New perturbative solutions of relativistic hydrodynamics

Bálint Kurgyis, Máté Csanád

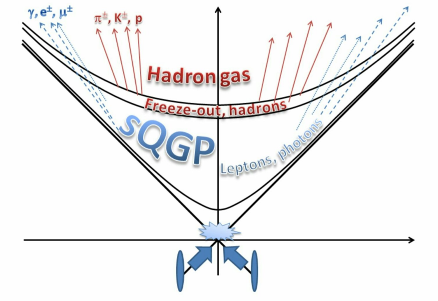
Eötvös Loránd University, Budapest

Day of Femtoscopy
November 2, 2017 Gyöngyös

Special thanks for
Sándor Lökös and Tamás Csörgő

Time evolution of the sQGP

- Strongly interacting QGP discovered at RHIC, created at LHC
- A hot, expanding, strongly interacting, perfect QG fluid
- Hadrons created at the freeze-out
- Leptons, photons "shine through"



Known solutions of relativistic hydrodynamics

- Many numerical solutions
- Exact, analytic solutions important: connect initial/final state
- Famous 1+1D solutions:
  Landau-Khalatnikov and Hwa-Bjorken

L. D. Landau, Izv. Akad. Nauk Ser. Fiz. **17**, 51 (1953)
I.M. Khalatnikov, Zhur. Eksp. Teor. Fiz. **27**, 529 (1954)
R. C. Hwa, Phys. Rev. **D 10**, 2260 (1974)

J. D. Bjorken, Phys. Rev. **D 27**, 140 (1983)

- Many new solutions: mostly 1+1D, few 1+3D
- First truly 3D relativistic solution: Hubble-flow

Csörgő, Csernai, Hama, Kodama, Heavy Ion Phys. **A21**, 73 (2004), nucl-th/0306004

- Multipole solutions also known

Csanád, Szabó , Phys. Rev. **C 90**, 054911 (2014)

Equations of relativistic hydrodynamics

Looking for (u^μ, p, ϵ, n or σ) fields

Assumptions:

- perfect fluid
- local energy-momentum conservation

Properties:

- $u_\mu u^\mu = 1$
- $u_\nu \partial_\mu u^\nu = 0$

Continuity of entropy density σ can also be considered

Continuity equation of the energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu}$$

Continuity equation

$\partial_\mu(n u^\mu) = 0$ or $\partial_\mu(\sigma u^\mu) = 0$

Equation of State (EoS)

$$\epsilon = \kappa p$$

Decomposition of energy-momentum conservation

Two equations:

- Lorentz-perpendicular to u^μ
- Lorentz-paralell to u^μ

Euler equation

$$(\kappa+1)pu^\nu\partial_\nu u^\mu = (g^{\mu\nu} - u^\mu u^\nu)\partial_\nu p$$

Energy equation

$$\kappa u^\mu\partial_\mu p + (\kappa+1)p\partial_\mu u^\mu = 0$$

Perturbative handling

Perturbed fields:

- start from a known solution (u^μ, p, n or σ)
- $u^\mu \to u^\mu + \delta u^\mu$
- $p \to p + \delta p$
- $n \to n + \delta n$
- or $\sigma \to \sigma + \delta\sigma$
- orthogonality:

$$u^\mu \delta u_\mu = 0 \qquad (1)$$

Perturbed equations:

- substitute perturbations into equations
- substract 0^{th} order equations
- neglect 2^{nd} or higher order perturbations
- remainder: perturbed equation
- solution yields perturbations $\delta u^\mu, \delta n, \delta p$

Perturbative equations

Euler equation

$$(\kappa+1)\delta p u^\mu \partial_\mu u^\nu + (\kappa+1) p \delta u^\mu \partial_\mu u^\nu + (\kappa+1) p u^\mu \partial_\mu \delta u^\nu = (g^{\mu\nu} - u^\mu u^\nu)\partial_\mu \delta p - \delta u^\mu u^\nu \partial_\mu p - u^\mu \delta u^\nu \partial_\mu p \quad (2)$$

Energy equation

$$\kappa \delta u^\mu \partial_\mu p + \kappa u^\mu \partial_\mu \delta p + (\kappa+1)\delta p \partial_\mu u^\mu + (\kappa+1) p \partial_\mu \delta u^\mu = 0 \quad (3)$$

Continuity equation

$$u^\mu \partial_\mu \delta n + \delta n \partial_\mu u^\mu + \delta u^\mu \partial_\mu n + n \partial_\mu \delta u^\mu = 0 \quad (4)$$

Perturbations on a standing fluid: waves

Known solution: Standing fluid

- $u^\mu = (1, 0, 0, 0)$
- $p =$ const.
- $n =$ const.

