

Model-independent shape analysis of BE correlations and differential cross-section of elastic scattering



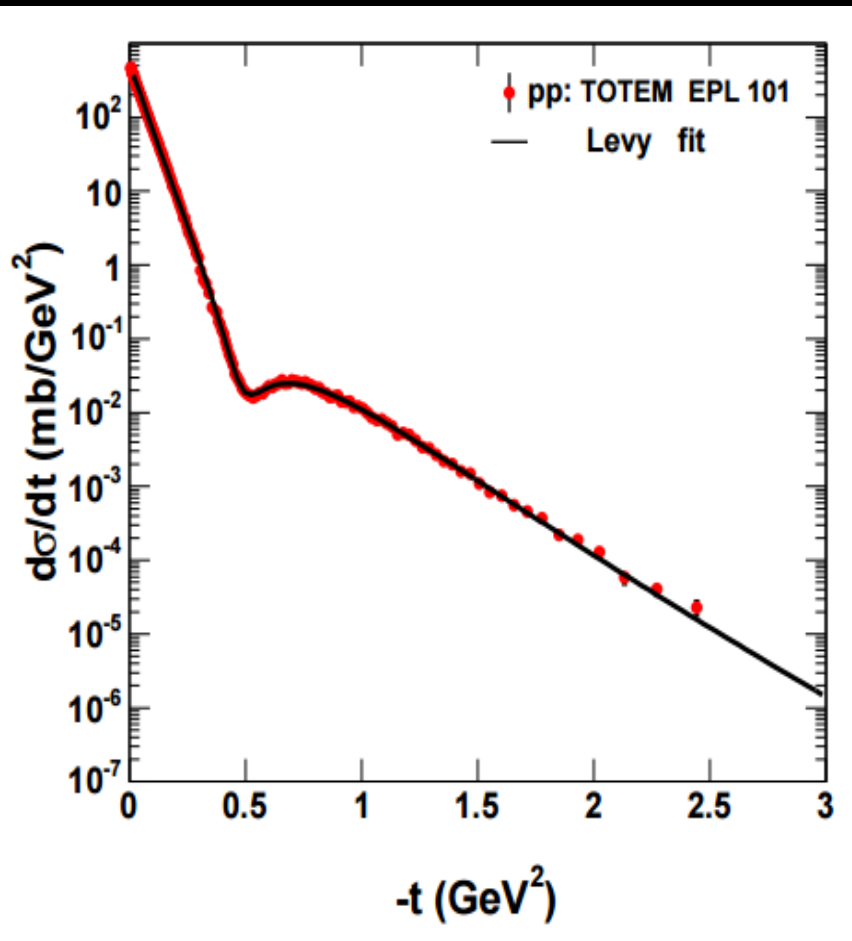
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OUTLINE

Model-independent shape analysis:

- General introduction
- Edgeworth, Laguerre
- Levy expansions
- Application in elastic pp scattering

Summary

MODEL - INDEPENDENT SHAPE ANALYSIS I.

Model-independent method, proposed to analyze Bose-Einstein correlations IF experimental data satisfy

- The measured data *tend to a constant for large values* of the observable Q .
- There is a *non-trivial structure* at some definite value of Q , shift it to $Q = 0$.

Model-independent, but experimentally testable:

- $t = QR$
- dimensionless scaling variable
- approximate form of the correlations $w(t)$
- *Identify $w(t)$ with a measure in an abstract Hilbert-space*

$$\int dt w(t) h_n(t) h_m(t) = \delta_{n,m},$$

$$f(t) = \sum_{n=0}^{\infty} f_n h_n(t),$$

$$f_n = \int dt w(t) f(t) h_n(t).$$

e.g. $t = Q_I R_I$

MODEL - INDEPENDENT SHAPE ANALYSIS II.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1) N_1(\mathbf{k}_2)},$$

$$R_2(\mathbf{k}_1, \mathbf{k}_2) = C_2(\mathbf{k}_1, \mathbf{k}_2) - 1.$$

Let us assume, that the function $g(t) = R_2(t)/w(t)$ is also an element of the Hilbert space H . This is possible, if

$$\int dt w(t) g^2(t) = \int dt [R_2^2(t)/w(t)] < \infty, \quad (6)$$

Then the function g can be expanded as

$$g(t) = \sum_{n=0}^{\infty} g_n h_n(t),$$
$$g_n = \int dt R_2(t) h_n(t).$$

From the completeness of the Hilbert space, if $g(t)$ is also in the Hilbert space:

$$R_2(t) = w(t) \sum_{n=0}^{\infty} g_n h_n(t).$$

MODEL - INDEPENDENT SHAPE ANALYSIS III.

