Fuchsia
and
Differential Equations for Multi-Scale Master Integrals

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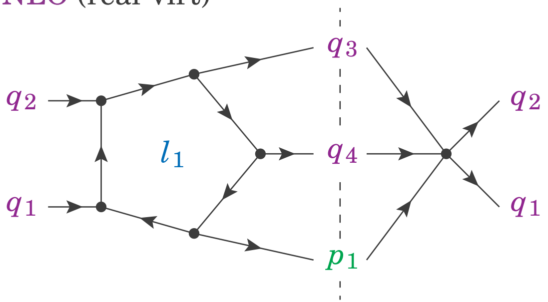
Agenda

Feynman Integrals Calculus

$$F(\vec{s_{ij}}, \vec{x}, \vec{m}, \epsilon) = \int \prod_{l=1}^{n_l} \frac{1}{D_l^{n_l}} \quad \prod_{i=1}^{n_i} \mathrm{d}^d l_i \quad \prod_{j=1}^{n_j} \mathrm{d}^d p_j \delta(p_j^2) \quad \prod_{k=1}^{n_k} \delta\left(x_k - f_k(\vec{p}, \vec{q})\right)$$

- loop momenta
- external momenta with rational projectors
- $s_{ij} = q_i \cdot q_j$ — kinematic invariants

Example: $2 \to 2$ at NNLO (real-virt)



$$q_4 = q_1 + q_2 - q_3 - p_1 \quad \text{and} \quad \vec{s_{ij}} = \{s, t, u\} \quad \text{and maybe} \quad x_1 = (q_1 + q_2) \cdot p_1$$

Feynman Integrals Calculus

$$F(\vec{s_{ij}},\vec{x},\vec{m},\epsilon) = \int \prod_{l=1}^{n_l} \frac{1}{D_l^{n_l}} \prod_{i=1}^{n_i} \mathrm{d}^d l_i \prod_{j=1}^{n_j} \mathrm{d}^d p_j \delta(p_j^2) \prod_{k=1}^{n_k} \delta\Big(x_k - f_k(\vec{p},\vec{q})\Big)$$

— UNDER CONTROL —

IBP Reduction

- only master integrals to calculate
- easy to write *differential equations*

GPL functions

— LESS UNDER CONTROL —

Functions beyond GPL

- Elliptic polylogarithms
- holonomic functions

Method of Differential Equations

Numerical methods

- Sector Decomposition Borowka's talk
- Mellin-Barnes talks by Dubovyk, Flieger, Prausa, Usovitsch
- Subtraction Schemes (e.g., CoLoRFulNNLO)

Analytical methods

- Feynman/Schwinger parametrization
  - `HyperInt` Panzer '14
- Integration-By-Parts reduction Chetyrkin Tkachov '81
  - Laporta algorithm Laporta '00: `AIR`, `FIRE`, `Kira`, `Reduze`
  - Symbolic reduction: `LiteRed` Lee '12
  - private code: `Crusher` Marquard; Laporta; ...
- Method of Differential Equations Kotikov '91 Remiddi '97
  - Epsilon form Henn '13
    - leading singularity Henn '14
    - Lee algorithm '14: `Fuchsia` OG Magerya '16, `Epsilon` Prausa '17
    - Meyer algorithm '16: `Canonica` Meyer '17

Method of Differential Equations

Construct System of ODE

- from IBP rules
  - `AIR`, `FIRE`, `Kira`, `LiteRed`, `Reduze`
- from definition
  - Holonomic Functions
- from cuts Papadopoulos's talk

Solve System of ODE

- epsilon form Henn '13
  - `Fuchsia`, `Epsilon`, `Canonica`
  - works well with *hyperlogarithms*
  - also (to some extend) with *elliptic functions* Weinzierl's talk
- by other means
  - series expansion Smirnov's talk

Find Constants of Integration

- no systematic approach
- usually non-trivial

— INPUT —

- Initial System of Ordinary Differential Equations

$$\frac{\partial \vec{F}}{\partial x_1} = \mathbb{M}(\vec{x},\epsilon)\,\vec{F}(\vec{x},\epsilon)$$

— OUTPUT —

- Epsilon Form, i.e., Equivalent System

$$\frac{\partial \vec{G}}{\partial x_1} = \epsilon\,\tilde{\mathbb{M}}(\vec{x})\,\vec{G}(\vec{x},\epsilon)$$

- Transformation to a new basis

$$\vec{F}(\vec{x},\epsilon) = \mathbb{T}(\vec{x},\epsilon)\,\vec{G}(\vec{x},\epsilon)$$

- Other Useful Operations: variable change, convert to block-diagonal form, . . .

