Four-loop form factor in N = 4

/w Tobias Huber and Gang Yang,
arXiv:1705.03444 & arXiv:1711.08449

Rutger Boels
University of Hamburg

in short

fun result: a four loop form factor in N=4

$$\ldots + (0.0007 \pm 0.0186)\epsilon^{-3} + (1.60 \pm 0.19)\epsilon^{-2} + (-17.98 \pm 3.47)\epsilon^{-1}$$

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- at -2,-1: != 0 → speculation in literature
- can be computed at all → methods

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our approach is almost certainly wrong…

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perturbative QFT talks cheatsheet

“preprocessing”

generate integrand

reduce to simpler integrals

explicit integration

“postprocessing”

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- govern IR/UV divergences [long literature, ~70 - today]

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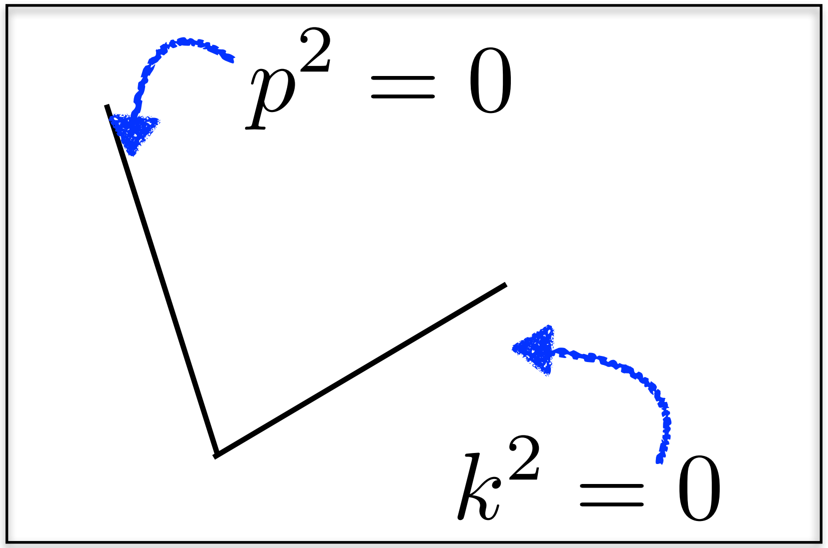
- example: light-like cusp anomalous dimension

I anomalous dimension of light like cusped Wilson line
II leading infrared divergence of amplitudes
III logarithmic growth of high-spin Wilson operators
IV related to gluon Regge trajectory
V appears in AdS/CFT (N=4)



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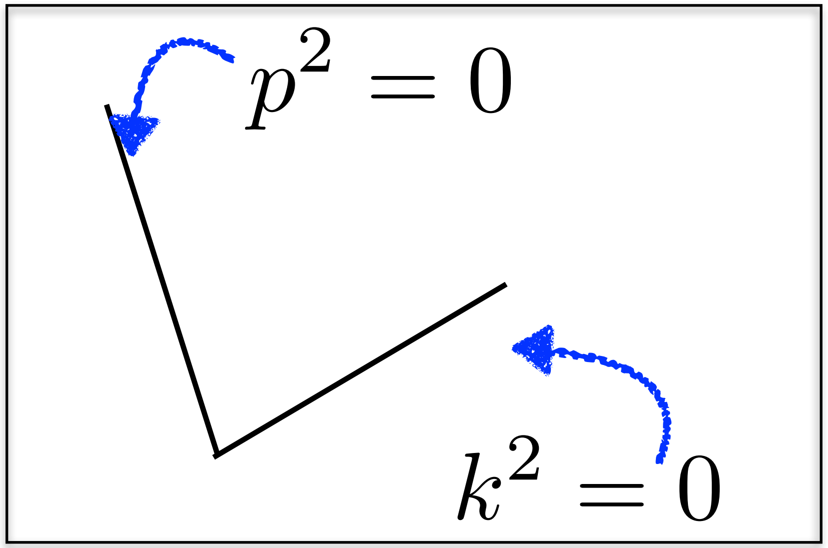

→ ample motivation to compute it!
(many approaches to compute it…)

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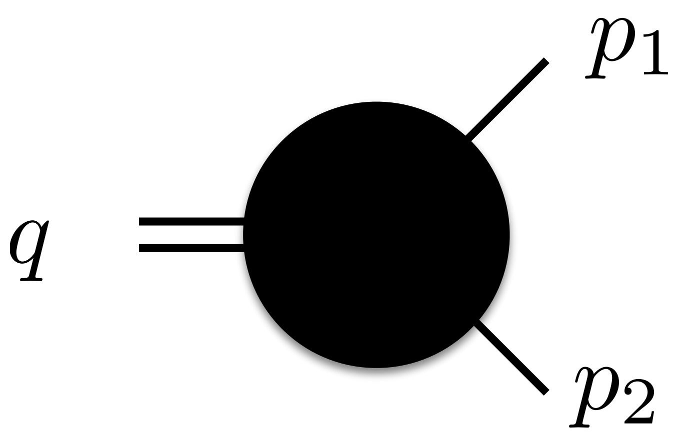

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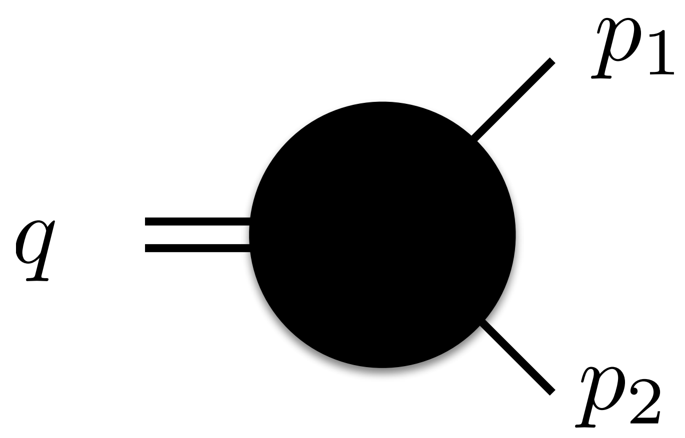

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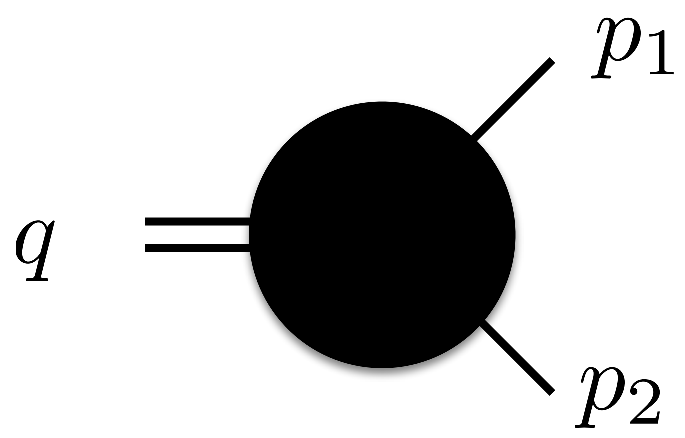

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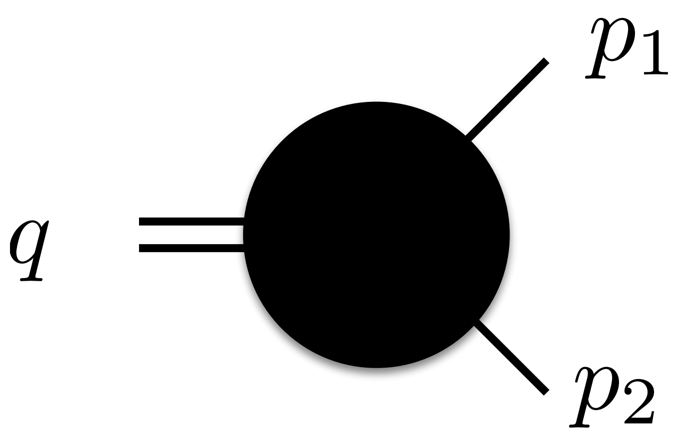

