RV + RR \to$ Antenna-S, Colorfull-S, Sector-improved-RS, q_T, N-jetiness

A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002

OPP at two loops

coefficients of MI $\oplus$ spurious terms

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007)

OPP at two loops

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

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ISP-irreducible integrals $\to$ use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

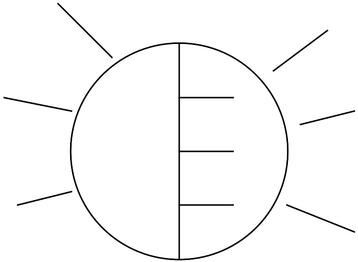
P. Mastrolia, T. Peraro and A. Primo, arXiv:1605.03157 [hep-ph].

J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

H. Ita, arXiv:1510.05626 [hep-th].

C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

Two-loop graph



- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$
$$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: `SecDec`, Weinzierl

IBPI: The current approach

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F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

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S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087

C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046

C. Studerus, Comput. Phys. Commun. **181** (2010) 1293

A. V. Smirnov, Comput. Phys. Commun. **189** (2014) 182

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Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

IBPI: The current approach

- Find a better IBP algorithm ... Generating function technique, Baikov ?

  P. Maierhoefer, J. Usovitsch and P. Uwer, "Kira - A Feynman Integral Reduction Program," arXiv:1705.05610 [hep-ph].

  P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

  V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

$$F_{a_1 \ldots a_N} = \sum_{i=\text{masters}} c^{(i)}_{a_1 \ldots a_N} G_i$$

- Baikov polynomial $\leftrightarrow$ LZ construction
- Sector $\leftrightarrow$ cut

$$\delta\left((k+p)^2 - m^2\right) \leftrightarrow \oint_{z=0} dz \frac{1}{z^{n=1}}$$

- Cut with higher powers in denominator

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K. J. Larsen and Y. Zhang, Phys. Rev. D **93** (2016) no.4, 041701

A. Georgoudis, K. J. Larsen and Y. Zhang, arXiv:1612.04252 [hep-th].

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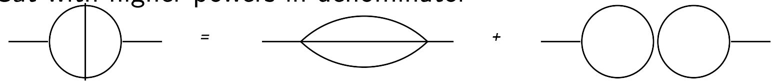

$$F_{11111} = \frac{(3d-10)(3d-8)}{(d-4)^2 (p^2)^2} F_{10011} + \frac{(3d-10)(3d-8)}{(d-4)^2 (p^2)^2} F_{01101} - 2\frac{(d-3)}{(d-4)\, p^2} F_{11110}$$

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1, \ldots, b_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

$\star$ f not MI! J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

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Differential Equations Approach

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Differential Equations Approach

- Iterated Integrals — K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n\left(x\right)$$

- Shuffle algebra

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

$$\mathcal{G}\left(a_1, a_2; x\right) \mathcal{G}\left(b_1; x\right) = \mathcal{G}\left(a_1, a_2, b_1; x\right) + \mathcal{G}\left(a_1, b_1, a_2; x\right) + \mathcal{G}\left(b_1, a_1, a_2; x\right)$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k+p_1+p_2)^2 \ldots (k+p_1+p_2+\ldots+p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \ldots) \rightarrow x \otimes (q_1 = p_1, q_2 = p_2, \ldots)$$

- Now the integral as a function of x, allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

The Simplified Differential Equations Approach

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$$G_{11\dots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k + p_1 + p_2)^2 \dots (k + p_1 + p_2 + \dots + p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \dots) \to x \otimes (q_1 = p_1, q_2 = p_2, \dots)$$

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- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \ldots) \rightarrow x \otimes (q_1 = p_1, q_2 = p_2, \ldots)$$

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5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

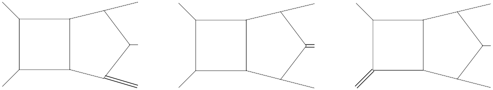

FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

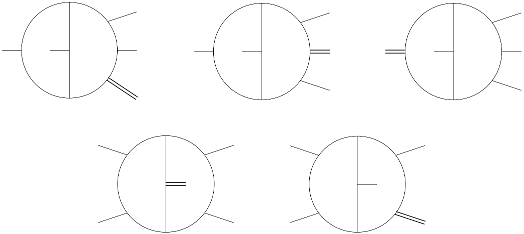

FIGURE : The five non-planar families with one external massive leg.

5BOX - ONE LEG OFF-SHELL: P1

$$p(q_1)p'(q_2) \to V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

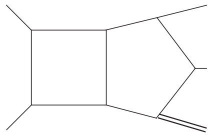

FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1. All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x)$$
$$q_{12}^2 = s_{12}x^2 \;\; q_{23}^2 = s_{45}(1-x) + s_{23}x \;\; q_{34}^2 = (s_{34} - s_{12}(1-x))\,x \;\; q_{45}^2 = s_{45} \;\; q_{51}^2 = s_{51}x$$