Project Fuchsia

Open-source OG Magerya '16 '17

- http://github.com/gituliar/fuchsia

Implemented in Python

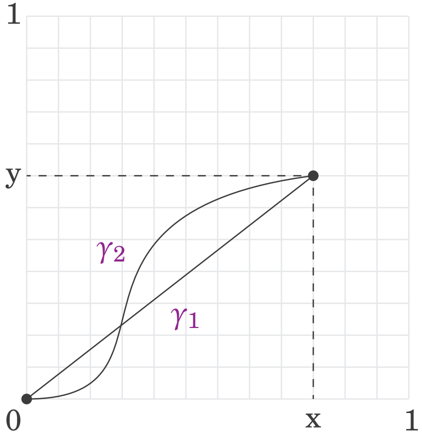
- SageMath
- Maxima
- Maple (optional)

Lee algorithm

- **Fuchsification**
  - construct Fuchsian form Moser '59
- **Normalization**
  - balance eigenvalues to $\alpha\,\epsilon$ form Lee, Pomeransky '17
  - eliminate resonances Smirnov's talk
- **Factorization**
  - construct epsilon form
- **Block-triangular optimization**
- **Multivariate Systems**

Multivariate Solutions

$$\frac{\vec{F}\big(x(t), y(t), \epsilon\big)}{\mathrm{d}t} = \frac{\partial \vec{F}}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{\partial \vec{F}}{\partial y}\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \left(\mathbb{M}_x \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \mathbb{M}_y \frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)\vec{F} = \mathbb{M}_t(x, y, t, \epsilon)\,\vec{F}$$



γ_1 (line)

$$x_1(t) = t\,x$$

$$y_1(t) = t\,y$$

γ_2 (cubic Bézier curve)

$$x_2(t) = \big(2t(1-t)^2 + t^3\big)\,x$$

$$y_2(t) = \big(2t^2(1-t) + t^3\big)\,y$$

γ_3 (your favorite curve)

$$x_3(t) = \dots$$

$$y_3(t) = \dots$$

Multivariate Solutions

$$\frac{\vec{F}\big(x(t),y(t),\epsilon\big)}{\mathrm{d}t} = \frac{\partial \vec{F}}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{\partial \vec{F}}{\partial y}\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \left(\mathbb{M}_x \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \mathbb{M}_y \frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)\vec{F} = \mathbb{M}_t(x,y,t,\epsilon)\,\vec{F}$$

`CANONICA`

- Meyer algorithm
  - Rational ansatz
- Output:
  - ϵ-form for every $\mathbb{M}_x$, $\mathbb{M}_y$, ...
- Single $\mathbb{T}$ for all paths
  - May spoil Fuchsian form
  - Fix with Moser reduction

`Fuchsia`

- Lee algorithm
  - Residues properties
- Output:
  - ϵ-form for $\mathbb{M}_t$ only
- Different $\mathbb{T}$ for every path

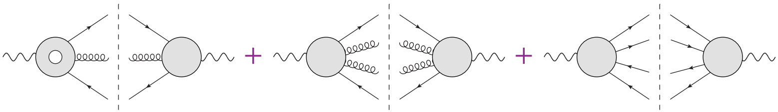
Energy-Energy Correlations in QCD

Definition

$$\Sigma(\xi) = \sum_{a,b} \int \mathrm{dPS}\, \frac{E_a\, E_b}{Q^2}\, \sigma\left(e^+ + e^- \to a + b + X\right) \delta\left(\xi - \cos\theta_{ab}\right)$$

- infrared-safe quantity (UV and IR poles cancel)

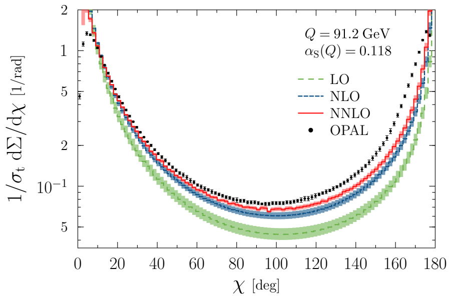
NLO contributions (virtual + real)