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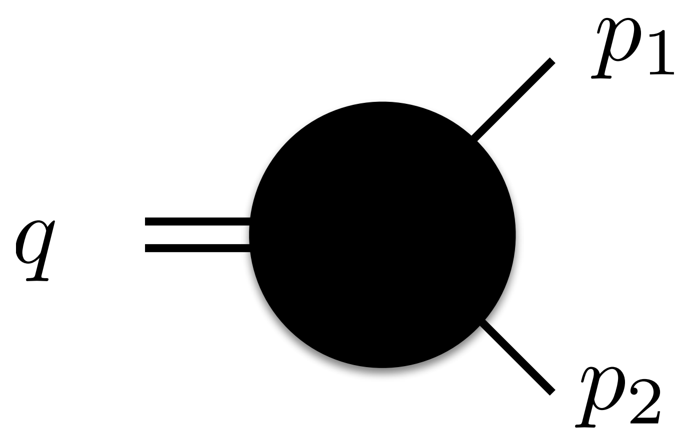

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- planar limit known exactly [Beisert-Eden-Staudacher, 06]

4 loop Sudakov form factor

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(see also more recent: [Moch-Ruijl-Ueda-Vermaseren-Vogt, 17], [Grozin-Henn-Stahlhofen, 17])

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- non-planar correction to the ‘Sudakov’ form factor in N=4 at four loops to at least leading divergent term: $\epsilon^{-2}, \epsilon^{-1}$
- is it zero? → numerics (may) suffice
- quite… → long-standing conjecture

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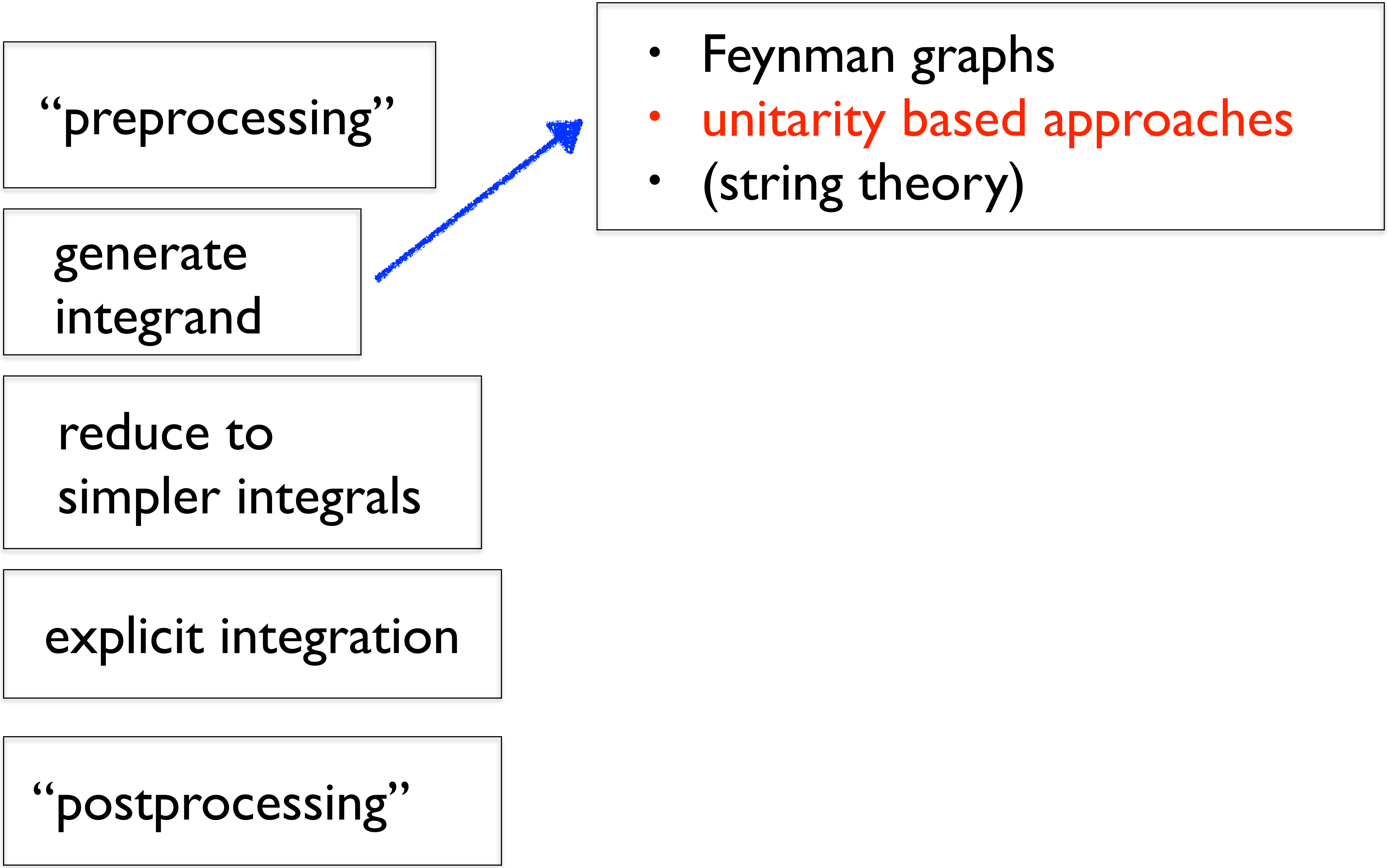

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- Feynman graphs
- unitarity based approaches
- (string theory)

Intermezzo: color kinematic duality

[Bern-Carrasco-Johannson, 08,10]

- write a gauge theory tree amplitude as:

$$\mathcal{A}_n = g_{ym}^{n-2} \sum_{\Gamma_i} \frac{n_i c_i}{s_i}$$

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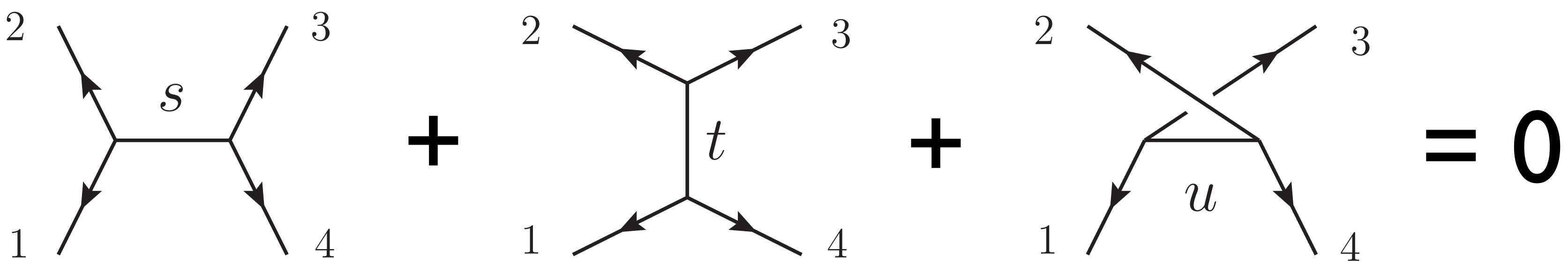

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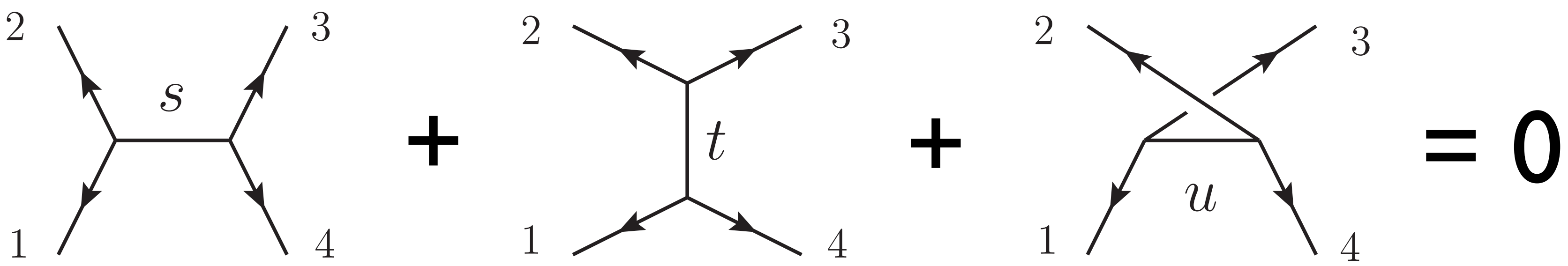

