Top quark pair production at threshold and four-loop on-shell renormalization

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DESY

in collaboration with

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Outline

1. Top pair production at threshold
2. $\overline{\rm MS}$ – on-shell relation
3. wave function renormalization: $Z_2^{\rm OS}$
4. Renormalon ambiguity of the pole mass
5. Conclusions

Framework

Physics of bound states of heavy particles and threshold phenomena best described within an effective field theory:
Non-Relativistic QCD (NRQCD) [Caswell,Lepage'86; Bodwin,Braaten,Lepage'95]
and potential Non-Relativistic QCD (pNRQCD)

[Beneke,Smirnov'97; Pineda,Soto'98; Brambilla,Pineda,Soto,Vairo'00]

or vNRQCD [Luke,Manohar,Rothstein'00; Hoang,Stewart'03]

Prominent applications are

- production of $t\bar{t}$ pairs
- decays of $b\bar{b}$ bound states
- $b\bar{b}$ sum rules
- positronium spectra

Master Formula

$$R = \frac{\sigma_{\text{tot}}(e^+e^- \to t\bar{t})}{\sigma_0}$$
$$= \frac{18\pi}{m_t^2} \text{Im} \left\{ c_v \left[c_v - \frac{E}{m_t} \left(c_v + \frac{d_v}{3} \right) \right] G(E) + \cdots \right\}$$

Ingredients:

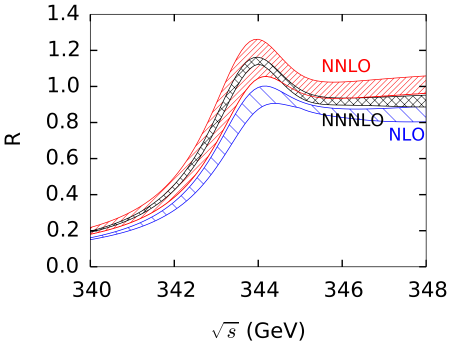
1. NRQCD matching coefficients: c_v, d_v

   [Kallen,Sabry'55;Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97;Marquard,Piclum,Seidel,Steinhauser'06'08'14]

2. pNRQCD Green function: $G(E)$

   [Beneke,Kiyo,Penin'07; Beneke,Kiyo'08; Beneke,Kiyo,Schuller'13]

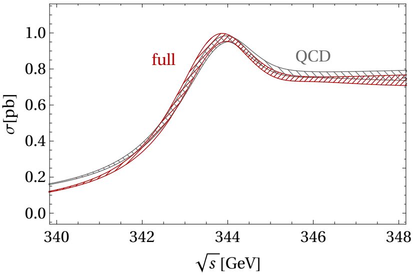
Total cross section – pure QCD

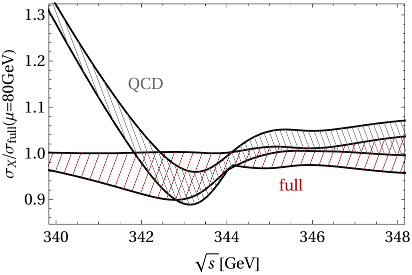


$m_t = 171.5\,\mathrm{GeV}$, $\Gamma_t = 1.33\,\mathrm{GeV}$,
$50\,\mathrm{GeV} \leq \mu \leq 350\,\mathrm{GeV}$

- very good convergence of perturbative series below threshold
- above threshold -8% shift driven by large negative corrections to c_v

QCD + EW





[Beneke,Maier,Rauh,Ruiz-Femenia '17]

Prelim. studies indicate an error $\pm 45\mathrm{MeV}$ from the theory prediction.