Exploit these fields:

- $\partial_\mu u^\mu = 0$
- $\partial_\mu p = 0$
- $u^\mu \partial_\mu = \partial_0$
- $Q^{\mu\nu} = (u^\mu u^\nu - g^{\mu\nu})$
- $Q^{\mu\nu}\partial_\mu = (0, \nabla)$

Perturbed Energy equation

$$\kappa\partial_0\delta p + (\kappa + 1)p\partial_\mu\delta u^\mu = 0$$

Perturbed Euler equation

$$(\kappa + 1)p\partial_0\delta u^\nu - Q^{\mu\nu}\partial_\mu\delta p = 0$$

Wave solution for pressure

$$\partial_0^2\delta p = \frac{1}{\kappa}\Delta\delta p$$

Known solution: Hubble-flow

Hubble-flow:

Csörgő, Csernai, Hama, Kodama , Heavy Ion Phys. **A21**, 73 (2004), nucl-th/0306004

- $u^\mu = \frac{x^\mu}{\tau}$
- $n = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}(S)$
- $p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$
- scaling variable must satisfy $u_\mu \partial^\mu S = 0$
- scaling variable can be arbitrary function of $r^n \cdot \tau^m \cdot t^{-n-m}$
- describes well hadronic data and photons

Csanád, Vargyas, Eur. Phys. J. **A 44**, 473 (2010)

Csanád, Májer, Central Eur. J. Phys. **10** (2012)

- Multipole solutions also possible

Csanád, Szabó , Phys. Rev. **C 90**, 054911 (2014)

Perturbative equations for Hubble-flow solution

Euler equation:
$$\frac{\partial_\mu \delta p}{(\kappa+1)p}\left[g^{\mu\nu} - u^\mu u^\nu\right] = \frac{\kappa-3}{\tau\kappa}\delta u^\nu + u^\mu \partial_\mu \delta u^\nu \tag{5}$$

Energy equation:
$$\kappa u^\mu \partial_\mu \delta p + \frac{3(\kappa+1)}{\tau}\delta p = -(\kappa+1)p\partial_\mu \delta u^\mu \tag{6}$$

Continuity equation:
$$\delta u^\mu n \frac{\mathcal{N}'(S)}{\mathcal{N}(S)}\partial_\mu S + u^\mu \partial_\mu \delta n + \frac{3\delta n}{\tau} + n\partial_\mu \delta u^\mu = 0 \tag{7}$$

To find a solution:

- choose test functions for $\delta p, \delta u^\mu, \delta n$
- choose a scaling variable S
- satisfy all the restrictions

The general form of solution

$$u^\mu = \frac{x^\mu}{\tau}$$

$$p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}(S)$$

The perturbations:

$$\delta u^\mu = \delta \cdot \left(\tau + c{\tau_0}^{\frac{\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}}\right) \partial^\mu S \cdot \chi(S) \quad (8)$$

$$\delta p = \delta \cdot p_0 \pi(S) \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}} \quad (9)$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau}\right)^3 h(x,y,z,t)\nu(S) \quad (10)$$

Relationship between $\mathcal{N}(S), \chi(S), \nu(S), \pi(S), h(x,y,z,t), S$

$$\frac{\chi'(S)}{\chi(S)} = -\frac{\partial_\mu \partial^\mu S}{\partial_\mu S \partial^\mu S} \quad (11)$$

$$\pi'(S) = \frac{(\kappa-3)(\kappa+1)}{\kappa} \cdot \chi(S) \quad (12)$$

$$\frac{\nu(S)}{\chi(S)\mathcal{N}'(S)} = -\frac{(\tau + c{\tau_0}^{\frac{\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}})\partial_\mu S \partial^\mu S}{u^\mu \partial_\mu h(x,y,z,t)} \quad (13)$$