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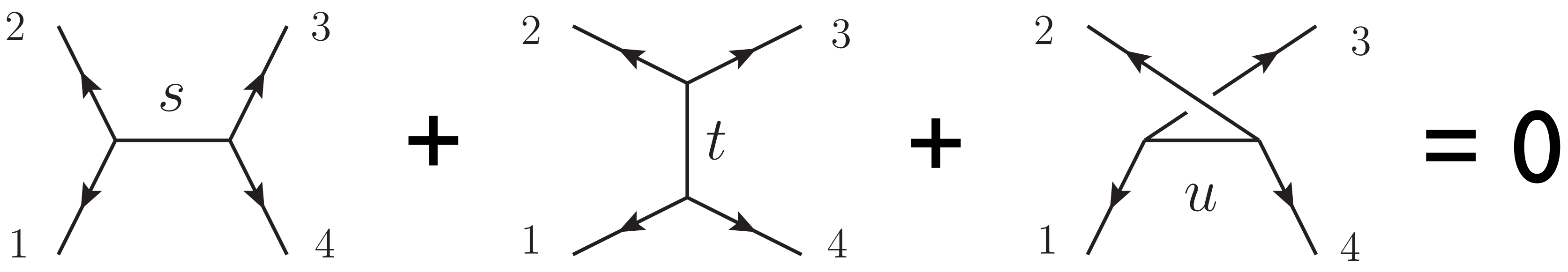

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- always possible at tree level, very similar looking loop level conjecture, see review in [Isermann, 13]

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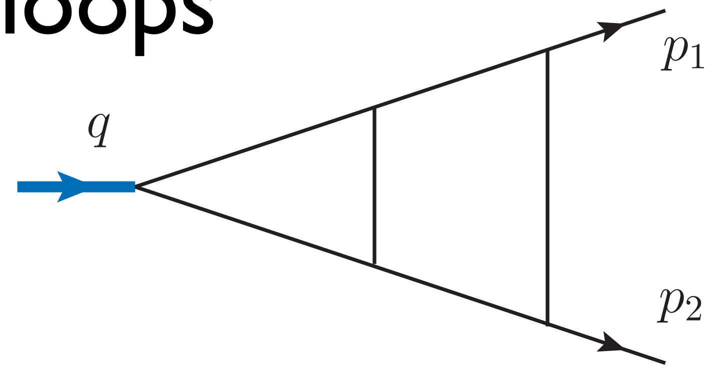

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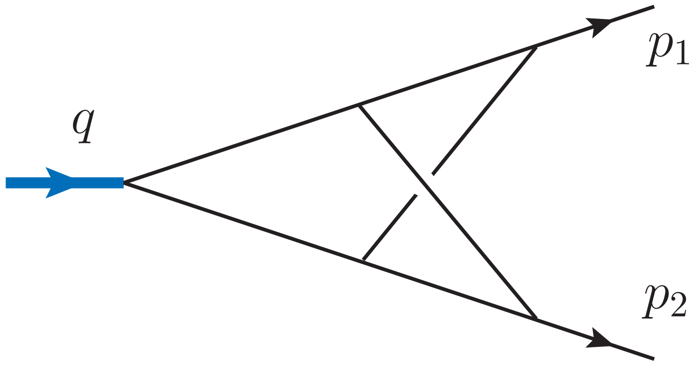

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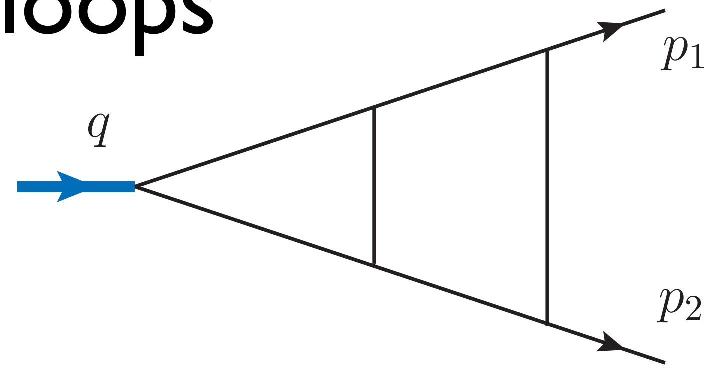

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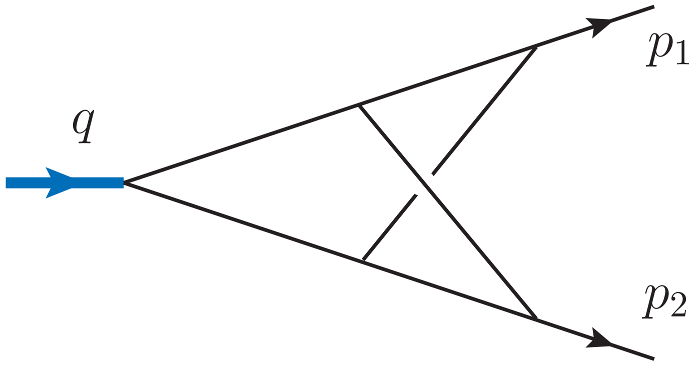

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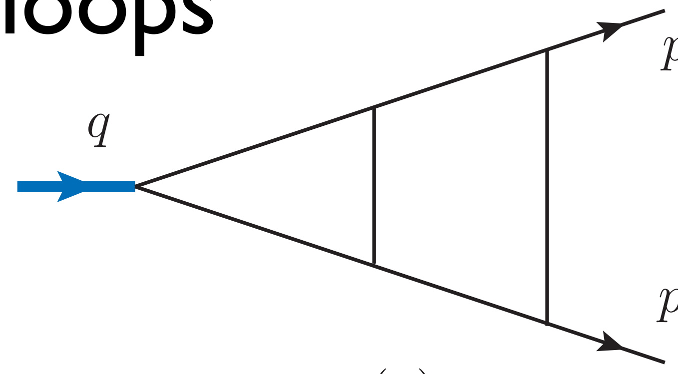

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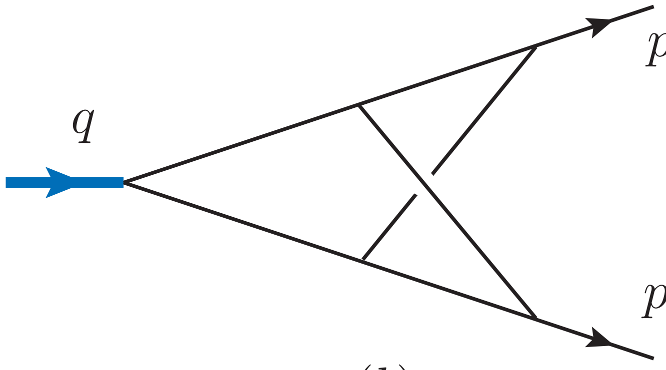

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[RB-Kniehl-Tarasov-Yang]

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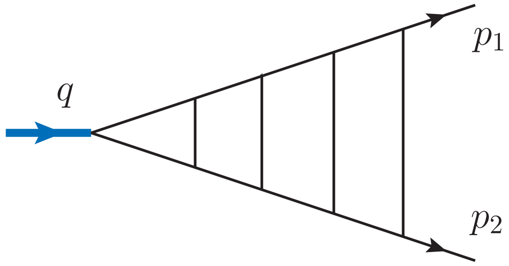

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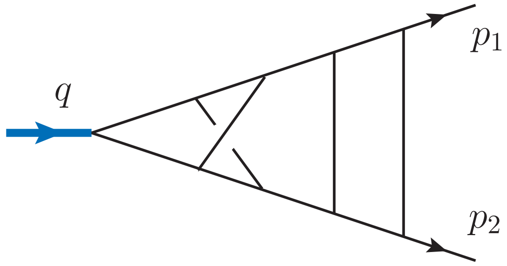