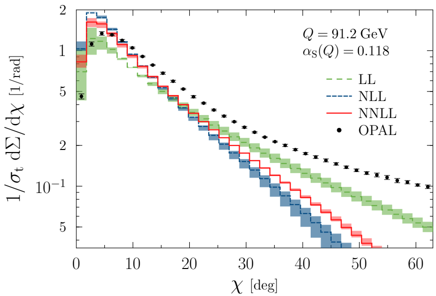
- correlated particles — a, b (e.g., qg, $\bar{q}g$, $q\bar{q}$, gg)
- angle variable — θ_{ab} or ξ
- energies — E_a, E_b

Status

Fixed Order



Resummation



Numerical Results

- 2010 — NNLL de Florian Grazzini ’05, Becher Neuberg ’10
- 2016 — NNLO Del Duca et al. ’16
- 2017 — NNLL + NNLO Tulipánt Kardos Somogyi ’17

Analytical Results

- 1978 — LO Basham et al. ’78
- 2018 — NLO Dixon et al. ’18

I.0 Preparation

Definition

$$\Sigma(\xi) = \sum_{a,b} \int \mathrm{dPS}\, \frac{E_a E_b}{Q^2}\, \sigma\big(e^+ + e^- \to a + b + X\big)\, \delta\big(\xi - \cos\theta_{ab}\big)$$

Process

$$e^+ + e^- \to q(p_1) + \bar{q}(p_2) + g(p_3) + g(p_4), \qquad p_i^2 = 0$$

Parametrization

- phase-space

$$\mathrm{dPS} = \mathrm{d}^m p_1\, \delta(p_1^2)\, \mathrm{d}^m p_2\, \delta(p_2^2)\, \mathrm{d}^m p^3\, \delta(p_3^2)\, \delta\big((q - p_1 - p_2 - p_3)^2\big)$$

- new *angle variable* (more convenient)

$$z = \frac{1-\xi}{2} = \frac{p_1 \cdot p_3}{2\, q\cdot p_1\, q\cdot p_3}$$

Linearization

$$\delta\left(z - \frac{p_1\cdot p_3}{2\, q\cdot p_1\, q\cdot p_3}\right) \to \int_0^1 \mathrm{d}x\, x\, q\cdot p_3\, \delta\big(x - 2\, q\cdot p_1\big)\, \delta\big(x\, z\, q\cdot p_3 - p_1\cdot p_3\big)$$

I.1 IBP Reduction

Cross-section

- $\sigma\left(e^+ + e^- \to q + \bar{q} + g + g\right)$

Cutkosky rules for Cut Propagators

$$\delta\left(p_1^2\right)\delta\left(p_2^2\right)\delta\left(p_3^2\right)\delta\left((q-p_1-p_2-p_3)^2\right)\delta\left(x-2\,q\cdot p_1\right)\delta\left(x\,z\,q\cdot p_3 - p_1\cdot p_3\right)$$

Reduction

- `LiteRed` Lee '12
  - 11 masters $\vec{F}(x,z,\epsilon)$ (max 3×3 coupled subsystem)

$$\begin{array}{lllll} F_1=\{\} & F_2=\{2\} & F_3=\{2,2\} & F_4=\{2,6\} & F_5=\{1,2\} \\ F_6=\{5\} & F_7=\{1,4,5\} & F_8=\{2,3,4\} & F_9=\{2,5\} & F_{10}=\{3,5\} \\ F_{11}=\{2,4,5\} & & & & \end{array}$$

  - denominators

$$\begin{array}{lll} D_1=(k_2+k_3)^2 & D_2=(q-k_2)^2 & D_3=(q-k_1-k_2)^2 \\ D_4=(q-k_1-k_3)^2 & D_5=(q-k_2-k_3)^2 & D_6=\left(k_1^2\right)_{\mathrm{x}} \end{array}$$

I.2 Differential Equations for $F(x,z,\epsilon)$

ODE Definition

$$\frac{\partial \vec{F}(x,z,\epsilon)}{\partial z} = \mathbb{M}(x,z,\epsilon)\,\vec{F}(x,z,\epsilon)$$