The requirements for the scaling variable

We have to choose an $h(x,y,z,t)$ function:

$$h(x,y,z,t) = \ln\left(\frac{\tau}{\tau_0}\right) + c\frac{\kappa}{3-\kappa}\left(\frac{\tau}{\tau_0}\right)^{\frac{3}{\kappa}-1} ; \kappa = 3 : h = (1+C)\ln\left(\frac{\tau}{\tau_0}\right)$$

The scaling variable must obey the following:

- $u_\mu \partial^\mu S = 0$
- $\frac{\partial_\mu \partial^\mu S}{\partial_\mu S \partial^\mu S}$ must be a function of the scaling variable
- $\tau^2 \partial_\mu S \partial^\mu S$ also a function of the scaling variable

The scaling variables found so far:

$$S = \frac{r^m}{t^m} \qquad S = \frac{r^m}{\tau^m} \qquad S = \frac{\tau^m}{t^m}$$

$$S = \frac{x^m}{t^m} \qquad S = \frac{y^m}{t^m} \qquad S = \frac{z^m}{t^m}$$

Scaling variable $S = \frac{r^m}{t^m}$, $h(x,y,z,t) = \ln\left(\frac{\tau}{\tau_0}\right) + c\frac{\kappa}{3-\kappa}\left(\frac{\tau}{\tau_0}\right)^{\frac{3}{\kappa}-1}$

The scaling variable

$$S = \frac{r^m}{t^m} \qquad (14)$$

The perturbations

$$\delta p = \delta \cdot p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}} \pi(S)$$

$$\delta u^\mu = \delta \cdot \left(\tau + c\tau_0^{\frac{\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}}\right) \partial^\mu S \chi(S)$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau}\right)^3 \left(\ln\left(\frac{\tau}{\tau_0}\right) + c\frac{\kappa}{3-\kappa}\left(\frac{\tau}{\tau_0}\right)^{\frac{3}{\kappa}-1}\right) \nu(S)$$

The solution using this scaling variable:

$$\chi(S) = \left(\frac{r}{t}\right)^{-m-1} \qquad (15)$$

$$\pi(S) = -\frac{(\kappa+1)(\kappa-3)}{\kappa} m \left(\frac{r}{t}\right)^{-1} \qquad (16)$$

A concrete solution for $S = t/r$, $\mathcal{N}(\mathcal{S}) = exp(-S^{-2})$

Special case of $\frac{r^m}{t^m}$, $m = -1$

The scaling variable

$$S = \frac{t}{r} \tag{18}$$

Let us choose a $\mathcal{N}(S)$

$$\mathcal{N}(S) = e^{-\frac{r^2}{t^2}} = e^{-S^{-2}}$$

The perturbations

$$\delta p = \delta \cdot p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}} \pi(S)$$

$$\delta u^\mu = \delta \cdot \left(\tau + c\tau_0^{\frac{\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}}\right) \partial^\mu S \chi(S)$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau}\right)^3 \left(\ln\left(\frac{\tau}{\tau_0}\right) + c\frac{\kappa}{3-\kappa}\left(\frac{\tau}{\tau_0}\right)^{\frac{3}{\kappa}-1}\right) \nu(S)$$

The $\nu(S), \chi(S), \pi(S)$ functions:

$$\chi(S) = 1 \tag{19}$$

$$\pi(S) = \frac{(\kappa+1)(\kappa-3)}{\kappa}\left(\frac{t}{r}\right) \tag{20}$$

$$\nu(S) = 2\left(\frac{t}{r}\right)^{-3}\left(1-\left(\frac{t}{r}\right)^2\right)^2 \mathcal{N}\left(\frac{t}{r}\right) \tag{21}$$

The pressure and velocity perturbations for $S = t/r$

The original velocity: $u^\mu = \frac{x^\mu}{\tau}$

The perturbation:

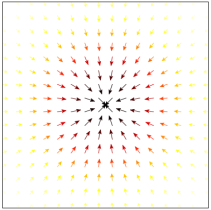
$\delta u^\mu = \delta \cdot \tau \partial^\mu \left(\frac{t}{r}\right)$

The original pressure: $p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$

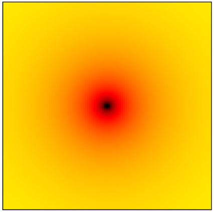
The perturbation:

$\delta p = \delta \cdot p_0 \frac{(\kappa+1)(\kappa-3)}{\kappa} \frac{t}{r} \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$

tau=1.0



tau=1.0



Singel-particle momentum distribution

For thermalized medium the source function:

$$S(x,p) = Nn\exp\left(-\frac{p_\mu u^\mu}{T}\right)H(\tau)\frac{p_\mu u^\mu}{u^0}d^3xd\tau \qquad (22)$$

With respect to the perturbations:

$$S(x,p) = Nn\exp\left(-\frac{p_\mu u^\mu}{T}\right)\delta(\tau-\tau_0)\frac{p_\mu u^\mu}{u^0}\cdot$$
$$\cdot\left[1+\frac{\delta u^0}{u^0}+\frac{p_\mu \delta u^\mu}{p_\nu u^\nu}-\frac{p_\mu \delta u^\mu}{T}+\frac{p_\mu u^\mu \delta T}{T^2}+\frac{\delta n}{n}\right]d\tau dx^3$$

Momentum distribution:

$$N_1(p) = \int S(x,p)d^3xd\tau \qquad (23)$$

Single-particle momentum distribution for $S = t/r$

Let us choose the $S = t/r$ scaling variable and $\mathcal{N}(S) = e^{-b\frac{x^2+x^2+z^2}{R_0^2 t^2}}$
Only free parameters: c and δ

Figure: Different values of c, $\delta = 0.5$

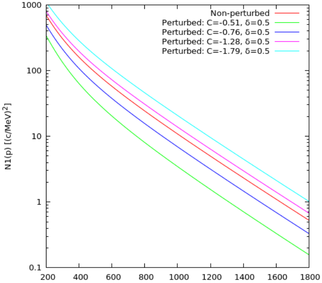
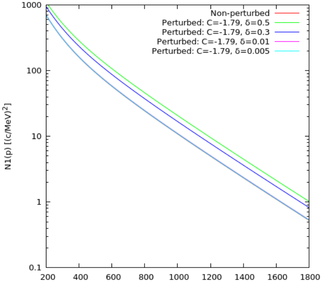


Figure: Different values of δ, c=-1.79



Summary

Hubble-flow

$$u^\mu = \frac{x^\mu}{\tau}$$

$$p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^{3} \mathcal{N}(S)$$

Perturbations

$$\delta u^\mu = \delta \cdot \left(\tau + c\tau_0^{\frac{-\kappa-3}{\kappa}} \tau^{\frac{3}{\kappa}}\right) \partial^\mu S \cdot \chi(S)$$

$$\delta p = \delta \cdot p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}} \pi(S)$$

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau}\right)^{3} h(x,y,z,t)\nu(S)$$

Accomplishments

- New relativistic perturbative solutions
- Broad family, new particular interesting solutions possible
- Many possibilities for scaling variables
- Single perturbation scale δ

Outlook

- Other scaling variable S, and $h(x, y, z, t)$
- Apply the same method for other known solutions

Thank you for your attention!

Choosing pressure perturbation field

Pressure: $p = p_0 \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}}$

Pressure perturbation

$$\delta p = \delta \cdot p_0 \pi(S) \left(\frac{\tau_0}{\tau}\right)^{3+\frac{3}{\kappa}} \qquad (24)$$

Test function for pressure substituted to (5) and (6) yields:

Energy equation

$$\partial_\mu \delta u^\mu = 0 \qquad (25)$$

Euler equation

$$\frac{\kappa - 3}{\kappa \tau} \delta u^\nu + u^\mu \partial_\mu \delta u^\nu = \frac{1}{(\kappa+1)\delta} \pi'(S) \partial^\nu S \qquad (26)$$

Solving the energy equation

Velocity: $u^\mu = \frac{x^\mu}{\tau}$

Test function for four-velocity perturbation

$$\delta u^\mu = \delta \cdot F(\tau)\partial^\mu S \cdot \chi(S) \quad (27)$$

- Satisfies orthogonality ($\delta u_\mu u^\mu = 0$)
- Energy equation:

$$\delta \cdot F' u_\mu \partial^\mu S \cdot \chi(S) + \delta \cdot F \partial_\mu \partial^\mu S \cdot \chi(S) + \delta \cdot F \partial^\mu S \cdot \chi'(S) \partial_\mu S = 0$$