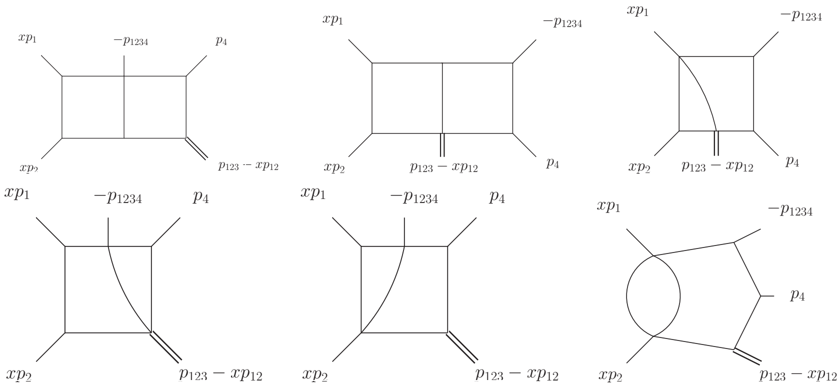

FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure 1, that are contained in the family P_1. All external momenta are incoming.

5BOX - ONE LEG OFF-SHELL: P1

$$G^{P_1}_{a_1\cdots a_{11}}(x,s,\epsilon) := e^{2\gamma_E \epsilon}\int \frac{d^d k_1}{i\pi^{d/2}}\frac{d^d k_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{(k_1+p_{1234})^{2a_5}k_2^{2a_6}(k_2-xp_1)^{2a_7}(k_2-xp_{12})^{2a_8}(k_2-p_{123})^{2a_9}(k_2-p_{1234})^{2a_{10}}(k_1+k_2)^{2a_{11}}},$$

$P_1(74)$: {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100, 10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011, 10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102, 00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011, 11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111, 01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100, 10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111, 10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111, 11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101, 1110m101011, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111, 111001m0111, 11100101111, 111001m1111, 111m0101111},

m= -1

$$\partial_x \mathbf{G} = \mathbf{M}\left(\left\{s_{ij}\right\}, \varepsilon, x\right) \mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}'\mathbf{G}' \qquad \mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1} + \left(\partial_x \mathbf{T}\right)\mathbf{T}^{-1} \qquad \mathbf{G}' = \mathbf{T}\mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ} M_{II}\left(\varepsilon = 0\right), I, J = 1 \ldots 74$$

$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}$.

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0, \quad 1, \quad \frac{s_{45}}{s_{45}-s_{23}}, \quad \frac{s_{45}}{s_{12}}, \quad 1-\frac{s_{34}}{s_{12}}, \quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}}, \quad \frac{s_{45}-s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}}, \quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})}, \quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})}, \quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12}+s_{23}},$$

5BOX P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\left\{s_{ij}\right\}, \varepsilon, x\right)\mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}'\mathbf{G}' \qquad \mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1} + \left(\partial_x \mathbf{T}\right)\mathbf{T}^{-1} \qquad \mathbf{G}' = \mathbf{T}\mathbf{G}$$

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$$1 - \frac{s_{34} - s_{51}}{s_{12}}, \quad \frac{s_{45} - s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23} + s_{45} + s_{51}}, \quad \frac{s_{45}}{s_{34} + s_{45}},$$

$$\frac{s_{12}s_{23} - 2s_{12}s_{45} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}\left(s_{23} - s_{45} - s_{51}\right)}, \quad \frac{s_{12}s_{23} - s_{12}s_{45} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_2}}{2s_{12}\left(s_{23} - s_{45} - s_{51}\right)},$$

$$\frac{s_{12}s_{23} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}\left(s_{23} + s_{34} - s_{51}\right)}, \quad \frac{s_{12}s_{45} \pm \sqrt{\Delta_3}}{s_{12}s_{34} + s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12} + s_{23}},$$

5box P1 - DE

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$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

5box P1 - DE

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$$1-\frac{s_{34}-s_{51}}{s_{12}}, \quad \frac{s_{45}-s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}}, \quad \frac{s_{45}}{s_{34}+s_{45}},$$

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$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)}, \quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j}+\sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\int_0^x dt\frac{1}{(t-a_n)^2}\mathcal{G}(a_{n-1},\ldots,a_1,t) \qquad \int_0^x dt\; t^m\;\mathcal{G}(a_{n-1},\ldots,a_1,t)$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\; G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j}+\sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\mathbf{G}\to(\mathbf{I}-\mathbf{K}_i)\,\mathbf{G},\quad \mathbf{M}\to(\mathbf{M}-\partial_x\mathbf{K}_i-\mathbf{K}_i\mathbf{M})\,(\mathbf{I}-\mathbf{K}_i)^{-1}\quad i=1,2,3$$

$$\partial_x\mathbf{G} = \left(\varepsilon\sum_{a=1}^{19}\frac{\mathbf{M}_a}{(x-l_a)}\right)\mathbf{G}$$

5box P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon), \ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1, 2, 3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

5BOX P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$\mathbf{M}(\varepsilon = 0)$ contains $(x-l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon=0))_{IJ} & I,J \neq 69,74 \\ 0 & I,J = 69,74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon=0))_{IJ} & I,J \neq 74 \\ 0 & I,J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon=0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, **44** (2009),1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x-l_a)} \right) \mathbf{G}$$

Fuchsian
$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\; G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})\,(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1, 2, 3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