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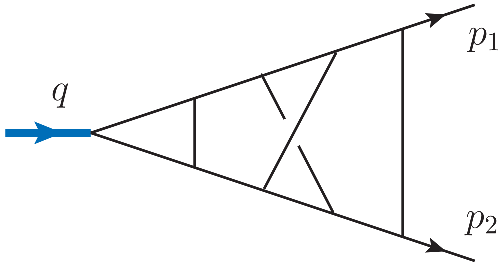

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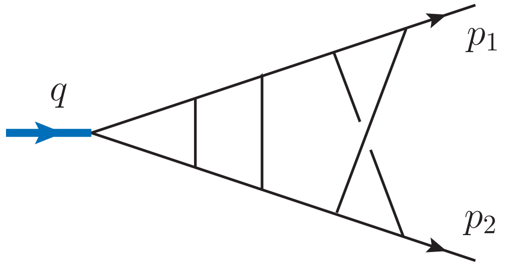

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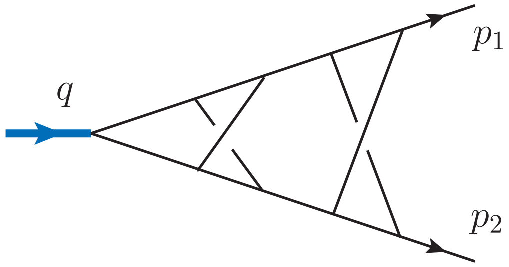

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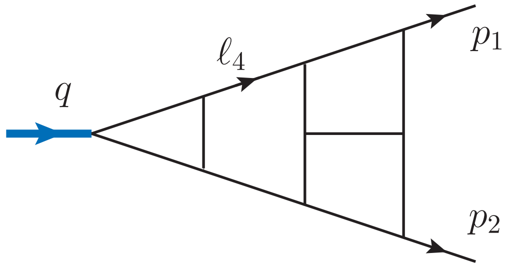

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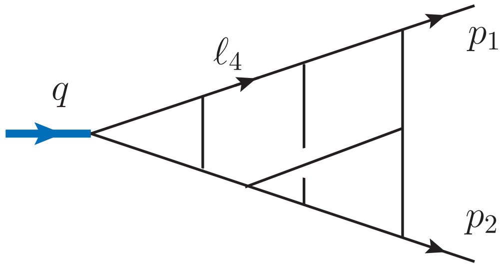

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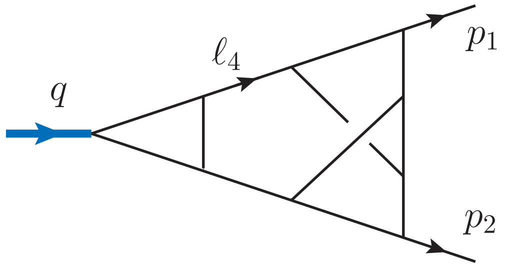

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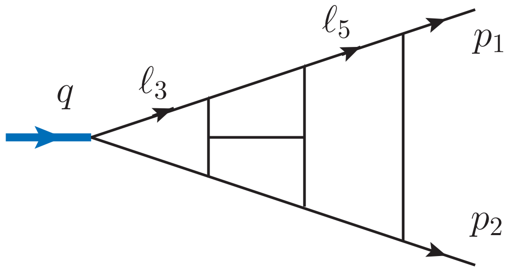

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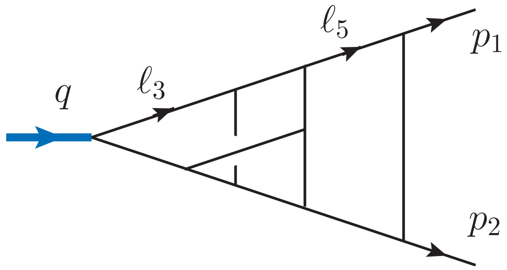

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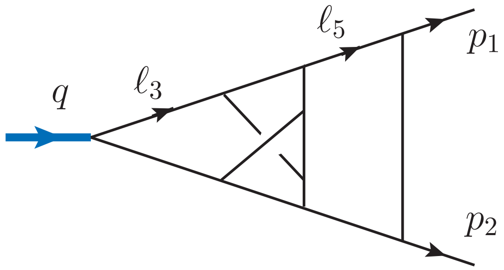

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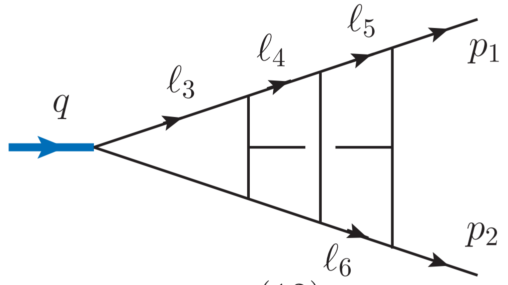

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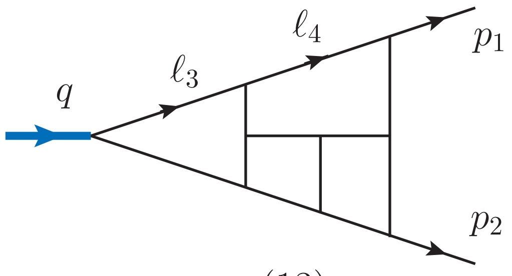

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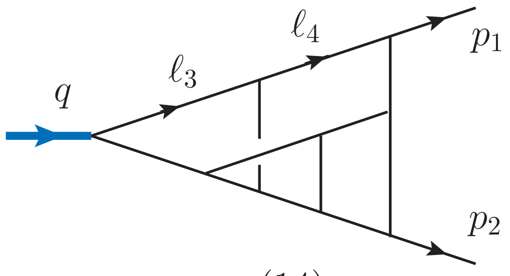

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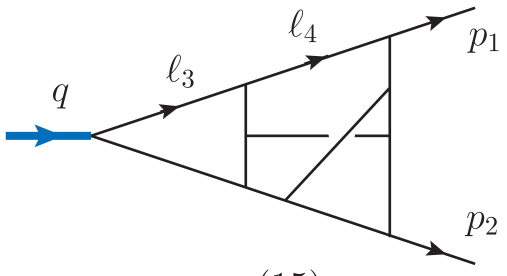

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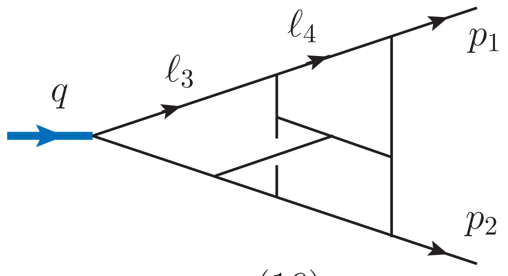

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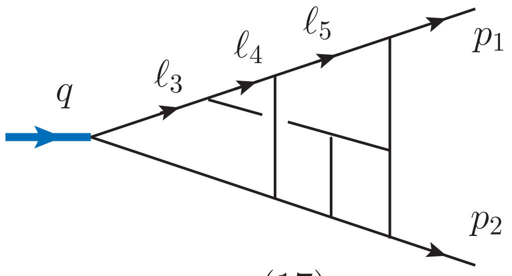

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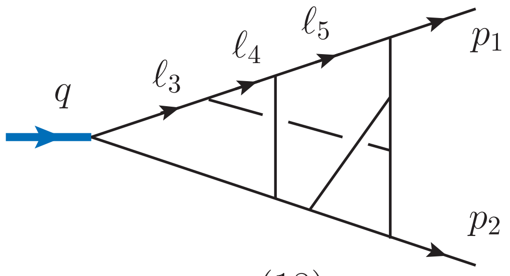

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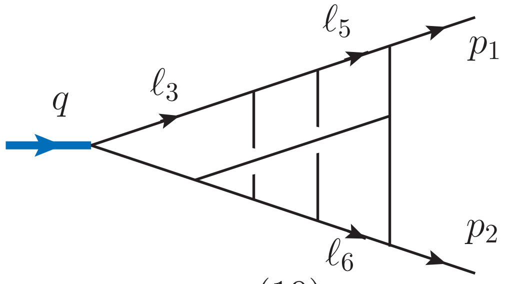