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1) N_1(\mathbf{k}_2)},$$

$$C_2(t) = \mathcal{N} \left\{ 1 + \lambda_w w(t) \sum_{n=0}^{\infty} g_n h_n(t) \right\}$$

Model-independent AND experimentally testable:

- method for any approximate shape $w(t)$
- the core-halo intercept parameter of the CF is
- coefficients by numerical integration (fits to data)
- condition for applicability: experimentally testable

$$\lambda_* = \lambda_w \sum_{n=0}^{\infty} g_n h_n(0)$$

$$g_n = \int dt R_2(t) h_n(t)$$

$$\int dt [R_2^2(t)/w(t)] < \infty$$

GAUSSIAN $w(t)$: EDGEWORTH EXPANSION

$$t = \sqrt{2}QR_E,$$

$$w(t) = \exp(-t^2/2),$$

$$\int_{-\infty}^{\infty} dt \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m},$$

$$H_n(t) = \exp(t^2/2) \left(-\frac{d}{dt} \right)^n \exp(-t^2/2).$$

$$H_1(t) = t,$$

$$H_2(t) = t^2 - 1,$$

$$H_3(t) = t^3 - 3t,$$

$$H_4(t) = t^4 - 6t^2 + 3, \dots$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[1 + \frac{\kappa_3}{3!} H_3(\sqrt{2}QR_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2}QR_E) + \dots \right] \right\}.$$

3d generalization straightforward

- Applied by NA22, L3, STAR, PHENIX, ALICE, CMS (LHCb)

EXPONENTIAL $w(t)$: LAGUERRE EXPANSIONS

Model-independent but experimentally tested:

- $w(t)$ exponential
- t dimensionless
- Laguerre polynomials

$$t = QR_L,$$
$$w(t) = \exp(-t)$$

$$\int_0^{\infty} dt \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m},$$

$$L_n(t) = \exp(t) \frac{d^n}{dt^n} (-t)^n \exp(-t).$$

$$L_0(t) = 1,$$
$$L_1(t) = t - 1,$$

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_L \exp(-QR_L) \left[1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + \dots \right] \right\}$$

First successful tests

- NA22, UA1 data
- convergence criteria satisfied
- intercept parameter ~ 1

$$\int_0^{\infty} dt R_2^2(t) \exp(+t) < \infty,$$

$$\lambda_* = \lambda_L [1 - c_1 + c_2 - \dots],$$
$$\delta^2 \lambda_* = \delta^2 \lambda_L [1 + c_1^2 + c_2^2 + \dots] + \lambda_L^2 [\delta^2 c_1 + \delta^2 c_2 + \dots]$$

STRETCHED $w(t)$: LEVY EXPANSIONS

$$w(t|\alpha) = \exp(-t^\alpha) = \exp(-Q^\alpha R^\alpha)$$

Model-independent but:

- Levy: stretched exponential
- generalizes exponentials and Gaussians
- ubiquitous in nature
- How far from a Levy?
- Need new set of polynomials orthonormal to a Levy weight

$$L_1(x|\alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x|\alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx x^r f(x|\alpha) = \frac{1}{\alpha} \Gamma\left(\frac{r+1}{\alpha}\right)$$

STRETCHED $w(t)$: LEVY EXPANSIONS

In case of $\alpha = 1$,
in 1 dimension Laguerre expansion is recovered

$$\begin{aligned}L_0(t | \alpha = 1) &= 1, \\L_1(t | \alpha = 1) &= t - 1, \\L_2(t | \alpha = 1) &= t^2 - 4t + 2.\end{aligned}$$

These reduce to the
Laguerre expansions and
Laguerre polynomials.

STRETCHED $w(t)=\exp(-t^\alpha)$: LEVY EXPANSIONS

In case of $\alpha = 2$,
a new formulae for one-sided Gaussians:

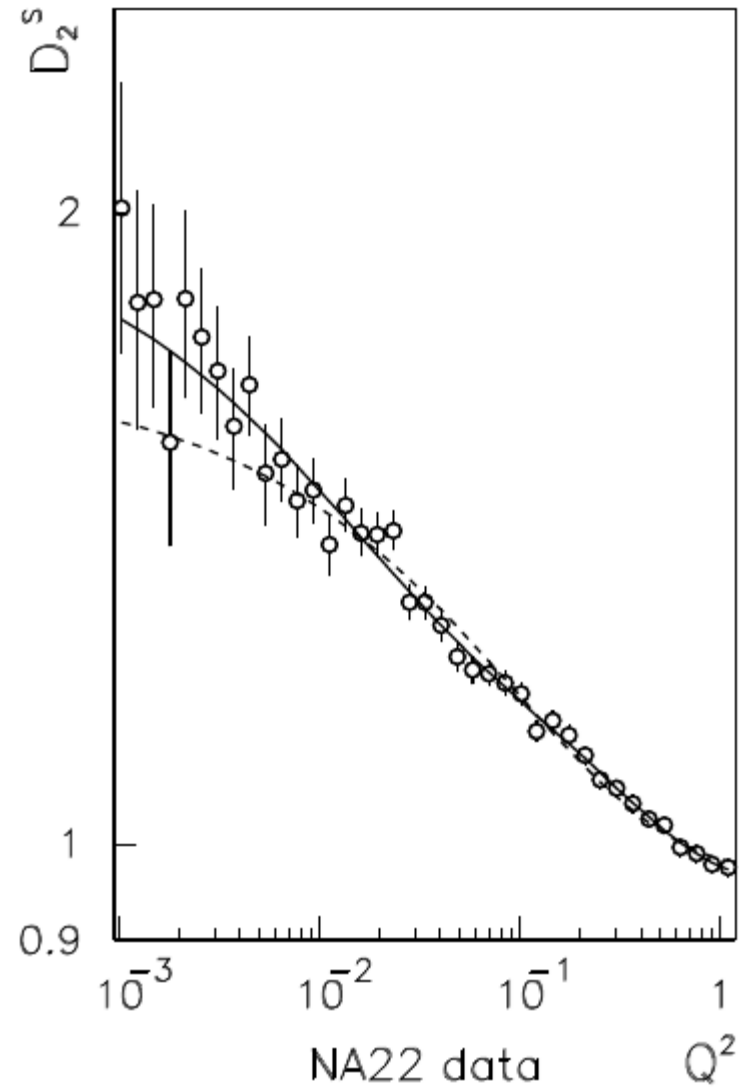
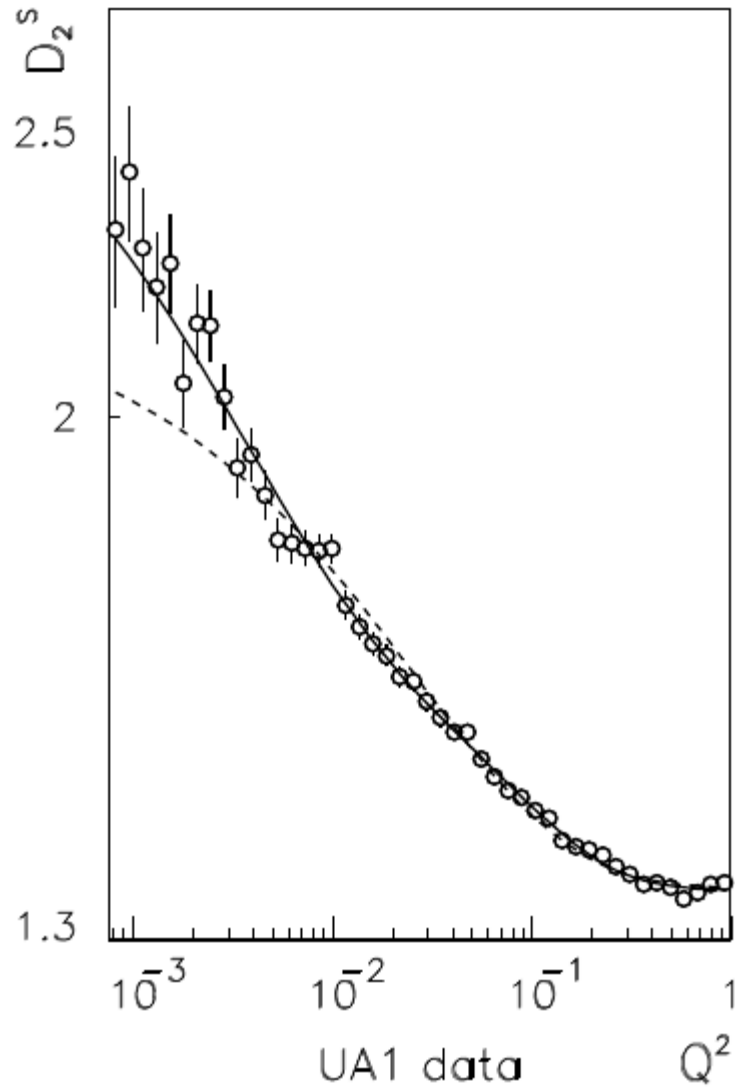
$$\begin{aligned}L_0(t | \alpha = 2) &= \frac{\sqrt{\pi}}{2}, \\L_1(t | \alpha = 2) &= \frac{1}{2} \{ \sqrt{\pi t} - 1 \}, \\L_2(t | \alpha = 2) &= \frac{1}{32} \left\{ (\pi - 2)t^2 - \sqrt{\pi t} + 2 - \frac{\pi}{2} \right\}.\end{aligned}$$

Provides a new expansion around a Gaussian shape that is defined for the non-negative values of t only.

Edgeworth expansion different, its around two-sided Gaussian, includes non-negative values of t also.