5BOX P1 - SOLUTION

- Solution:

$$
\begin{aligned}
\mathbf{G} \;=\;& \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
+\;& \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
+\;& \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
+\;& \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
+\;& \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x\to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

$\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

5box P1 - solution

- Solution:

$$\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
&+ \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x\to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

  $\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

Baikov representation

Baikov, P. A., Phys. Lett. B385 (1996)

$$F_{\alpha_1\ldots\alpha_N} = \int \left(\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right) \frac{1}{D_1^{\alpha_1}\ldots D_N^{\alpha_N}}$$

$$D_a = \sum_{i=1}^{L}\sum_{j=i}^{M} A_a^{ij} s_{ij} + f_a = \sum_{i=1}^{L}\sum_{j=i}^{L} A_a^{ij} k_i\cdot k_j + \sum_{i=1}^{L}\sum_{j=L+1}^{M} A_a^{ij} k_i\cdot p_{j-L} + f_a, \quad a=1,\ldots,N$$

$$F_{\alpha_1\ldots\alpha_N} = C_N^L (G(p_1,\ldots,p_E))^{(-d+E+1)/2} \int \frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}} P_N^L(x_1-f_1,\ldots,x_N-f_N)^{(d-M-1)/2}$$

$$C_N^L = \frac{\pi^{-L(L-1)/4-LE/2}}{\prod_{i=1}^{L}\Gamma\left(\frac{d-M+i}{2}\right)} \det\left(A_{ij}^a\right)$$

$$P_N^L(x_1,x_2,\ldots,x_N) = G(k_1,\ldots,k_L,p_1,\ldots,p_E)\Big|_{s_{ij}=\sum_{a=1}^{N} A_{ij}^a x_a \ \&\ s_{ji}=s_{ij}}$$

Baikov representation

$$d^dk_1d^dk_2\cdots d^dk_L = d^{M-1}k_{1||}d^{d-M+1}k_{1\perp}d^{M-2}k_{2||}d^{d-M+2}k_{2\perp}\cdots d^{M-L}k_{L||}d^{d-M+L}k_{L\perp}$$

$$\begin{aligned}
d^{M-1}k_{1||} &= \frac{ds_{12}ds_{13}\cdots ds_{1M}}{G^{1/2}(k_2,\ldots,k_L,p_1,\ldots,p_E)}\,,\\
d^{M-2}k_{2||} &= \frac{ds_{23}ds_{24}\cdots ds_{2M}}{G^{1/2}(k_3,\ldots,k_L,p_1,\ldots,p_E)}\,,\\
\ldots\\
d^{M-L}k_{L||} &= \frac{ds_{L,L+1}ds_{L,L+2}\cdots ds_{LM}}{G^{1/2}(p_1,\ldots,p_E)}\,,
\end{aligned}$$

$$\begin{aligned}
d^{d-M+1}k_{1\perp} &= \frac{1}{2}\Omega_{d-M+1}\left(\frac{G(k_1,\ldots,k_L,p_1,\ldots,p_E)}{G(k_2,\ldots,k_L,p_1,\ldots,p_E)}\right)^{(d-M-1)/2} ds_{11}\,,\\
d^{d-M+2}k_{2\perp} &= \frac{1}{2}\Omega_{d-M+2}\left(\frac{G(k_2,\ldots,k_L,p_1,\ldots,p_E)}{G(k_3,\ldots,k_L,p_1,\ldots,p_E)}\right)^{(d-M)/2} ds_{22}\,,\\
\ldots\\
d^{d-M+L}k_{L\perp} &= \frac{1}{2}\Omega_{d-M+L}\left(\frac{G(k_L,p_1,\ldots,p_E)}{G(p_1,\ldots,p_E)}\right)^{(d-M+L-2)/2} ds_{LL}\,.
\end{aligned}$$

Baikov representation – IBP

P. A. Baikov, Nucl. Instrum. Meth. A **389**, 347 (1997) [hep-ph/9611449].

$$O_{ij} P_N^L = 0 \tag{2.5}$$

with the operators O_{ij} given by $(i = 1, \ldots, L)$

$$j \leq L \, (q_j = k_j) \qquad O_{ij} = d\delta_{ij} + \sum_{a=1}^{N} \sum_{b=1}^{N} \sum_{m=1}^{M} A_a^{mi} A_{mj}^b \, (1 + \delta_{mi}) \, (x_b - f_b) \frac{\partial}{\partial x_a} \tag{2.6}$$

and

$$j > L \, (q_j = p_{j-L}) \quad O_{ij} = \sum_{a=1}^{N} \left(\sum_{m=1}^{L} \sum_{b=1}^{N} A_a^{mi} A_{mj}^b \, (1 + \delta_{mi}) \, (x_b - f_b) + \sum_{m=L+1}^{M} A_a^{mi} s_{mj} \right) \frac{\partial}{\partial x_a} \tag{2.7}$$