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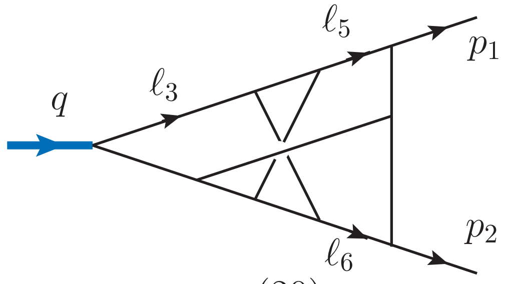

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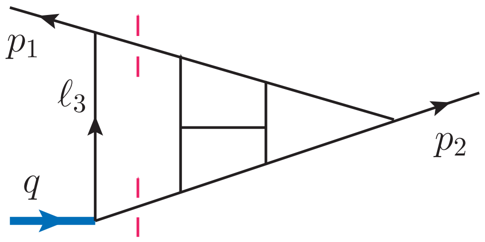

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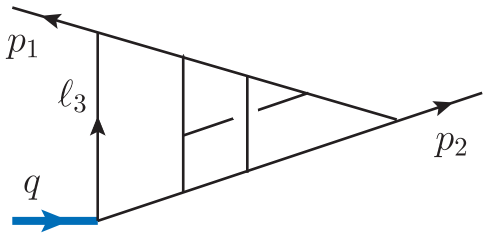

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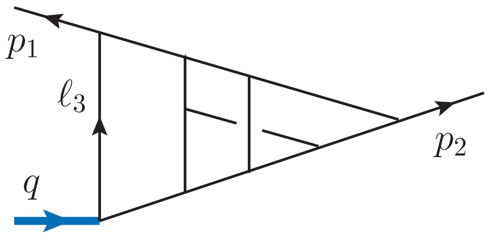

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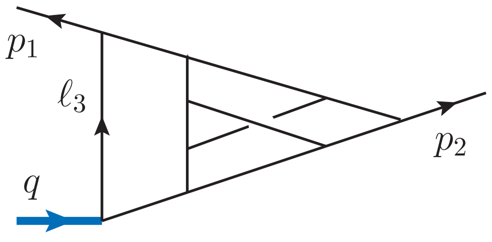

(34)

Application: form factors [RB-Kniehl-Tarasov-Yang]

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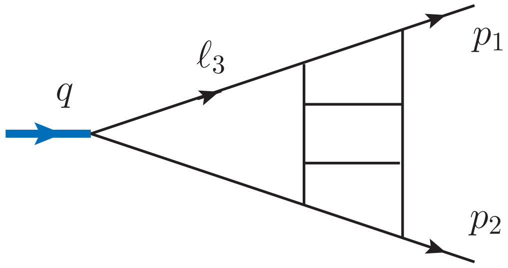

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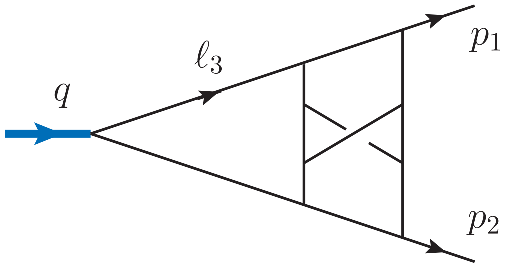

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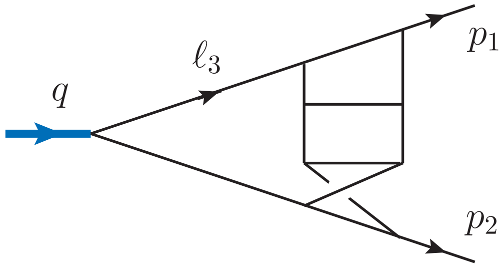

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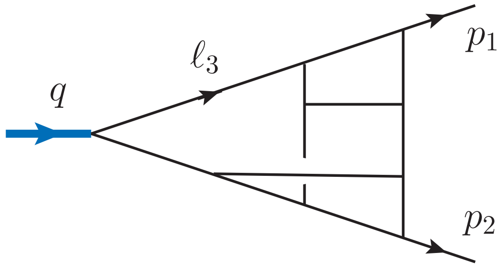

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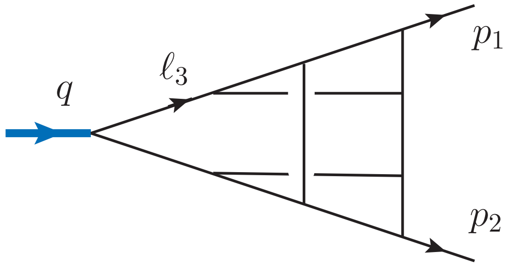

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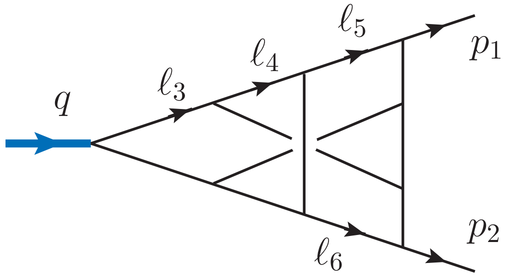

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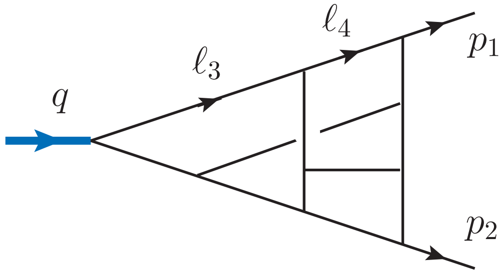

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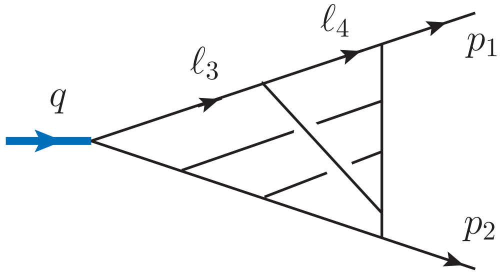

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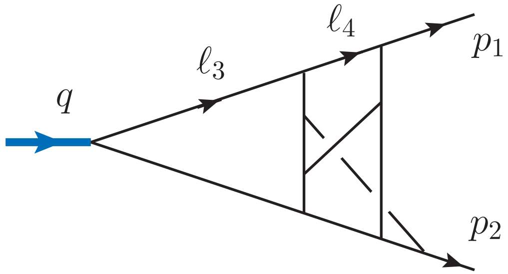

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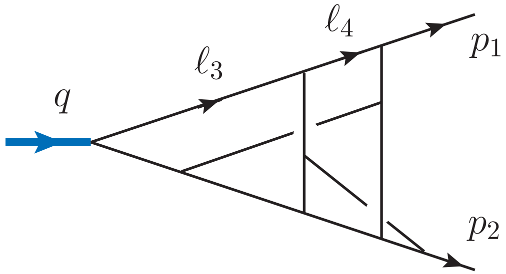

(30)