Renormalization constants and Quark masses

- Quark masses can be measured with high precision using different renormalization schemes
  $\Rightarrow$ have to be able to translate between them
- pole masses suffer from (infrared) renormalon problems
  $\Rightarrow$ try to quantify the effect
- Renormalization constants are on there own fundamental quantities of the given theory
- Central role: $Z_m^{\mathrm{MS}}/Z_m^{\mathrm{OS}}$, most threshold masses derived from pole mass

Setup of the calculation

- Need to calculate mass renormalization constant Z_m^{OS} by calculating four-loop on-shell integrals
- Together with the renormalization constant in the $\overline{\text{MS}}$-scheme $Z_m^{\overline{\text{MS}}}$

  [Chetyrkin '97; Larin, van Rittbergen, Vermaseren '97; Baikov,Chetyrkin,Kühn '14]

  we get for the relation between the two schemes

$$\left.\begin{aligned} m_{\text{bare}} &= Z_m^{\text{OS}} M \\ m_{\text{bare}} &= Z_m^{\overline{\text{MS}}} m \end{aligned}\right\} \Rightarrow m = M \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}}$$

  - 1-loop [Tarrach'81]
  - 2-loop [Gray,Broadhurst,Grafe,Schilcher'90]
  - 3-loop [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]

Scheme definitions

One considers the renormalized quark propagator

$$S_F(q) = \frac{-i\,Z_2}{\not{q} - Z_m m + \Sigma(q,m)}$$

with $\Sigma(q,m)$ the quark two-point function.
For the $\overline{\text{MS}}$ scheme we require

$$S_F(q) \quad \text{finite}$$

and for the on-shell scheme we require a pole at the position of the mass

$$S_F(q) \overset{q^2 \to M^2}{\longrightarrow} \frac{-i}{\not{q} - M}$$

Setup of the calculation

expand the quark two-point function

$$\Sigma(q,M) \approx M\,\Sigma_1(M^2,M) + (\not{q} - M)\left(2M^2\frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2} + \Sigma_2(M^2,M)\right) + \ldots$$

renormalization constants are then given by

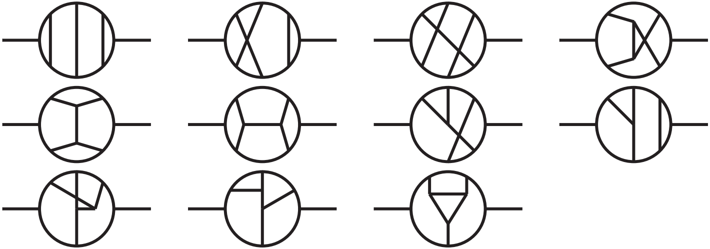
$$Z_m^{\rm OS} = 1 + \Sigma_1(M^2,M)\,,$$
$$\left(Z_2^{\rm OS}\right)^{-1} = 1 + 2M^2\frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2} + \Sigma_2(M^2,M)\,.$$

best to apply the projector

$$\mathrm{Tr}\left\{\frac{\not{Q} + M}{4M^2}\Sigma(q,M)\right\} = \Sigma_1(q^2,M) + t\Sigma_2(q^2,M)$$
$$= \Sigma_1(M^2,M) + \left(2M^2\frac{\partial}{\partial q^2}\Sigma_1(q^2,M)\Big|_{q^2=M^2} + \Sigma_2(M^2,M)\right)t$$

Setup of the calculation cont'd

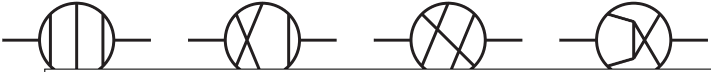
Need to calculate 4-loop on-shell diagrams of the form

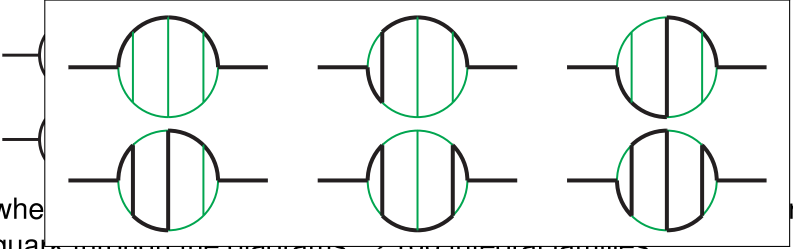


where we have to consider all possible ways to route the external quark through the diagrams ⇒ 100 integral families.

Setup of the calculation

Need to calculate 4-loop on-shell diagrams of the form





whe … nal
quark through the diagrams ⇒ 100 integral families.