Baikov representation – Integration Limits

H. Frellesvig and C. G. Papadopoulos, arXiv:1701.07356 [hep-ph].

$$F_{\alpha_1\ldots\alpha_{N-1}0} = C_N^1 G(p_1,\ldots,p_{N-1})^{(N-d)/2} \int \frac{dx_1\ldots dx_{N-1}}{x_1^{\alpha_1}\ldots x_{N-1}^{\alpha_{N-1}}} \int_{x_N^-}^{x_N^+} dx_N {P_N^1}^{(d-N-1)/2}$$
$$= C_{N-1}^1 G(p_1,\ldots,p_{N-2})^{(N-1-d)/2} \int \frac{dx_1\ldots dx_{N-1}}{x_1^{\alpha_1}\ldots x_{N-1}^{\alpha_{N-1}}} {P_{N-1}^1}^{(d-(N-1)-1)/2} \quad (2.10)$$

where $P_N^1\left(x_N^+\right) = P_N^1\left(x_N^-\right) = 0$ and

$$\int_{x_N^-}^{x_N^+} dx_N {P_N^1}^{(d-N-1)/2} = \frac{2\pi^{1/2}\Gamma\left(\frac{d-N+1}{2}\right)}{\Gamma\left(\frac{d-N+2}{2}\right)} G\left(p_1,\ldots,p_{N-1}\right)^{(d-N)/2} G\left(p_1,\ldots,p_{N-2}\right)^{(N-1-d)/2}$$

$P_N^1 = \frac{1}{4}G\left(p_1,\ldots,p_{N-2}\right)\left(x_N^+ - x_N\right)\left(x_N - x_N^-\right)$ and $\left(x_N^+ - x_N^-\right)^2 = 16\frac{G(p_1,\ldots,p_{N-1})}{G(p_1,\ldots,p_{N-2})^2}P_{N-1}^1$

J. Bosma, M. Sogaard and Y. Zhang, arXiv:1704.04255 [hep-th].

M. Harley, F. Moriello and R. M. Schabinger, arXiv:1705.03478 [hep-ph].

S. Abreu, R. Britto, C. Duhr and E. Gardi, arXiv:1702.03163 [hep-th].

Differential equations

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} = \left(\frac{-d+E+1}{2}\right)\left(\frac{1}{G}\frac{\partial G}{\partial X}\right)F_{\alpha_1\ldots\alpha_N} + C_N^L G^{(-d+E+1)/2}\int\frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots\, x_N^{\alpha_N}}P_N^{L\,(d-M-1)/2}\left[\left(\frac{d-M-1}{2}\right)\frac{1}{P_N^L}\frac{\partial P_N^L}{\partial X}\right] \tag{4.1}$$

$$b\frac{\partial P_N^L}{\partial X}+\sum_a c_a\frac{\partial P_N^L}{\partial x_a}=0$$

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} = \left(\frac{-d+E+1}{2}\right)\frac{1}{G}\frac{\partial G}{\partial X}F_{\alpha_1\ldots\alpha_N} + C_N^L G^{(-d+E+1)/2}\int\frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots\, x_N^{\alpha_N}}\left(-\sum_a\frac{c_a}{b}\frac{\partial}{\partial x_a}P_N^{L\,(d-M-1)/2}\right)$$

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} = \left(\frac{-d+E+1}{2}\right)\frac{1}{G}\frac{\partial G}{\partial X}F_{\alpha_1\ldots\alpha_N} + C_N^L G^{(-d+E+1)/2}\int dx_1\ldots dx_N P_N^{L\,(d-M-1)/2}\left\{\sum_a\frac{\partial}{\partial x_a}\left(\frac{c_a}{b}\frac{1}{x_1^{\alpha_1}\ldots\, x_N^{\alpha_N}}\right)\right\} \tag{4.4}$$

syzygy equation: equivalent to standard determinant equations

Cuts

Definition:

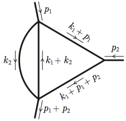
$$F_{\alpha_1\ldots\alpha_N}|_{n\times\text{cut}} \equiv C_N^L (G)^{(-d+E+1)/2} \left(\prod_{a=n+1}^{N} \int dx_a\right) \left(\prod_{c=1}^{n} \oint_{x_c=0} dx_c\right) \frac{1}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}} P_N^{L\,(d-M-1)/2}$$

$$\frac{\partial}{\partial X_j} F_i = \sum_{l=1}^{I} M_{il}^{(j)} F_l \qquad\qquad \frac{\partial}{\partial X_j} F_i|_{n\times\text{cut}} = \sum_{l=1}^{I} M_{il}^{(j)} F_l|_{n\times\text{cut}}$$

cut-integrals satisfy the same DE.