EXAMPLE, LAGUERRE EXPANSIONS

Laguerre expansion fit



EXAMPLE, LEVY EXPANSIONS

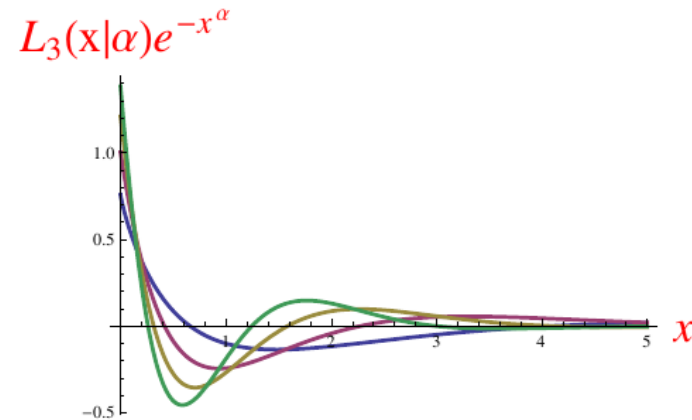
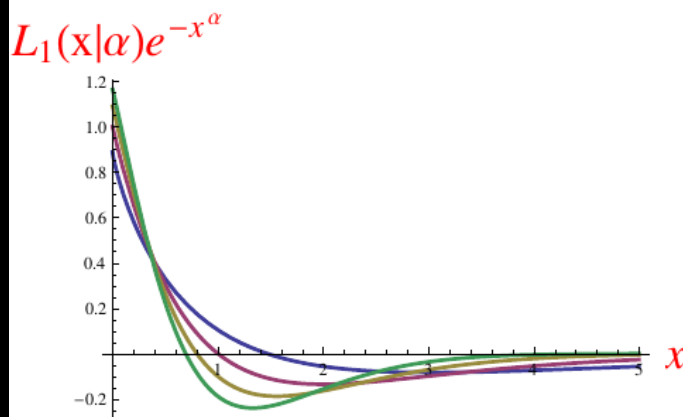
Model-independent but:

- Levy generalizes exponentials and Gaussians
- ubiquitous in nature
- How far from a Levy?
- Not necessarily positive definit !

$$L_1(x | \alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} \\ 1 & x \end{pmatrix}$$

$$L_2(x | \alpha) = \det \begin{pmatrix} \mu_{0,\alpha} & \mu_{1,\alpha} & \mu_{2,\alpha} \\ \mu_{1,\alpha} & \mu_{2,\alpha} & \mu_{3,\alpha} \\ 1 & x & x^2 \end{pmatrix}$$

$$\mu_{r,\alpha} = \int_0^\infty dx x^r f(x | \alpha) = \frac{1}{\alpha} \Gamma\left(\frac{r+1}{\alpha}\right)$$



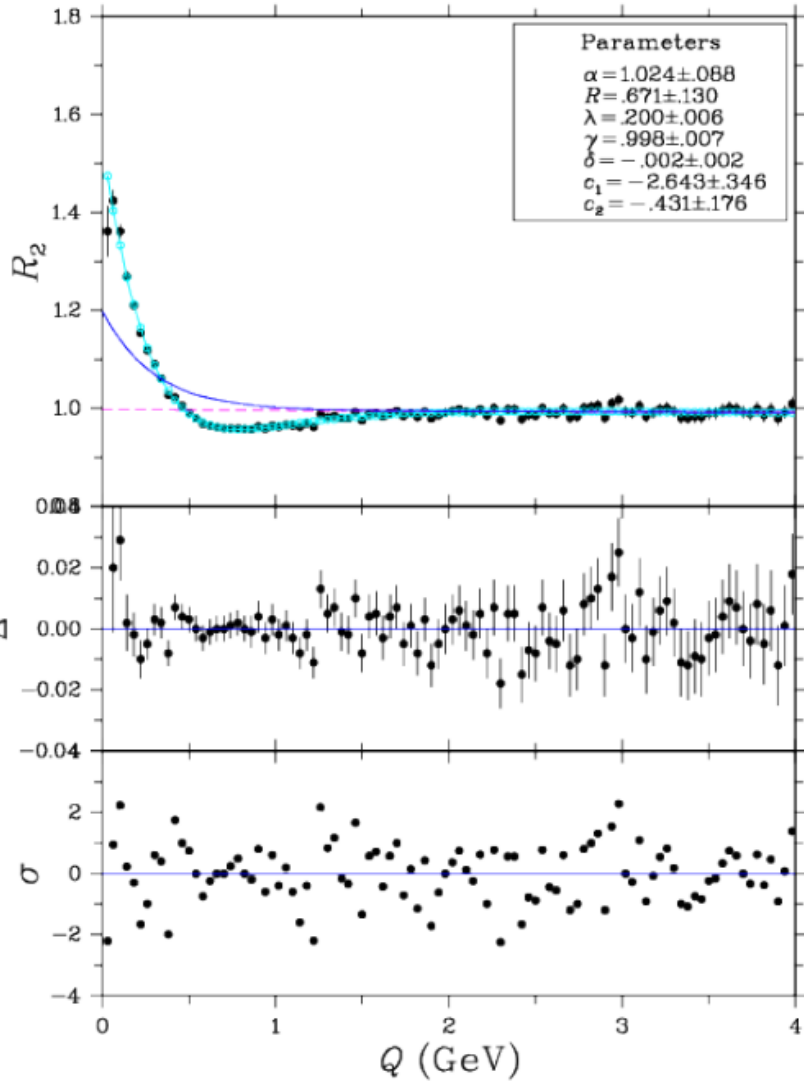
Lévy polynomials of first and third order times the weight function e^{-x^α} for $\alpha = 0.8, 1.0, 1.2, 1.4$.