Application: form factors

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[Yang, 16]: five loop case

constructing the form factor integrand

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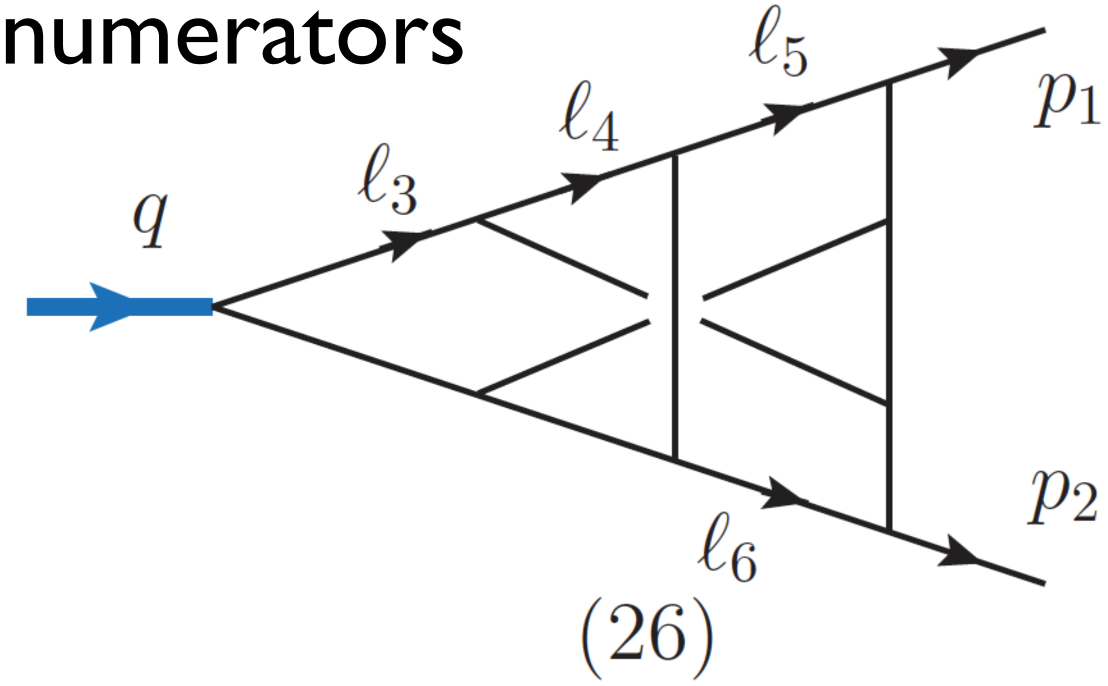
- 34 integrals
- 13 have a non-planar color part
- 10 are purely non-planar color

- mostly quadratic in 6 irreducible numerators
- topology 26: no internal boxes



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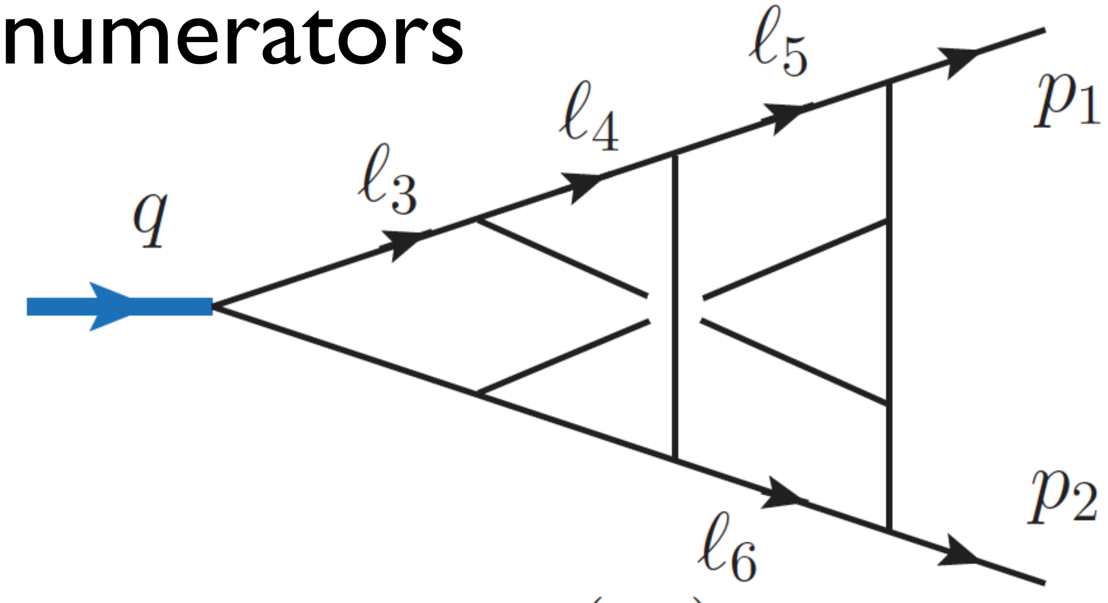


(26)

→ way to complicated!

constructing the form factor integrand

recent progress planar, Nf dependent pieces:

[Henn-Smirnov^2-Steinhauser, 16]
[Henn-Lee-Smirnov^2-Steinhauser, 16]
[Von Manteuffel-Schabinger, 16]
[Lee-Smirnov^2-Steinhauser, 17]
[Ahmed-Henn-Steinhauser, 17]
[Grozin-Henn-Stahlhofen, 17]

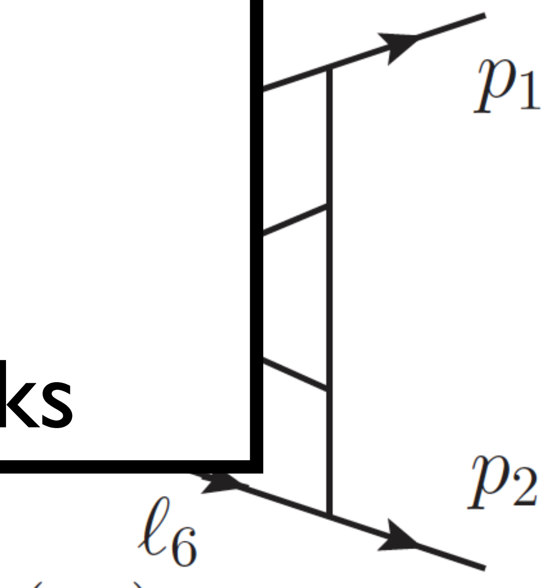
→ Volodya Smirnov, Peter Marquard's talks

Inte

- 3
- 1
- 1

- m
- to

→ way to complicated!



(26)

perturbative QFT talks cheatsheet

“preprocessing”

generate integrand

reduce to simpler integrals

explicit integration

“postprocessing”

perturbative QFT talks cheatsheet

“preprocessing”

generate integrand

reduce to simpler integrals

- integration-by-parts identities
- graph symmetries
- dimensional shifts
- choice of basis

explicit integration

“postprocessing”

simplification?

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- integration-by-parts identities

$$\int d^D l_1 \dots d^D l_L \ \frac{\partial}{\partial l_i^\mu}\,(\text{integrand}) = 0$$

- LARGE systems of linear equations, solution in terms of choice of master integrals
- Laporta algorithm, implemented in e.g. LiteRed, FIRE, Reduze, Kira, Air, private → Volodya's talk

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- here: Reduze ([Von Manteuffel-Studerus, 12]) in [Boels-Kniehl-Yang, 15]

two problems:

- too many hard master integrals
- epsilon dependent coefficients

$$\mathrm{FF} = \ldots + \left(\frac{\sim 1}{\epsilon^4} + \frac{\sim 10}{\epsilon^3} + \ldots \right) I_{\mathrm{master}}$$

good master integrals?

key idea: uniformly transcendental integrals are good

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in expansions of Feynman integrals certain constants always appear: multiple zeta values

e.g. $\zeta(n)$

not that many constants:

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- cf. [Gehrman-Henn-Huber, 11] at 3 loops

when is an integral uniformly transcendental?

example from [Henn-Smirnov-Smirnov-Steinhauser, 16]:

$$I_{12\text{ prop}} = \frac{1}{576} + \epsilon^2 \frac{1}{216}\pi^2 + \epsilon^3 \frac{151}{864}\zeta_3 + \epsilon^4 \frac{173}{10368}\pi^4 + \epsilon^5 \left[\frac{505}{1296}\pi^2\zeta_3 + \frac{5503}{1440}\zeta_5\right] + \\ + \epsilon^6 \left[\frac{6317}{155520}\pi^6 + \frac{9895}{2592}\zeta_3^2\right] + \epsilon^7 \left[\frac{89593}{77760}\pi^4\zeta_3 + \frac{3419}{270}\pi^2\zeta_5 - \frac{169789}{4032}\zeta_7\right]$$

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(differential equations)

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can you tell an integral is UT without integrating it?