A. Primo and L. Tancredi, Nucl. Phys. B **916** (2017) 94 [arXiv:1610.08397 [hep-ph]].

Canonical Basis



$$I_1 = \epsilon R_{12} F_{11210} \qquad I_2 = \left(s F_{1221-1} - \tfrac{1}{2}\epsilon(p_1^2 - p_2^2 - s) F_{11210} \right)$$

$$\begin{aligned} I_{1\,|4\times\mathrm{cut}} &= \frac{2^{4\epsilon-3}\epsilon\cos(\pi\epsilon)\Gamma\left(\epsilon+\frac{1}{2}\right)}{\pi^2\Gamma\left(\frac{3}{2}-\epsilon\right)} \left(p_1^2\right)^{-2\epsilon} x^{-\epsilon}(x+1)^{-\epsilon}(y-1)(xy+1)^{-\epsilon} \\ &\times\ {}_2F_1(1-\epsilon,\epsilon+1;2-2\epsilon;1-y) \end{aligned}$$

$$I_{2\,|4\times\mathrm{cut}} = \frac{4^{2\epsilon-1}}{\pi\Gamma\left(\frac{1}{2}-\epsilon\right)^2}\left(p_1^2\right)^{-2\epsilon} x^{-\epsilon}(x+1)^{-\epsilon}(xy+1)^{-\epsilon}\,{}_2F_1(-\epsilon,\epsilon;-2\epsilon;1-y)$$

$$\begin{aligned} N_\epsilon I_{1\,|4\times\mathrm{cut}} &= \epsilon\log(y) + \epsilon^2\left(-2\,\mathrm{Li}_2(1-y) - \log^2(y)\right) + \epsilon^3\Big(-4\,\mathrm{Li}_3(1-y) - 2\,\mathrm{Li}_3(y) \\ &- \mathrm{Li}_2(y)\log(y) + \frac{2}{3}(\log(y) - 3\log(1-y))\log^2(y) + 2\,\zeta(3)\Big) + \mathcal{O}(\epsilon^4) \quad \text{(B.29)} \end{aligned}$$

$$\begin{aligned} N_\epsilon I_{2\,|4\times\mathrm{cut}} &= 1 - \frac{1}{2}\epsilon\log(y) + \frac{1}{2}\epsilon^2\left(\log^2(y) - \pi^2\right) + \frac{1}{12}\epsilon^3\Big(36\,\mathrm{Li}_3(y) + 18\,\mathrm{Li}_2(1-y)\log(y) \\ &- 4\log^3(y) + 18\log(1-y)\log^2(y) - 3\pi^2\log(y) - 92\,\zeta(3)\Big) + \mathcal{O}(\epsilon^4) \qquad \text{(B.30)} \end{aligned}$$

with $N_\epsilon = e^{2\gamma_E\epsilon}(p_1^2)^\epsilon x^\epsilon (x+1)^\epsilon (xy+1)^\epsilon$.

Canonical Basis

R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, F. Moriello and V. A. Smirnov, JHEP **1612** (2016) 096 [arXiv:1609.06685 [hep-ph]].

$$F_{\text{box-triangle}} = \int \frac{d^d k_1\, d^d k_2}{(i\pi^{d/2})^2} \frac{1}{x_1 x_2 x_3 x_4 x_5 x_6} \tag{B.31}$$

with

$$\begin{aligned} &x_1 = k_1^2 - m^2 && x_2 = (k_1+p_1)^2 - m^2 && x_3 = (k_1+p_1+p_2)^2 - m^2 \\ &x_4 = (k_2-p_4)^2 - m^2 && x_5 = k_2^2 - m^2 && x_6 = (k_1-k_2)^2 \\ &x_7 = (k_1-p_4)^2 \end{aligned} \tag{B.32}$$

$$F_1(z) = m^4 - 2m^2 p_4^2 + p_4^4 - 2m^2 z - 2p_4^2 z + z^2$$
$$F_2(z) = s(m^4 s + 2m^2(2tu + s(t-z)) + s(t-z)^2)$$

$$F_{\text{box-triangle}|6\times\text{cut}} = C \int_{r_-}^{r_+} \frac{\mathrm{d}z}{\sqrt{F_1(z)F_2(z)}} + \mathcal{O}(\epsilon)$$

$$F_{\text{box-triangle}|6\times\text{cut}} = \frac{2iC}{\sqrt{X}} K\left(\frac{-16m^2\sqrt{-p_4^2 stu}}{X}\right) + \mathcal{O}(\epsilon)$$

$$X = s(p_4^2 - t)^2 - 4m^2\left(p_4^2 s - tu + 2\sqrt{-p_4^2 stu}\right)$$

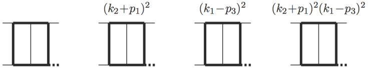


Figure 5. The four master integrals of the elliptic sector $I^A_{1,1,1,1,1,1,1,0,0}$.

$$F_{\text{ell. double-box}} = \int \frac{d^d k_1\, d^d k_2}{(i\pi^{d/2})^2} \frac{1}{x_1 x_2 x_3 x_4 x_5 x_6 x_7} \tag{B.40}$$

$$\begin{aligned} &x_1 = k_1^2 - m^2 && x_2 = (k_1+p_1)^2 - m^2 && x_3 = (k_1+p_1+p_2)^2 - m^2 \\ &x_4 = (k_2+p_1+p_2)^2 - m^2 && x_5 = (k_2-p_4)^2 - m^2 && x_6 = k_2^2 - m^2 \\ &x_7 = (k_1-k_2)^2 && x_8 = (k_1-p_4)^2 && \end{aligned} \tag{B.41}$$

$$F_{\text{ell. double-box}} = \frac{-\pi^{-3}}{\Gamma^2\left(\frac{d-3}{2}\right)} \frac{\det(A^{-1})}{\sqrt{-G_1}} \int \frac{1}{x_1\cdots x_7} \frac{\lambda_{22}^{(d-5)/2}\,\lambda_{11}^{(d-5)/2}}{\sqrt{-G_2}}\, d^8x \tag{B.42}$$