$$\text{1st-order Lévy polynomial} \quad \gamma \left[1 + \lambda e^{-R^\alpha Q^\alpha} [1 + c_1 L_1(Q|\alpha, R)] \right]$$

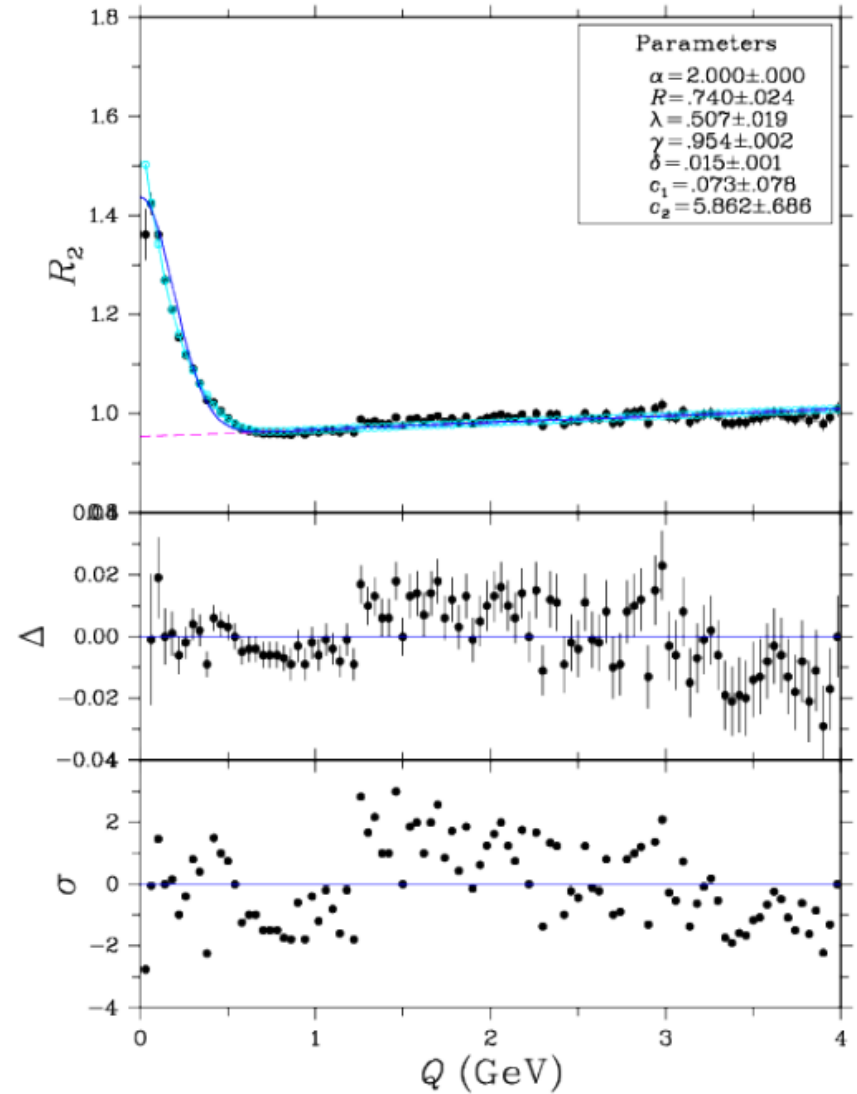
$$\text{3rd-order Lévy polynomial} \quad \gamma \left[1 + \lambda e^{-R^\alpha Q^\alpha} [1 + c_1 L_1(Q|\alpha, R) + c_3 L_3(Q|\alpha, R)] \right]$$

Example, L_3 BE correlation function

FIT7 $\chi^2/\text{DoF}=91.3/93$ CL=52% $.0 < Q < 4.0$, $.10.10 < m_t < 4.02$ 4.02 , $\Delta m_t < 9.0$
 this plot: $.14.14 < m_t < 4.02$ 4.02 $\Delta m_t < 9.00$ $\chi^2=93.6$, 100 pts



FIT7 $\chi^2/\text{DoF}=170.4/94$ CL= 2.5E-06 $.0 < Q < 4.0$, $.10.10 < m_t < 4.02$ 4.02 , $\Delta m_t < 9.0$
 this plot: $.14.14 < m_t < 4.02$ 4.02 $\Delta m_t < 9.00$ $\chi^2=172.4$, 100 pts



$$C_2(t) = N \left\{ 1 + \lambda \exp(-t^\alpha) \left[1 + \sum_{n=1}^{\infty} c_n L_n(t|\alpha) \right] \right\} \quad t = QR$$

LEVY EXPANSIONS for POSITIVE DEFINITE FORMS

experimental conditions:

(i) The correlation function tends to a constant for large values of the relative momentum Q .

(ii) The correlation function deviates from its asymptotic, large Q value in a certain domain of its argument.

(iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

Model-independent but:

- Assumes that Coulomb can be corrected
- No assumptions about analyticity yet
- For simplicity, consider 1d case first
- For simplicity, consider factorizable x k
- Normalizations :
 - density
 - multiplicity
 - single-particle spectra

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{N_2(\mathbf{k}_1, \mathbf{k}_2)}{N_1(\mathbf{k}_1) N_1(\mathbf{k}_2)}$$

$$S(x, k) = f(x) g(k)$$

$$\int dx f(x) = 1, \quad \int dk g(k) = \langle n \rangle,$$

$$N_1(k) = \int dx S(x, k) = g(k).$$

MINIMAL MODEL ASSUMPTION: LEVY

Model-independent but:

- not assumes analyticity
- C_2 measures a modulus squared Fourier-transform vs relative momentum
- Correlations non-Gaussian
- Radius not a variance
- $0 < \alpha \leq 2$

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int dx \exp(iq_{12}x) f(x),$$

$$C(q; \alpha) = 1 + \lambda \exp(-|qR|^\alpha).$$

UNIVARIATE LEVY EXAMPLES

Include some well known cases:

- $\alpha = 2$

- Gaussian source, Gaussian C_2

$$f(x) = \frac{1}{(2\pi R^2)^{1/2}} \exp \left[-\frac{(x - x_0)^2}{2R^2} \right]$$

$$C(q) = 1 + \exp(-q^2 R^2)$$

- $\alpha = 1$

- Lorentzian source, exponential C_2

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x - x_0)^2},$$

$$C(q) = 1 + \exp(-|q R|).$$

- asymmetric Levy:

- asymmetric support
- Stretched exponential

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x - x_0)^{3/2}} \exp \left(-\frac{R}{8(x - x_0)} \right)$$

$$x_0 < x < \infty,$$

$$C(q) = 1 + \exp \left(-\sqrt{|q R|} \right).$$

Multivariate, nearly Levy correlations

If the BE correlation function is

$$C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

then

$$t = \left(\sum_{i,j=\text{side,out,long}} R_{i,j}^2 q_i q_j \right)^{1/2},$$
$$C_2(t) = N \left\{ 1 + \lambda \exp(-t^\alpha) \left| 1 + \sum_{n=1}^{\infty} (a_n + ib_n) L_n(t|\alpha) \right|^2 \right\}$$

where $\{c_n = a_n + ib_n\}_{n=1}^{\infty}$ are now complex valued expansion coefficients

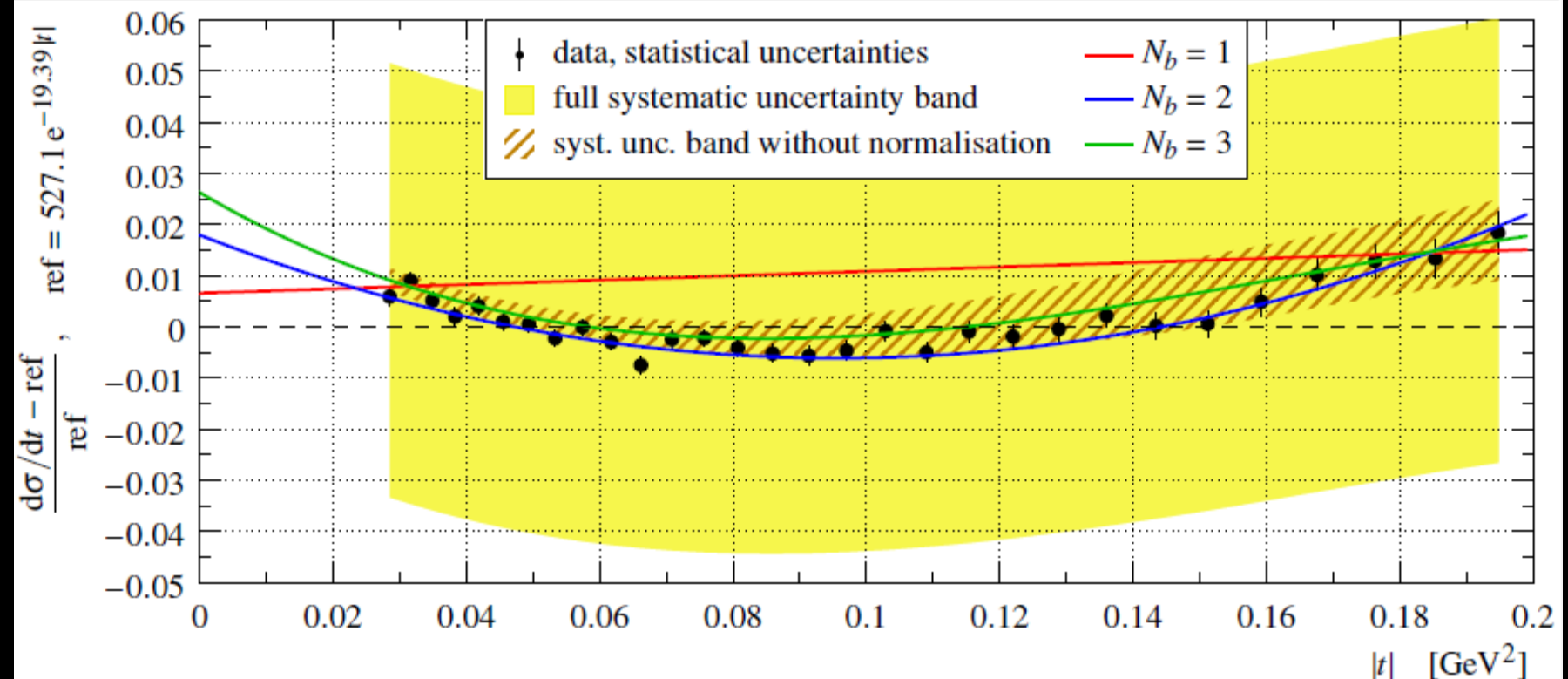
but for a positive definite example let us take
differential cross-section of TOTEM