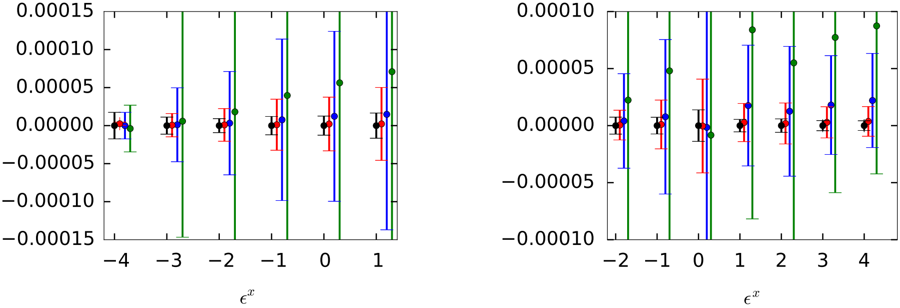
Setup of the calculation

Follow the *standard* procedure for multi-loop calculations

- exploit that the appearing integrals are not linear independent but related through Integration-By-Parts identities [Chetyrkin, Tkachov '81]
- reduce all appearing integrals to a small set of basis integrals using `FIRE` [Smirnov] or `Crusher` [PM,Seidel]
- evaluate the remaining basis integrals ($\mathcal{O}(350)$) using analytic or numerical techniques (Mellin-Barnes or Sector Decomposition)[`MB.m` [Czakon] `FIESTA` [Smirnov]]

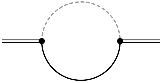
Numerics: Convergence of SD

Check the convergence of the results using Sector Decomposition when increasing statistics 500k $\rightarrow$ 500M points

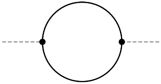


Numerics: Mellin-Barnes

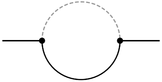
- Building blocks



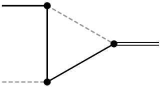
0-dim



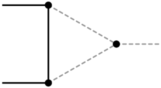
1-dim



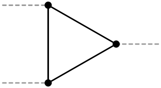
1-dim



2-dim



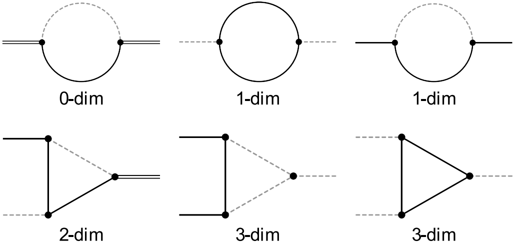
3-dim



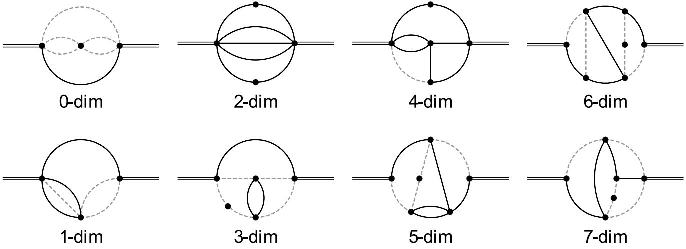
3-dim

Numerics: Mellin-Barnes

- Building blocks

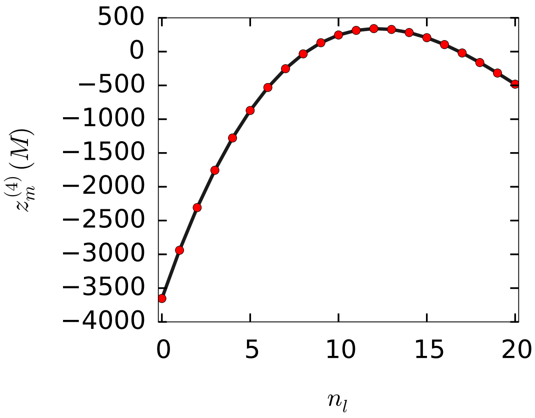


- Example Integrals



$\overline{\text{MS}}$–on-shell relation at four-loop order

$$
\begin{aligned}
z_m^{(4)} &= -3654.15 \pm 1.64 + (756.942 \pm 0.040) n_l \\
&\quad -43.4824 n_l^2 + 0.678141 n_l^3 \,.
\end{aligned}
$$