$$F_{\text{ell. double-box}|7\times\text{cut}} = \frac{C}{\sqrt{s(s-4m^2)}} \int_{r_-}^{r_+} \frac{dz}{z\sqrt{f(z)}} + \mathcal{O}(\epsilon) \tag{B.43}$$

$$f(z) = s(4m^2tu + s(t-z)^2) \qquad r_{\mp} = t \mp 2\sqrt{-m^2stu}/s$$

$$F_{\text{ell. double-box}|7\times\text{cut}} = \frac{-i}{4\pi^3} \frac{1}{s\sqrt{(4m^2-s)t(st+4m^2u)}} + \mathcal{O}(\epsilon)$$

One loop

S. Abreu, R. Britto, C. Duhr and E. Gardi, JHEP **1706** (2017) 114

$$\mathcal{C}_C I_n = (2\pi i)^{\lfloor c/2 \rfloor} \frac{2^{1-c} e^{\gamma_E \epsilon}}{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{D-n+1}{2}\right)} \frac{1}{\sqrt{\mu^c Y_C}} \left(\mu \frac{Y_C}{\mathrm{Gram}_C}\right)^{\frac{D-c}{2}} [f_c^n]_C \,.$$

$$\mathcal{C}_{[n-1]} I_n = (2\pi i)^{\lfloor \frac{n-1}{2} \rfloor} \frac{2^{D-2n+1} e^{\gamma_E \epsilon} \Gamma\left(\frac{D-n+1}{2}\right)}{\pi^{\frac{n-1}{2}} \Gamma(D-n+1)} \frac{1}{\sqrt{-Y_{[n]}}} \left(\frac{Y_{[n-1]}}{\mathrm{Gram}_{[n-1]}}\right)^{\frac{D-n}{2}}$$
$$ {}_2F_1\left(\frac{1}{2}, \frac{D-n}{2}; \frac{D-n+2}{2}; \frac{\mathrm{Gram}_{[n]} Y_{[n-1]}}{\mathrm{Gram}_{[n-1]} Y_{[n]}}\right).$$

Box-triangle

J. Bosma, M. Sogaard and Y. Zhang, arXiv:1704.04255 [hep-th].

$$F_{\text{box-triangle}} = \int \frac{d^dk_1\, d^dk_2}{(i\pi^{d/2})^2} \frac{1}{x_1x_2x_3x_4x_5x_6} \tag{B.31}$$

with

$$\begin{aligned} &x_1 = k_1^2 - m^2 && x_2 = (k_1+p_1)^2 - m^2 && x_3 = (k_1+p_1+p_2)^2 - m^2 \\ &x_4 = (k_2-p_4)^2 - m^2 && x_5 = k_2^2 - m^2 && x_6 = (k_1-k_2)^2 \\ &x_7 = (k_1-p_4)^2 \end{aligned} \tag{B.32}$$

$$F_1(z) = m^4 - 2m^2p_4^2 + p_4^4 - 2m^2z - 2p_4^2z + z^2$$
$$F_2(z) = s(m^4s + 2m^2(2tu + s(t-z)) + s(t-z)^2)$$

$$F_{\text{box-triangle}|6\times\text{cut}} = C\int_{r_-}^{r_+} \frac{\mathrm{d}z}{\sqrt{F_1(z)F_2(z)}} + \mathcal{O}(\epsilon)$$

$$F_{\text{box-triangle}|6\times\text{cut}} = \frac{2iC}{\sqrt{X}} K\left(\frac{-16m^2\sqrt{-p_4^2stu}}{X}\right) + \mathcal{O}(\epsilon)$$

$$X = s(p_4^2-t)^2 - 4m^2\left(p_4^2s - tu + 2\sqrt{-p_4^2stu}\right)$$

Double-box

$$
\begin{aligned}
J[a,b]\big|^{(\mathrm{I})}_{\mathrm{m.c.}} &= \frac{2^{D-10}}{\pi^4 st\Gamma(D-4)(s+t)} \int_0^{\chi s} dz_8 \int_0^{\frac{s(\chi s - z_8)}{s+z_8}} dz_9 F^{\frac{D-6}{2}} z_8^a z_9^b \\
&= \frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(a+\frac{D}{2}-2\right)\Gamma\left(b+\frac{D}{2}-2\right) s^{a+b+D-7}\chi^{a+b+D-5}}{16\pi^4\Gamma(D-4)} \\
&\times\ {}_2\tilde{F}_1\left(a+D-4, b+D-4; a+b+\frac{3D}{2}-6; -\chi\right),
\end{aligned}
$$

$$
\begin{aligned}
J[a,b]\big|^{(\mathrm{II})}_{\mathrm{m.c.}} &= \frac{(-1)^b\Gamma\left(\frac{D}{2}-2\right)\Gamma(-a-D+5)\Gamma\left(b+\frac{D}{2}-2\right) s^{a+b+D-7}\chi^{a+\frac{D}{2}-3}(\chi+1)^{2-\frac{D}{2}}}{16\pi^4\Gamma(D-4)} \\
&\times\ {}_2\tilde{F}_1\left(-a-D+5, b+\frac{D}{2}-2; -a+b+1; -\frac{1}{\chi}\right).
\end{aligned} \tag{5.21}
$$