- z as free variable
- IBP rules (step I.1)
- 11×11 matrix
- alphabet:

$$z, \quad z-1, \quad z-\frac{1}{x}, \quad z-\frac{1}{x\,(x-2)}$$

Epsilon form

- multivariate problem
- `Fuchsia` (Lee algorithm)

$$\frac{\partial \hat{\vec{F}}(x,z,\epsilon)}{\partial z} = \epsilon\,\hat{\mathbb{M}}(x,z)\,\hat{\vec{F}}(x,z,\epsilon), \quad \text{where} \quad \vec{F}(x,z,\epsilon) = \boxed{\mathbb{T}(x,z,\epsilon)}\,\hat{\vec{F}}(x,z,\epsilon)$$

Solutions for $\hat{\vec{F}}(x,z,\epsilon)$

- any order in ϵ

I.2 Example: $F_4(x,z,\epsilon)$

$$F_4(x,z,\epsilon) = \frac{1}{15\,\epsilon^2}\Big(C_3^0(x) - 2C_2^0(x)\Big) + \frac{1}{30\,\epsilon}\Bigg(\Big(15C_1^0(x) + 4C_2^0(x) - 6xC_3^0(x) - 2xC_4^0(x)\Big)\ \mathrm{H}_0(z)$$
$$+\Big(\frac{15}{1-x}C_1^0(x) + 2C_3^0(x) - 2xC_4^0(x)\Big)\ \mathrm{H}_1(z)$$
$$+\Big(\frac{15(x-2)}{1-x}C_1^0(x) + 20C_2^0(x) + 2(3x-7)C_3^0(x) + 4xC_4^0(x)\Big)\mathrm{H}_{1/x}(z)$$
$$-\frac{15(1-2z)}{xz(1-z)}C_1^0(x) + \frac{4(13xz-1)}{xz}C_2^0(x) + \frac{2\big(13xz^2 - 17xz + 3x + z\big)}{xz(1-z)}C_3^0(x)$$
$$+\frac{2(xz-2z+1)}{z(1-z)}C_4^0(x) - 4C_2^1(x) + 2C_3^1(x)\Bigg) + \mathcal{O}\big(\epsilon^0\big)$$

where

- Hyperlogarithms (Panzer '15 for overview)

$$\mathrm{H}_{a,\vec{w}}(z) = \int_0^z \frac{\mathrm{d}z'}{z'-a}\ \mathrm{H}_{\vec{w}}(z')$$

- Integration constants
  - $C_1^0(x), C_2^0(x), C_3^0(x), C_4^0(x), C_2^1(x), C_3^1(x)$ — unknown functions

I.3 Integration Constants for $F(x,z,\epsilon)$

$$F_4^\star(x,\epsilon) = \int_0^1 \mathrm{d}z\, f_4(z)\, F_4(x,z,\epsilon)$$

Calculate RHS

- *direct integration*
  - `HyperInt` Panzer '14
  - Mellin moments

i	1	2	3	4	5	6	7	8	9	10	11
f_i	1	1	z	$z\,(1-z)$	1	z^2	$1-z$	z	z	z	$z\,(1-z)$

Calculate LHS

- *IBP reduction*
- *differential equations*

$$F_4^\star(x,\epsilon) = \int_0^1 \mathrm{d}z\, z\,(1-z) \int \frac{\mathrm{dPS}(3;\, x,z)}{D_2\, D_6}$$

  - phase space before x-integration

$$\mathrm{dPS}(3;\, x,z) = x\ q\cdot k_3\ \delta\left(1-x-(q-k_1)^2\right)\ \delta\left(x\,z\,q\cdot k_3 - k_1\cdot k_3\right)\ \mathrm{dPS}(3)$$

  - phase space after x-integration

$$\mathrm{dPS}(3;\, x) = \int_0^1 \mathrm{d}z\ \mathrm{dPS}(3;\, x,z) = \mathrm{dPS}(3)\,\delta\left(1-x-(q-k_1)^2\right)$$

II.1 IBP Reduction for $F^{\star}(x,\epsilon)$

Cutkosky rules for Cut Propagators

$$\delta\big(p_1^2\big)\,\delta\big(p_2^2\big)\,\delta\big(p_3^2\big)\,\delta\big((q-p_1-p_2-p_3)^2\big)\,\delta\big(x-2\,q\cdot p_1\big)\,\xcancel{\delta\big(x\,z\,q\cdot p_3-p_1\cdot p_3\big)}$$