- dLog form exist: certainly UT
- conjecture: constant ‘leading singularity’ integrals are UT

[Bern-Hermann-Litsey-Stankowicz-Trnka, 14]
[Henn-Smirnov-Smirnov-Steinhauser, 16]

finding UT integrals

express all loop momenta in a four D basis:

$$l^i = \alpha_1^i p_1 + \alpha_2^i p_2 + \alpha_3^i q_1 + \alpha_4^i q_2$$

consider integrand $I\left(\vec{\alpha}^i\right)$ in D=4

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integral property from integrand

- many multi residues possible (4*4=16 variables)
- pick random sequences: non-UT integrals tend to fail quickly
  → double or higher poles

finding UT integrals: algorithm

if non-simple pole appears in taking multi-residues:
integral certainly not UT

- take a set of integrals
- find any higher-pole-generating sequence of residues
- derive constraint on set of integrals to evade higher order residue → smaller set of integrals
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(some topologies have no candidates!)

express form factor in terms of UT integrals

- now known: form factor & a set of candidate UT integrals & IBP solution → in principle enough information

result: full form factor expressed in UT-candidate integrals

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  - advantage: fits easily in laptop memory!
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form factor is (likely) maximally transcendental

express form factor in terms of UT integrals

- want to put UT integrals into product form for easy input into FIESTA / MB

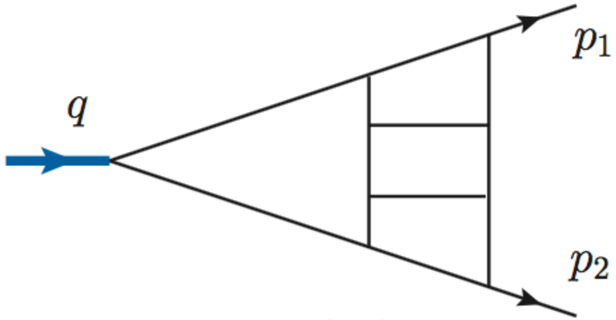
$$\sum c_i \mathrm{UTC}_i = (\text{quadratic in } l_i)\,(\text{quadratic in } l_i)$$

- brute force using Mathematica
- aim to minimise number of integrals for form factor

- found choice of 23 / 34 UT integrals non-planar/planar

- all passing >10.000 random residue checks separately

- dLog:



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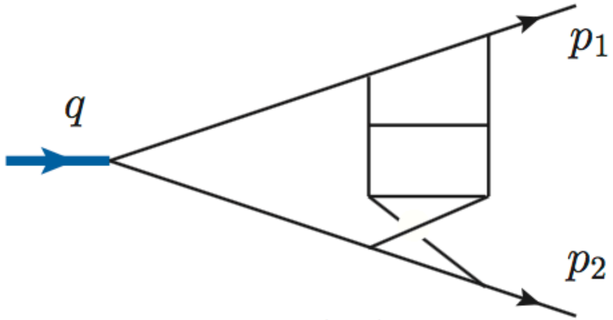

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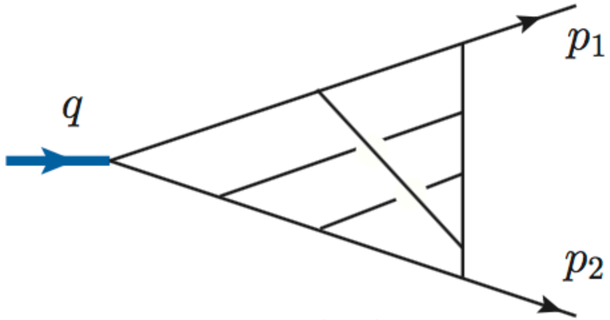


(28)

perturbative QFT talks cheatsheet

“preprocessing”

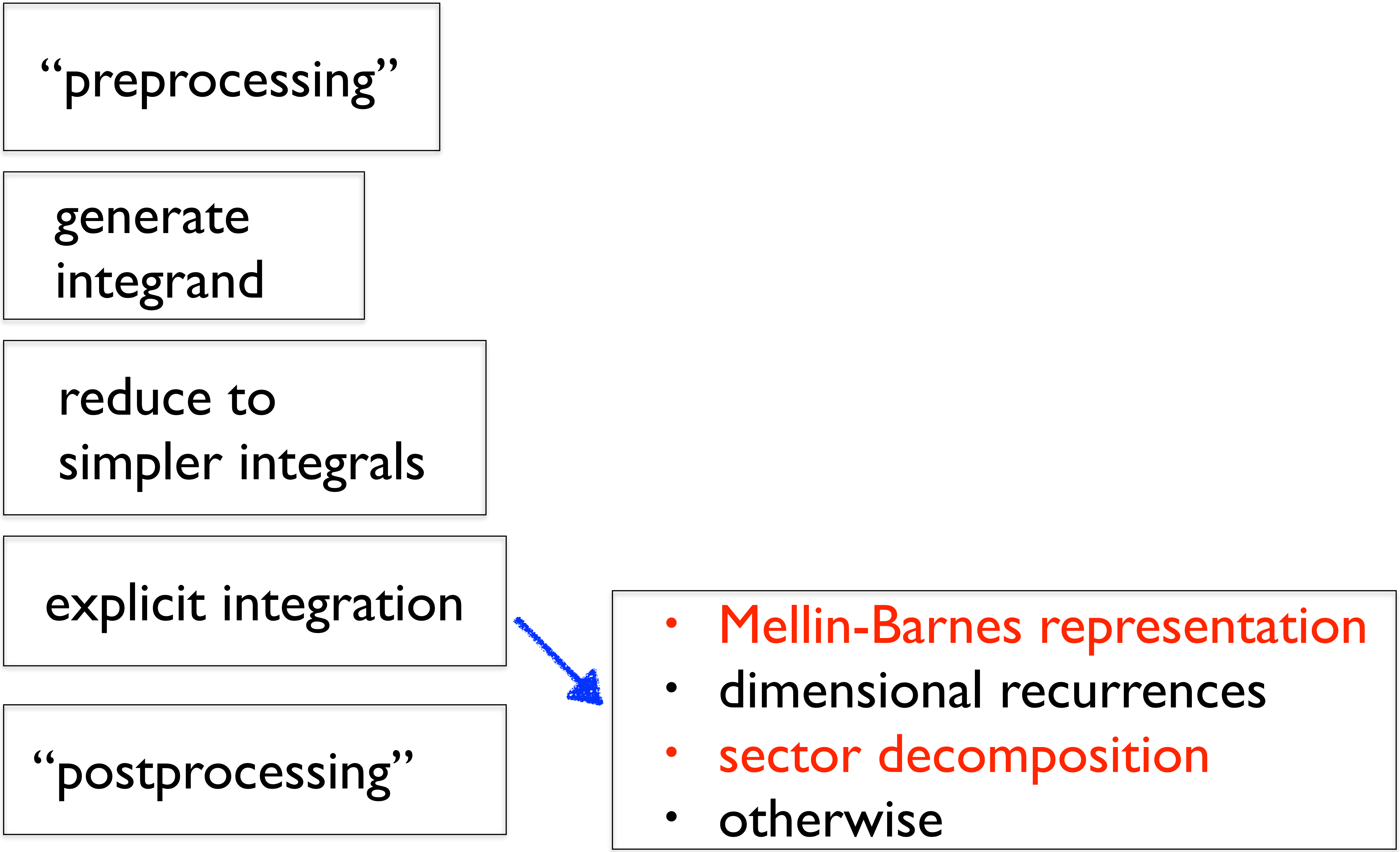
generate integrand

reduce to simpler integrals

explicit integration

“postprocessing”

perturbative QFT talks cheatsheet



“preprocessing”

generate integrand

reduce to simpler integrals

explicit integration

“postprocessing”

- Mellin-Barnes representation
- dimensional recurrences
- sector decomposition
- otherwise

numerical integration, non-planar

important observation: UT integrals are simpler to integrate numerically than non-UT ones!

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- for some integrals derived low-dimensional valid MB representation by hand & inspection → precise results
- automated tools used include: [Czakon, 05], [Smirnov^2, 07], [Gluza, Kajda, Riemann, 07 / 11], [Blümlein et.al, 14]
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- FIESTA uses sector decomposition (cf Sophia Borowska’s talk)

numerical integration, non-planar

important observation: UT integrals are simpler to integrate numerically than non-UT ones!