Double-box with one massive leg

$$\chi = t/s,\ \kappa = m_1^2/s,$$

$$J[a,b]\Big|_{\text{m.c.}}^{(\text{I})} = \frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(a+\frac{D}{2}-2\right)\Gamma\left(b+\frac{D}{2}-2\right)s^{a+b-7}\chi^{a+b+d-5}\left(\frac{1}{1-\kappa}\right)^{a+d-4}}{16\pi^4\chi^5\Gamma(D-4)}$$
$$\times\ {}_2\tilde{F}_1\left(a+D-4, b+D-4; a+b+\frac{3D}{2}-6; \frac{\chi}{\kappa-1}\right). \qquad (5.36)$$

$$J[a,b]\Big|_{\text{m.c.}}^{(\text{II})} = \frac{(-1)^b(1-\kappa)^{-a+b}\Gamma\left(\frac{D}{2}-2\right)\chi^{a+\frac{D}{2}-3}\Gamma(-a-D+5)\Gamma\left(b+\frac{D}{2}-2\right)s^{a+b+D-7}}{16\pi^4\Gamma(D-4)}$$
$$\times(-\kappa+\chi+1)^{2-\frac{D}{2}}\,{}_2\tilde{F}_1\left(-a-D+5, b+\frac{D}{2}-2; -a+b+1; \frac{\kappa-1}{\chi}\right). \qquad (5.37)$$

Double-box with two massive legs

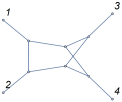
$$\chi = t/s,\ \kappa_1 = m_1^2/s,\ \kappa_2 = m_2^2/s.$$

$$J[a,b]\Big|^{(\mathrm{I})}_{\mathrm{m.c.}} = (\kappa_1\kappa_2)^{-a-\frac{D}{2}+2}\left((\kappa_1-\chi)(\kappa_2-\chi)+\chi\right)^{2-\frac{D}{2}} s^{a+b+D-7}\chi^{2a+b+2D-9}$$
$$\times\frac{\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(a+\frac{D}{2}-2\right)\Gamma(a+b+D-4)}{16\pi^4\Gamma(D-4)\Gamma(2a+b+2D-8)}$$
$$\times F_1\left(a+b+D-4;\frac{1}{2}(2a+D-4),\frac{1}{2}(2a+D-4);2a+b+2D-8;w_1,w_2\right).$$

$$\tilde{I} = \left(s^3\chi J[0,0], -s^2\sqrt{\kappa_1^2-2\kappa_2\kappa_1-2\kappa_1+\kappa_2^2-2\kappa_2+1}J[1,0], -s^2J[0,1]\right)^T.$$

$$\frac{\partial}{\partial\chi}\tilde{I} = (D-4)\tilde{B}_\chi\tilde{I}+\ldots,\quad \frac{\partial}{\partial\kappa_1}\tilde{I} = (D-4)\tilde{B}_{\kappa_1}\tilde{I}+\ldots,\quad \frac{\partial}{\partial\kappa_2}\tilde{I} = (D-4)\tilde{B}_{\kappa_2}\tilde{I}+\ldots$$

Non-planar double-box



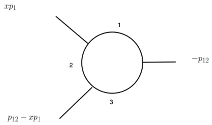
$$X[a,b]\big|^{(I)}_{\rm m.c.} = \frac{2^{D-5}}{\pi^4\Gamma(D-4)} s^{a+b+D-7}\chi^{a+b+D-5}(1+\chi)^{2-\frac{D}{2}}\Gamma\Big(a+\frac{D}{2}-2\Big)\Gamma\Big(b+\frac{D}{2}-2\Big) \times\Gamma\Big(\frac{D}{2}-2\Big)\,{}_2\tilde{F}_1\Big(3-\frac{D}{2},a+\frac{D}{2}-2;a+b+\frac{3D}{2}-6;-\chi\Big). \quad (6.8)$$

$$X[a,b]\big|^{(II)}_{\rm m.c.} = \frac{2^{D-5}}{\pi^4\Gamma(D-4)}(-1)^{b+D}s^{a+b+D-7}\chi^{a+b+\frac{3D}{2}-8}(1+\chi)^{2-\frac{D}{2}}\Gamma(10-a-b-2D) \times\Gamma\Big(\frac{D}{2}-2\Big)\Gamma\Big(b+\frac{D}{2}-2\Big)\,{}_2\tilde{F}_1\Big(10-a-b-2D,3-\frac{D}{2};6-a-D;-\frac{1}{\chi}\Big). \quad (6.9)$$

$$\frac{d}{d\chi}\begin{pmatrix}\tilde{J}_1\\ \tilde{J}_2\end{pmatrix} = (D-4)\begin{pmatrix}\frac{2\chi+1}{\chi(\chi+1)} & -\frac{4}{\chi}\\ \frac{1}{2(\chi+1)} & -\frac{2\chi+1}{2\chi(\chi+1)}\end{pmatrix}\begin{pmatrix}\tilde{J}_1\\ \tilde{J}_2\end{pmatrix},$$

$$\tilde{J}_1 \equiv s^2(1+\chi)J[1,0]\,,\quad \tilde{J}_2 \equiv -\frac{1}{8}s^3\chi J[0,0]+\frac{1}{8}s^2(2\chi+3)J[1,0] = sJ[1,1]\,.$$