Non-Exponential Differential Cross-Section

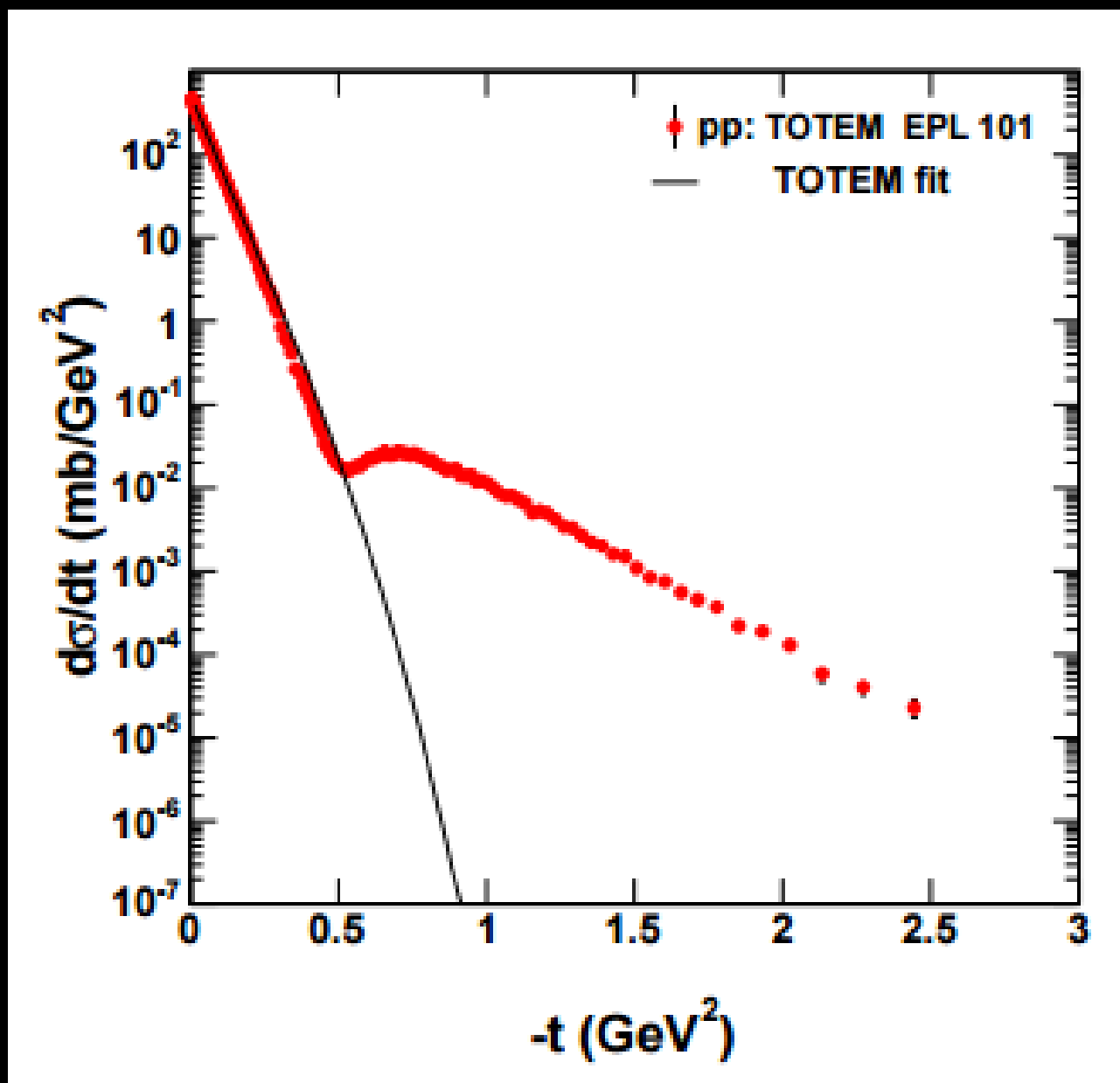
To study the detailed behaviour of the differential cross-section, a series of fits has been made using the parametrisation:

$$\frac{d\sigma}{dt}(t) = \frac{d\sigma}{dt}\Big|_{t=0} \exp\left(\sum_{i=1}^{N_b} b_i t^i\right), \quad (15)$$

which includes the pure exponential ($N_b = 1$) and its straight-forward extensions ($N_b = 2, 3$).