$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \to$ on-shell

$$
\begin{aligned}
m_t(m_t) &= M_t \left(1 - 0.4244\,\alpha_s - 0.9246\,\alpha_s^2 - 2.593\,\alpha_s^3 \right. \\
&\quad \left. -(8.949 \pm 0.018)\,\alpha_s^4\right) \\
&= 173.34 - 7.924 - 1.859 - 0.562 \\
&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann ’16]

$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \rightarrow$ on-shell

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&= 173.34 - 7.924 - 1.859 - 0.562 \\
&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

$$
\begin{aligned}
M_b &= m_b(m_b) \left(1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 \right. \\
&\quad \left. +(12.685 \pm 0.025)\,\alpha_s^4\right) \\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

Light quark mass dependence

- starts at 2 loops [Gray,Broadhurst,Grafe,Schilcher ’90]
- available up to 3 loops [Bekavac,Grozin,Seidel,Steinhauser ’07]
- for top quark
  - 11 MeV @ two loop
  - 16 MeV @ three loop
  - getting more important at higher orders due to renormalon enhancement [Hoang ’17]

Threshold masses, e.g. PS mass

$$m^{\text{PS}} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi}(1 + \alpha_s \ldots)$$

need static potential @ three loops

[Smirnov,Smirnov,Steinhauser ’09; Anzai ,Kiyo,Sumino ’09]

$$\begin{aligned} m_t^{\text{PS}}(\mu_f = 80\ \text{GeV}) &= 163.508 + (7.531 - 3.685) \\ &\quad + (1.607 - 0.989) + (0.495 - 0.403) \\ &\quad + (0.195 - 0.211 \pm 0.0004)\ \text{GeV} \\ &= 163.508 + 3.847 + 0.618 + 0.092 \\ &\quad - (0.016 \pm 0.0004)\ \text{GeV} \end{aligned}$$

- large cancellations between contributions from OS-MS and PS-OS
- good convergence

Z_2^{OS} Overview

- final missing building block to complete the on-shell renormalization procedure of QCD at four loops.
- gauge dependence starts only at 3 loops

[Broadhurst,Gray,Schilcher]

[Melnikov,van Ritbergen;Marquard,Mihaila,Piclum,Steinhauser]

- computationally more involved due to additional derivative and gauge dependence
- N.B.: all results are preliminary

Z_2^{OS} 4-loop results, QCD, i.e. $N_c = 3$

ξ^0

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
n_l^0	-1.7728 ± 0.00040	-27.666 ± 0.0041	-317.093 ± 0.029	-3142.15 ± 0.33	-28709.9 ± 3.2
n_l^1	0.460936 ± 0.000016	6.69143 ± 0.00023	74.6540 ± 0.0013	696.6612 ± 0.0076	6174.390 ± 0.084
n_l^2	-0.039931	-0.51572	-5.5055	-48.777	-418.93
n_l^3	0.0011574	0.012539	0.12676	1.0711	8.916

- Not showing $\mathcal{O}(\xi)$ terms
- As usual largish cancellations between hard 4-loop part and counter term contributions

$\overline{\text{MS}}$–on-shell relation beyond 4 loops

$$m_P = m(\mu_m)\left(1 + \sum_{n=1}^{\infty} c_n(\mu, \mu_m, m(\mu))\, \alpha_s^n(\mu)\right)$$

for large n

$$c_n(\mu, \mu_m, m(\mu_m)) \underset{n\to\infty}{\longrightarrow} N c_n^{(\text{as})}(\mu, m(\mu_m)) \equiv N \frac{\mu}{m(\mu_m)}\, \tilde{c}_n^{(\text{as})}\,,$$

[Beneke,Braun '94;Beneke '94 '99]

where

$$\tilde{c}_{n+1}^{(\text{as})} = (2b_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \cdots\right).$$

b_0, b, s_1, s_2: Combinations of coefficients of the β-function.