Reduction

- run `LiteRed`
  - 12 masters $G(x,\epsilon)$

$$\begin{array}{llll} G_1=\{\} & G_2=\{2\} & G_3=\{7\} & G_4=\{2,7\} \\ G_5=\{2,\cancel{6},7\} & G_6=\{1,2\} & G_7=\{2,3,4,7\} & G_8=\{5,7\} \\ G_9=\{2,4,5\} & G_{10}=\{2,4,5,7\} & G_{11}=\{3,5,7\} & G_{12}=\{1,4,5,7\} \end{array}$$

  - denominators

$$\begin{array}{llll} D_1=(k_2+k_3)^2 & D_2=(q-k_2)^2 & D_3=(q-k_1-k_2)^2 & \\ D_4=(q-k_1-k_3)^2 & D_5=(q-k_2-k_3)^2 & D_6=\big(k_1^2\big)_{\mathrm{x}} & D_7=q\cdot k_3 \end{array}$$

II.1 IBP Reduction for $F^\star(x,\epsilon)$

Cutkosky rules for Cut Propagators

$$\delta\big(p_1^2\big)\,\delta\big(p_2^2\big)\,\delta\big(p_3^2\big)\,\delta\big((q-p_1-p_2-p_3)^2\big)\,\delta\big(x-2\,q\cdot p_1\big)\,\cancel{\delta\big(x\,z\,q\cdot p_3-p_1\cdot p_3\big)}$$

Reduction

$$F_4^\star(x,\epsilon)=\frac{(2-3\epsilon)\left(x+5\epsilon x-2\epsilon^2\,(8-7x)\right)}{4\,\epsilon^2\,x^2\,(4-5x)}\,G_1(x,\epsilon)$$
$$+\frac{x\,(1-x)+\epsilon\left(16-33x+15x^2\right)-\epsilon^2\left(48-82x+26x^2\right)}{4\,\epsilon\,x^2\,(4-5x)}\,G_2(x,\epsilon)$$
$$+\frac{(1-2\epsilon)\left(x-2\epsilon\,(2-x)\right)}{4\,\epsilon\,x\,(4-5x)}\,G_3(x,\epsilon)-\frac{4-7x+2x^2+\epsilon\,x\,(4-2x)}{4\,x\,(4-5x)}\,G_4(x,\epsilon)$$
$$-\frac{3\,x\,(1-x)}{4\,(4-5x)}\,G_5(x,\epsilon)$$

II.2 Differential Equations for $G(x,\epsilon)$

ODE Definition

$$\frac{\mathrm{d}\vec{G}(x,\epsilon)}{\mathrm{d}x} = \mathbb{M}(x,\epsilon)\,\vec{G}(x,\epsilon)$$

- x as free variable
- IBP rules (step II.1)
- 12×12 matrix
- alphabet

$$x, \quad x-1, \quad x-2$$

Epsilon form

- `Fuchsia` (Lee algorithm)

$$\frac{\mathrm{d}\vec{\hat{G}}(x,\epsilon)}{\mathrm{d}x} = \epsilon\,\mathbb{M}(x)\,\vec{\hat{G}}(x,\epsilon), \quad \text{where} \quad \vec{G}(x,\epsilon) = \boxed{\mathbb{T}(x,\epsilon)}\,\vec{\hat{G}}(x,\epsilon)$$

Solutions for $\vec{\hat{G}}(x,\epsilon)$

- any order in ϵ

II.2 Example: $G_5(x,\epsilon)$

$$
\begin{aligned}
G_5(x,\epsilon) = {} & \frac{2}{x\,\epsilon^2}\left(30C_1^0+6C_2^0+6C_3^0+(14-35x)C_4^0-2C_5^0\right)+\frac{1}{x\,\epsilon}\Big(-390C_1^0+60C_1^1-78C_2^0 \\
& +12C_2^1-78C_3^0+12C_3^1-(182-455x)C_4^0+(28-70x)C_4^1+26C_5^0-4C_5^1 \\
& +\Big((60-120x)C_1^0+(132-144x)C_2^0-(48-36x)C_3^0-(112-84x)C_4^0 \\
& +(16-12x)C_5^0\Big)\mathrm{H}_0(x)+\Big((-480+120x)C_1^0-(96-144x)C_2^0-36(1+x)C_3^0 \\
& +(-224+336x)C_4^0+(-8+12x)C_5^0\Big)\mathrm{H}_1(x)\Big)+\mathcal{O}\left(\epsilon^0\right)
\end{aligned}
$$

where

- Hyperlogarithms

$$
\mathrm{H}_{a,\vec{w}}(z)=\int_0^z \frac{\mathrm{d}z'}{z'-a}\,\mathrm{H}_{\vec{w}}(z')
$$