- for some integrals derived low-dimensional valid MB representation by hand & inspection → precise results
- automated tools used include: [Czakon, 05], [Smirnov^2, 07], [Gluza, Kajda, Riemann, 07 / 11], [Blümlein et.al, 14]
- one integral known analytically
- rest: FIESTA + CUBA (mostly vegas) + complete cluster [Smirnov-Tentyukov,08][Smirnov^2-Tentyukov,09] [Smirnov,13,15] + [Hahn, 04]
- FIESTA uses sector decomposition (cf Sophia Borowska's talk)
- (some cross-checks for simple integrals)

numerical integration and results

limiting factor:

improvements of accuracy scale as $\sqrt{\mathrm{maxeval}}$

integration time scales as $\mathrm{maxeval}$

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other users of the local cluster after some time:

numerical integration and results

- integrals diverge as $\sim \dfrac{1}{\epsilon^8}$
- non-planar cusp is at $\sim \dfrac{1}{\epsilon^2}$ (transcendentality 6)

seven orders of expansion, first six should cancel

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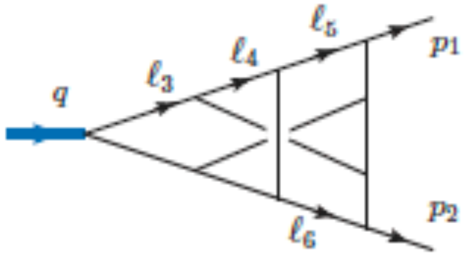
→ non-planar form factor cancels analytically down to $\sim \dfrac{1}{\epsilon^4}$

$$\ldots + (0.0007 \pm 0.0186)\epsilon^{-3} + (1.60 \pm 0.19)\epsilon^{-2} + (-17.98 \pm 3.47)\epsilon^{-1}$$

some sample result…

$$I_7^{(26)} =$$



$$\times \Big\{4\left[(\ell_4-\ell_5)(\ell_3-\ell_4+\ell_5-p_1)\right]\left[(\ell_4-\ell_6)(\ell_3-\ell_4+\ell_6-p_2)\right]$$

$$-\ell_5^2\,(\ell_6-p_2)^2-4\,(\ell_4-\ell_5)^2\,(\ell_3-\ell_4+\ell_6-p_2)^2-\ell_6^2\,(\ell_5-p_1)^2$$

$$-(\ell_3-\ell_4)^2\,(\ell_5+\ell_6-\ell_4)^2-\ell_4^2\,(\ell_3-\ell_4+\ell_5+\ell_6-p_1-p_2)^2\Big\}$$

$$= \frac{0.00347222}{\epsilon^8}-\frac{0.0000000013}{\epsilon^7}+\frac{0.0114231(17)}{\epsilon^6}+\frac{1.1631(3)}{\epsilon^5}+\frac{2.90880(35)}{\epsilon^4}$$

$$-\frac{12.2720(43)}{\epsilon^3}+\frac{29.708(57)}{\epsilon^2}+\frac{3185.60\pm 2.63}{\epsilon},$$

$$I_{7,\mathrm{PSLQ}}^{(26)} = \frac{1}{288\epsilon^8}+\frac{\zeta_2}{144\epsilon^6}+\frac{209\zeta_3}{216\epsilon^5}+\frac{43\zeta_4}{16\epsilon^4}+\mathcal{O}(\epsilon^{-3})\,.$$

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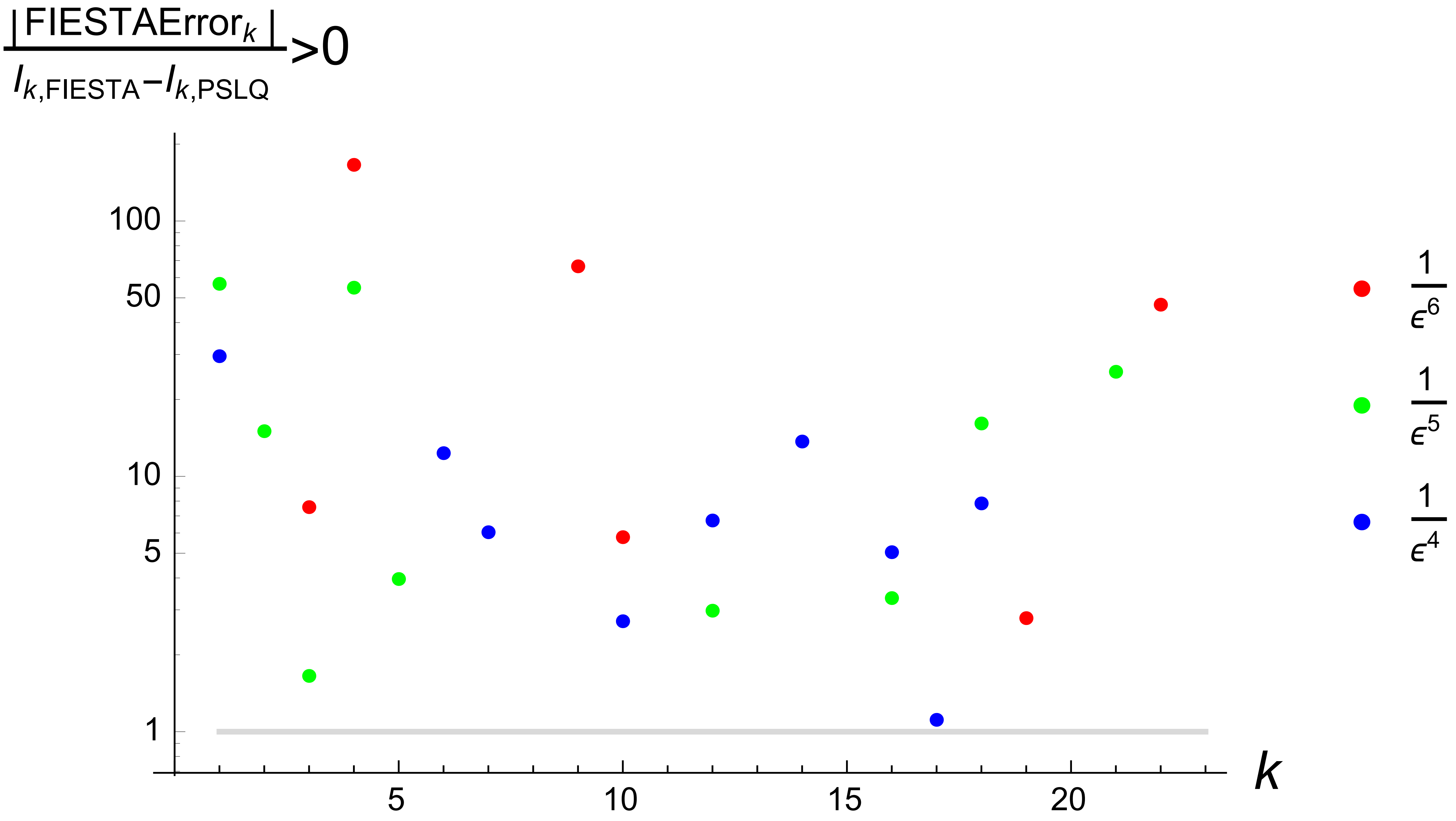
integration error is somewhat naive
(cf. [Marquard-Smirnov^2-Steinhauser-Wellmann, 16])

- MB, exact cross-checks
- PSLQ possible
- eps^-3 coefficient central value
- checked stability of central value with increasing points
- error dominated by very few integrals

the rug: example check

- take PSLQ result as exact result, and study numerical deviation



- uses number theory to check numerical computations!

in short

fun result: a four loop form factor in N=4

- at -2,-1: != 0 → speculation in literature
- can be computed at all → methods

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- extend UT finding to other integrals (five loops!)
- analytical results for integrals needed…
- QCD applications: nice choice of basis
- input for non-planar Beisert-Eden-Staudacher

THANKS FOR A NICE WORKSHOP!

Your Question Here?