Chapter 7. Transverse Coupling (Method by Edwards and Teng)

7.1 Initial Values

When the `COUPLE` option is set, the `TWISS` command uses a method similar to reference [22]. Consider the linear transfer map M in *two* degrees of freedom partitioned into four 2×2 blocks:

$$\mathrm{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{7.1}$$

The 4-dimensional phase space vector shall also be partitioned according to the horizontal and vertical planes. Edwards and Teng introduce a "symplectic rotation"

$$\mathrm{R} = \begin{pmatrix} I\cos\phi & \overline{R}\sin\phi \\ -R\sin\phi & I\cos\phi \end{pmatrix} \tag{7.2}$$

R is a 2×2 matrix with unit determinant, and $\overline{R}$ denotes its symplectic conjugate:

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad |R| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1, \qquad \overline{R} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{7.3}$$

This leaves three free parameters for the elements of R, and a fourth parameter ϕ. Edwards and Teng then determine R such that M conjugated with R becomes block diagonal:

$$\mathrm{RMR}^{-1} = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \tag{7.4}$$

If $|B + \overline{C}| < 0$ both ϕ and all elements of R become imaginary. This may be avoided by redefining

$$\mathrm{R} = \frac{1}{\sqrt{1 + |R|}} \begin{pmatrix} I & \overline{R} \\ -R & I \end{pmatrix}. \tag{7.5}$$

where all four elements of R are free parameters. The solutions is:

$$\begin{aligned} R &= -\left(\frac{1}{2}(\mathrm{Tr}\,A - \mathrm{Tr}\,D) + \mathrm{sign}(|B + \overline{C}|)\sqrt{|B + \overline{C}| + \frac{1}{4}(\mathrm{Tr}\,A - \mathrm{Tr}\,D)^2}\right)^{-1} \left(B + \overline{C}\right), \\ E &= A - BR, \qquad F = D + \overline{R}C. \end{aligned} \tag{7.6}$$

The block diagonal matrix can be parametrised as usual. From the eigenvectors of the conjugated system

$$V_1 = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix}, \qquad V_2 = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ \frac{\alpha_2}{\sqrt{\beta_2}} & \frac{2}{\sqrt{\beta_2}} \end{pmatrix} \tag{7.7}$$

one may find the eigenvectors of the coupled system:

$$\mathrm{V}_1 = \frac{1}{\sqrt{1 + |R|}} \begin{pmatrix} V_1 \\ \overline{R}V_1 \end{pmatrix}, \qquad \mathrm{V}_2 = \frac{1}{\sqrt{1 + |R|}} \begin{pmatrix} -RV_2 \\ V_2 \end{pmatrix}. \tag{7.8}$$