- Integration constants
  - $C_1^0, C_2^0, C_3^0, C_4^0, C_5^0, C_1^1, C_2^1, C_3^1, C_4^1, C_5^1$ — unknown constants

II.3 Integration Constants for $G(x,\epsilon)$

$$G_5^\star(\epsilon) = \int_0^1 \mathrm{d}x\, g_5(x)\, G_5(x,\epsilon)$$

Calculate RHS

- direct integration
  - `HyperInt`
  - Mellin moments

i	1	2	3	4	5	6	7	8	9	10	11	12
g_i	1	1	1	1	x	1	$(1-x)^2$	1	x	$x(1-x)$	$1-x$	$1-x$

Calculate LHS

- *IBP reduction*
- *direct integration*

$$G_5^\star(\epsilon) = \int_0^1 \mathrm{d}x\, x \int \frac{\mathrm{dPS}(3;\, x)}{D_2\, D_6\, D_7}$$

  - phase space before x-integration

$$\mathrm{dPS}(3;\, x) = \mathrm{dPS}(3)\, \delta\left(1 - x - (q - k_1)^2\right)$$

  - phase space after x-integration

$$\int_0^1 \mathrm{d}x\, \mathrm{dPS}(3;\, x) = \mathrm{dPS}(3)$$

III.1 IBP Reduction for $G^{\star}(\epsilon)$

Cutkosky rules for Cut Propagators

$$\delta\left(p_1^2\right)\delta\left(p_2^2\right)\delta\left(p_3^2\right)\delta\left((q-p_1-p_2-p_3)^2\right)\cancel{\delta\left(x-2\,q\cdot p_1\right)\delta\left(x\,z\,q\cdot p_3-p_1\cdot p_3\right)}$$

Reduction

- run `LiteRed`
  - 2 masters $\vec{H}(\epsilon)$

$$H_1=\{\}\quad H_2=\{1,2,7\}$$

  - denominators

$$D_1=(k_2+k_3)^2 \quad D_2=(q-k_2)^2 \quad D_3=(q-k_1-k_2)^2$$
$$D_4=(q-k_1-k_3)^2 \quad D_5=(q-k_2-k_3)^2 \quad D_6=\left(k_1^2\right)_{\mathrm{x}} \quad D_7=q\cdot k_3$$

- example

$$G_5^{\star}(\epsilon)=-\frac{2\,(2-3\epsilon)\,(3-4\epsilon)\left(1-7\epsilon+30\epsilon^2-36\epsilon^3\right)}{3\,\epsilon^2\left(1-5\epsilon+6\epsilon^2\right)}\,H_1(\epsilon)$$

Direct Calculation

- H_1 is known in closed-form GehrmannDe Ridder et al. '03

$$H_1(\epsilon) = \frac{1}{12} + \frac{59}{72}\epsilon + \left(\frac{2\,263}{432} - \frac{2}{3}\zeta_2\right)\epsilon^2 + \left(\frac{72\,023}{2\,592} - \frac{59}{9}\zeta_2 - \frac{13}{6}\zeta_3\right)\epsilon^3 + \left(\frac{2\,073\,631}{15\,552} - \frac{2\,263}{54}\zeta_2 - \frac{767}{36}\zeta_3 + \frac{1}{12}\zeta_4\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)$$

- H_2 is more effortful, but doable

$$H_2(\epsilon) = -\frac{4\zeta_3}{\epsilon} - 42\zeta_4 + \mathcal{O}(\epsilon)$$