C. G. Papadopoulos, JHEP **1407** (2014) 088 [arXiv:1401.6057 [hep-ph]].



$$T(q_1^2,q_2^2,q_3^2)=\int\frac{d^dk}{i\pi^{d/2}}\frac{1}{(-k^2)\left(-(k+q_1)^2\right)\left(-(k+q_1+q_2)^2\right)}$$

$$\begin{aligned}G_{111}(x)&=\int\frac{d^dk}{i\pi^{d/2}}\frac{1}{(-k^2)\left(-(k+x\,p_1)^2\right)\left(-(k+p_1+p_2)^2\right)}\\&=T(q_1^2=x^2m_1,q_2^2=(p_{12}-xp_1)^2,q_3^2=m_3)\end{aligned}$$

$$\frac{\partial}{\partial x}G_{111}=-\frac{1}{x}G_{111}+m_1xG_{121}+\frac{1}{x}G_{021}$$

$$\frac{\partial}{\partial x}MG_{111}=c_\Gamma\frac{1}{\varepsilon}\left(1-x\right)^{-1+\varepsilon}\left(-m_3+m_1x\right)^{-1+\varepsilon}\left(\left(-m_1x^2\right)^{-\varepsilon}-\left(-m_3\right)^{-\varepsilon}\right)$$

$$M=x\left(1-x\right)^{\frac{4-d}{2}}\left(-m_3+m_1x\right)^{\frac{4-d}{2}}$$

expressed in terms of GPs: $\ldots-2G\left(0,\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)+2G\left(\frac{m_3}{m_1},0,1,x\right)-2G\left(\frac{m_3}{m_1},0,\frac{m_3}{m_1},x\right)$ .....

$$I = C\int\prod_i dx_i \frac{1}{x_1x_2x_3}P^B = C\int\prod_i dx_i \frac{1}{x_1x_2x_3}P^{B-1}P$$

$$\sum_i c_i^{(3)}\frac{\partial P}{\partial x_i} + P = 0 \qquad C\int\prod_i dx_i \frac{1}{x_1x_2x_3}P^{B-1}\left(-\sum_i c_i^{(3)}\frac{\partial P}{\partial x_i}\right) = C\frac{1}{B}\int\prod_i dx_i P^B\left(\sum_i \frac{\partial}{\partial x_i}\left(\frac{c_i^{(3)}}{x_1x_2x_3}\right)\right)$$

$$\sum_i c_i^{(4)}\frac{\partial P}{\partial x_i} + \frac{\partial P}{\partial z} = 0 \qquad \frac{\partial I}{\partial z} - \left(\frac{1}{C}\frac{\partial C}{\partial z}\right)I = CB\int\prod_i dx_i \frac{1}{x_1x_2x_3}P^{B-1}\frac{\partial P}{\partial z} = C\int\prod_i dx_i P^B\left(\sum_i \frac{\partial}{\partial x_i}\left(\frac{c_i^{(4)}}{x_1x_2x_3}\right)\right)$$

$$C\frac{\partial}{\partial z}\left(C^{-1}I\right) - B\frac{A_4}{A_3}I = C\int\prod_i dx_i P^B\left(\sum_i \frac{\partial}{\partial x_i}\left(\frac{c_i^{(4)}}{x_1x_2x_3}\right)\right) - \frac{A_4}{A_3}\left(\sum_i \frac{\partial}{\partial x_i}\left(\frac{c_i^{(3)}}{x_1x_2x_3}\right)\right)$$

$$\frac{\partial}{\partial x}MG_{111} = c_\Gamma \frac{1}{\varepsilon}(1-x)^{-1+\varepsilon}(-m_3+m_1x)^{-1+\varepsilon}\left(\left(-m_1x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

$$\underbrace{C\int\prod dx_i \frac{x_3}{x_1x_2}\left(\frac{1}{4}G(p_1)(x_3^+ - x_3)(x_3 - x_3^-)\right)^B}_{G_{11-1}} = \frac{1}{2}(S_1 - S_2 - S_3)\underbrace{\bar{C}\int\prod dx_i \frac{1}{x_1x_2}\bar{P}^{\bar{B}}}_{G_{110}}$$

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Latest news

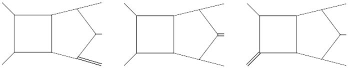

Figure 1. The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

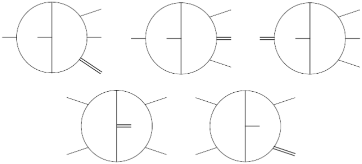

Figure 2. The five non-planar families with one external massive leg.

- Non-planar pentaboxes check canonicality
- Working with on-shell kinematics, reducing number of MI
- Using Baikov-R to integrate non-rational dlog terms, (semi-)numerical approach
- 8-denominator integrals to complete basis of MI at two loops

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