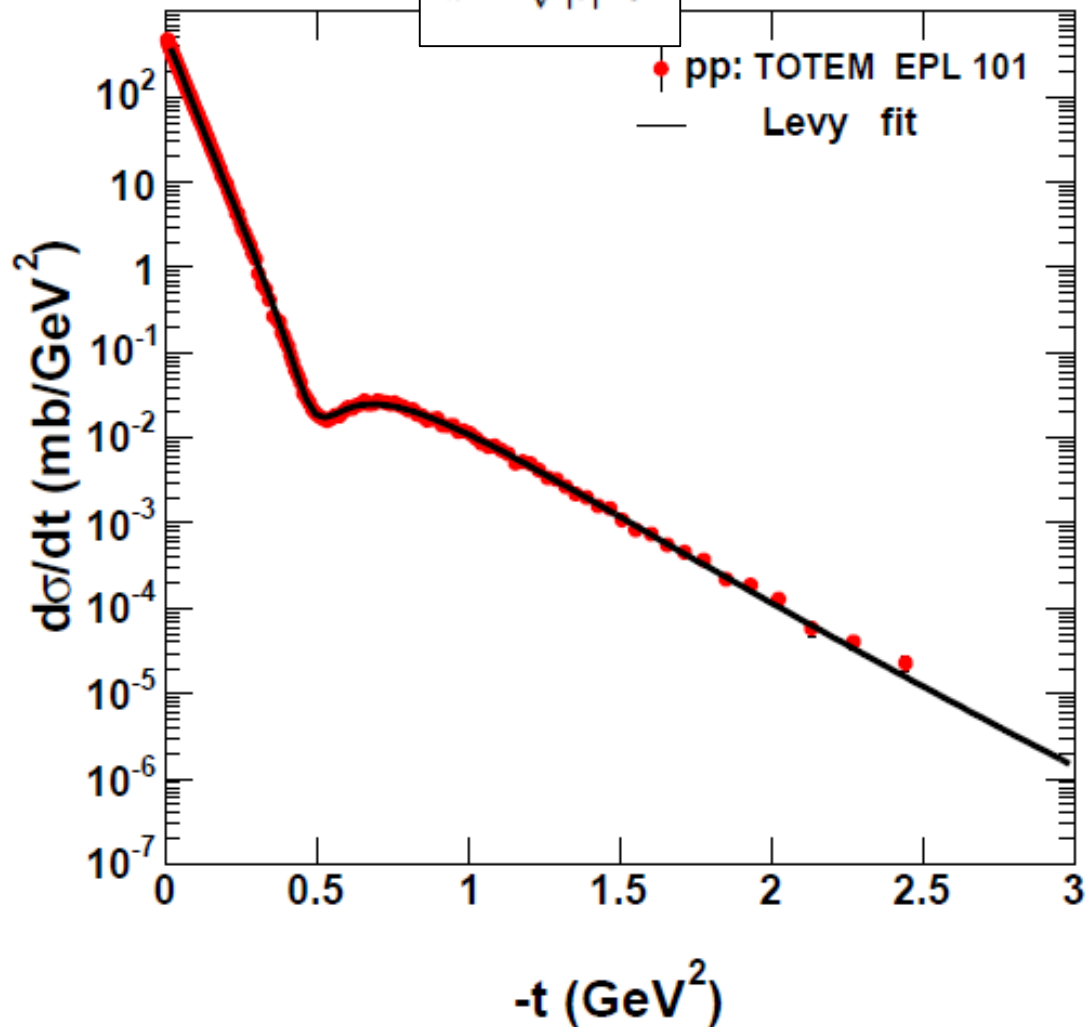
However, this method does not extrapolate well



LEVY EXPANSION FIT TO NON-EXPONENTIALS

$$\frac{d\sigma}{dt} = \frac{d\sigma}{dt} \Big|_{t=0} \exp(-z^\alpha) |1 + c_1 L_1(z|\alpha) + c_2 L_2(z|\alpha) + \dots|^2$$

$$z = \sqrt{|t|} R$$



SUMMARY AND CONCLUSIONS

Several model-independent methods:

- Based on matching an abstract measure in H to the approximate shape of data
- Gaussian: Edgeworth expansions
- Exponential: Laguerre expansions
- Levy ($0 < \alpha \leq 2$): Levy expansions
- TOTEM: excluded a purely exponential diff. cross-section at low $|t|$ at 8 TeV
- **Levy** expansion: indicate a non-exponential diff. cross-section up to $-t = 3.0 \text{ GeV}^2$ even at 7 TeV
- **Deviation** from exponential measured by 1 **parameter**: α