II.4 Results for $G(x,\epsilon)$

- now we know
  - $G^\star(\epsilon)$ functions (stage III)
  - integration constants
- final result

$$
\begin{aligned}
G_5(x,\epsilon) = \frac{1}{3x}\Bigg[& -\frac{1}{\epsilon^2} + \frac{\mathrm{H}_0(x)+4\,\mathrm{H}_1(x)}{\epsilon} - (7-6x)\,\mathrm{H}_{0,0}(x) - 2(5-3x)\,\mathrm{H}_{0,1}(x) \\
& - 2(2+3x)\,\mathrm{H}_{1,0}(x) - 2(5+3x)\,\mathrm{H}_{1,1}(x) - 2(1-3x)\,\zeta_2 + \Big((61-54x)\,\mathrm{H}_{0,0,0}(x) \\
& + (46-36x)\,\mathrm{H}_{0,0,1}(x) + 4\,\mathrm{H}_{0,1,0}(x) + 28\,\mathrm{H}_{0,1,1}(x) - 18x\,\mathrm{H}_{0,1,1}(x) + 4\,\mathrm{H}_{1,0,0}(x) \\
& + 18x\,\mathrm{H}_{1,0,0}(x) + 16\,\mathrm{H}_{1,0,1}(x) + 4\,\mathrm{H}_{1,1,0}(x) + 36x\,\mathrm{H}_{1,1,0}(x) + 10\,\mathrm{H}_{1,1,1}(x) \\
& + 54x\,\mathrm{H}_{1,1,1}(x) + \zeta_2\,\big(38\,\mathrm{H}_0(x) - 36x\,\mathrm{H}_0(x) - 16\,\mathrm{H}_1(x)\big) + (36-18x)\,\zeta_3 \Big)\,\epsilon \Bigg] + \mathcal{O}\big(\epsilon^2\big)
\end{aligned}
$$

I.4 Results for $F(x,z,\epsilon)$

- now we know
  - $F^{\star}(x,\epsilon)$ functions (stage II)
  - integration constants
- final result

$$
\begin{aligned}
F_{11}(x,z,\epsilon) = {} & \frac{4}{3(1-x)x^2(1-z)z}\Bigg[\frac{3+x-4xz}{2\epsilon^2}+\frac{1}{\epsilon}\Big(-2(3+x-4xz)\mathrm{H}_1(x)-(6-x \\
& -5xz)\mathrm{H}_0(x)-(3+x-4xz)\mathrm{H}_0(z)-(3-x-2xz)\mathrm{H}_1(z)+6(1-xz)\mathrm{H}_{1/x}(z)\Big) \\
& +9(1-x)\mathrm{H}_{0,1}(x)+8(3+x-4xz)\mathrm{H}_{1,1}(x)+(24-7x-17xz)\mathrm{H}_{0,0}(x)+\Big(2(6-x \\
& -5xz)\mathrm{H}_0(z)+4(3-x-2xz)\mathrm{H}_1(z)-15(1-xz)\mathrm{H}_{\frac{1}{x}}(z)-3(1-xz)\mathrm{H}_{\frac{1}{x(2-x)}}(z)\Big)\mathrm{H}_0(x) \\
& +\mathrm{H}_0(1-x)\Big(5(3+x-4xz)\mathrm{H}_0(x)+4(3+x-4xz)\mathrm{H}_0(z)+4(3-x-2xz)\mathrm{H}_1(z) \\
& -21(1-xz)\mathrm{H}_{\frac{1}{x}}(z)-3(1-xz)\mathrm{H}_{\frac{1}{x(2-x)}}(z)\Big)+2(3+x-4xz)\mathrm{H}_{0,0}(z)+2(3-x \\
& -2xz)\mathrm{H}_{0,1}(z)-12(1-xz)\mathrm{H}_{0,\frac{1}{x}}(z)+2(3-x-2xz)\mathrm{H}_{1,0}(z)+2(3-2x-xz)\mathrm{H}_{1,1}(z) \\
& -6(2-x-xz)\mathrm{H}_{1,\frac{1}{x}}(z)-9(1-xz)\mathrm{H}_{\frac{1}{x},0}(z)-6(1-xz)\mathrm{H}_{\frac{1}{x},1}(z)+15(1-xz)\mathrm{H}_{\frac{1}{x},\frac{1}{x}}(z) \\
& -3(1-xz)\mathrm{H}_{\frac{1}{x(2-x)},0}(z)+3(1-xz)\mathrm{H}_{\frac{1}{x(2-x)},\frac{1}{x}}(z)+2(3-5x+2xz)\zeta_2\Bigg]+\mathcal{O}(\epsilon)
\end{aligned}
$$

Summary

— Fuchsia project —

- http://github.com/gituliar/fuchsia
- arXiv: 1607.0079, 1701.04269, 1711.05549
- open-source (Python, SageMath, Maxima)

Fuchsia works with multiple variables

Thank you!