How well does the asypt. formula work?

Fit N to 1, . . . , 4-loop term and compare

n_l	$c_1/c_1^{(\mathrm{as})}$	$c_2/c_2^{(\mathrm{as})}$	$c_3/c_3^{(\mathrm{as})}$	$c_4/c_4^{(\mathrm{as})}$	Δ_{34}
−1000000	0.6953	0.9624	0.9349	0.9714	0.038
−10	0.4744	0.7152	0.6898	0.7005	0.015
0	0.4377	0.6357	0.6130	0.5977	0.025
3	0.3954	0.6150	0.5723	0.5370	0.064
4	0.3633	0.6120	0.5522	0.5056	0.088
5	0.3143	0.6119	0.5244	0.4616	0.127
6	0.2436	0.6089	0.4818	0.3942	0.200
7	0.1474	0.5378	0.4084	0.2786	0.378
8	0.0098	0.0379	0.2719	0.0564	1.312
10	0.2684	−0.0916	−0.1108	−1.7228	1.758

$$\Delta_{34} = 2\,\frac{|c_3/c_3^{(\mathrm{as})} - c_4/c_4^{(\mathrm{as})}|}{|c_3/c_3^{(\mathrm{as})} + c_4/c_4^{(\mathrm{as})}|}.$$

Beyond 4 loops

Fit N to 4-loop term and take higher orders from asymptotic formula

j	$\tilde{c}_j^{(\mathrm{as})}$	$\tilde{c}_j^{(\mathrm{as})} \alpha_s^j$
5	0.985499×10^2	0.001484
6	0.641788×10^3	0.001049
7	0.495994×10^4	0.000880
8	0.443735×10^5	0.000854
9	0.451072×10^6	0.000942
10	0.513535×10^7	0.001164

Asymptotic series $\Rightarrow$ converges only up to $\approx$ 8 loop

5+ loops and remaining ambiguity

An asymptotic series f can (sometimes) be summed by using the Borel transform $B[f]$

$$f(\alpha_s) = \sum_{n=0}^{\infty} c_n \alpha_s^n \qquad \Rightarrow \qquad B[f](t) = \sum_{n=0}^{\infty} c_{n+1} \frac{t^n}{n!}$$

the Borel integral

$$\int_0^{\infty} \mathrm{d}t\, e^{-t/\alpha_s} B[f](t)$$

has the same series expansion as $f(\alpha_s)$ and the same value.

5+ loops and remaining ambiguity cont'd

In our case we have

$$c_{n+1} = (2b_0)^n n! \qquad \Rightarrow \qquad \int_0^\infty \mathrm{d}t\, e^{-t/\alpha_s} \frac{1}{1 - 2b_0 t}$$

Not integrable due to pole at $1 - 2b_0 t = 0$
Possible prescription for the integral:
Take principle value and assign ambiguity Im/π

$$\delta^{(5+)} m_P \quad = \quad 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \pm 0.010\,(\alpha_s) \pm 0.071 \text{ (ambiguity) GeV}$$

5+ loops and remaining ambiguity: light mass effects

- take 2-loop and 3-loop mass effects into account
- 4 loop: massless value using five-flavour theory
- 5 loop: massless value using four-flavour theory
- 6+ loops: massless value using three-flavour theory
- final estimate depends on $\Lambda^{(3)}_{\mathrm{QCD}}$

$$\begin{aligned}\delta^{(5+)} m_P &= 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \\ &\quad \pm 0.009\,(\alpha_s) \pm 0.108\text{ (ambiguity) GeV}\end{aligned}$$

Cmp. analysis by [Hoang,Lepenik, Preisser '17] using different prescription

$$\frac{1}{2} \sum_{c_n < f c_n^{min}} c_n \to 250\mathrm{MeV} \qquad \text{with} \quad f = 5/4$$

Conclusions

- Presented current status of the theory prediction for $t\bar{t}$ production @ threshold
- $\overline{\mathrm{MS}}$–on-shell relation
- renormalon ambiguity of the top quark
- on-shell wave function renormalization constant