The final energy equation

$$\frac{\chi'(S)}{\chi(S)} = -\frac{\partial_\mu \partial^\mu S}{\partial_\mu S \partial^\mu S} \quad (28)$$

Right side must be a function of S

- restriction for scaling variable
- ordinary diff. eq. for $\chi(S)$

Solving the Euler equation

The Euler equation (substituting (11) into (10)):

$$\frac{\kappa-3}{\kappa}\frac{F(\tau)}{\tau}\delta\cdot\chi(S)\partial^\nu S+\delta\cdot u^\mu\partial_\mu(F(\tau)\cdot\chi(S)\partial^\nu S)=\frac{1}{\kappa+1}\delta\cdot\pi'(S)\partial^\nu S$$

$$\pi'(S)=\left(\frac{\kappa-3}{\kappa}\frac{F(\tau)}{\tau}-\frac{F(\tau)}{\tau}+F'(\tau)\right)(\kappa+1)\cdot\chi(S)$$

- $\pi(S)$ and $\chi(S)$ functions of S
- first term on the right: cannot depend on τ
- $F(\tau)=\tau+C\tau^{\frac{3}{\kappa}}$

The final equation from Euler equation

$$\pi'(S)=\frac{(\kappa+1)(\kappa-3)}{\kappa}\cdot\chi(S) \quad (29)$$

Solving the continuity equation

Particle density:
$n = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}(S)$

Test function for particle density perturbation

$$\delta n = \delta \cdot n_0 \left(\frac{\tau_0}{\tau}\right)^3 h(x,y,z,t)\nu(S) \qquad (30)$$

Substitute this into continuity (7):

The final equation from continuity equation

$$-\frac{\chi(S)\mathcal{N}'(S)}{\nu(S)} = \frac{1}{(\tau + C\tau^{\frac{3}{\kappa}})\partial_\mu S \partial^\mu S} u^\mu \partial_\mu h(x,y,z,t) \qquad (31)$$

Both sides must be functions of the scaling variable:

- restriction for $h(x,y,z,t)$ and S

Scaling variable $S = \frac{r^m}{\tau^m}$ and $S = \frac{\tau^m}{t^m}$

Scaling variable $S = \frac{r^m}{\tau^m}$

$$\chi(S) = \frac{S^{-\frac{m+1}{m}}}{\sqrt{S^{\frac{2}{m}} + 1}} \tag{32}$$

$$\pi(S) = \frac{(\kappa+1)(\kappa-3)}{\kappa}\left[-m\sqrt{1 + S^{-\frac{2}{m}}}\right] \tag{33}$$

Scaling variable $S = \frac{\tau^m}{t^m}$

$$\chi(S) = exp\left(\frac{2(m-2)\ln S - m(m+1)S^{\frac{2}{m}}}{2m(S^{\frac{2}{m}} - 1)}\right) \tag{34}$$

Equations of non-relativistic hydrodynamics

Looking for (u, p, ρ) fields

Assumptions:

- zero viscosity
- zero heat conductivity

Euler-equation

$$\frac{\partial u}{\partial t} + (u\nabla)u = -\frac{1}{\rho}\nabla p$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla(\rho u) = 0$$

Equation of state

$p - \rho$ relation

Perturbative equations

Perturbed fields

- $u \to u + \delta u$
- $p \to p + \delta p$
- $\rho \to \rho + \delta\rho$

Perturbed equations

- first order perturbation
- using another solution

Wave solution

Known solution: Standing fluid

- $u = 0$
- $p = \text{const.}$
- $\rho = \text{const.}$

Sound speed from equation of state:

$$\frac{\delta p}{\delta \rho} = c^2$$

Perturbed Euler-equation

$$\frac{\partial \delta u}{\partial t} = -\frac{1}{\rho}\nabla \delta p$$

Perturbed continuity equation

$$\frac{\partial \delta \rho}{\partial t} + \rho \nabla \delta u = 0$$

Wave solution for pressure

$$\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$$

Source function

$$S(x,p) = N\delta(\tau-\tau_0)d\tau d^3 x n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}(S)$$

$$\exp\left[-\frac{Et - xp_x - yp_y - zp_z}{\tau T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{3}{\kappa}}}\mathcal{N}(S)\right]\left(E - \frac{xp_x + yp_y + zp_z}{t}\right)\cdot$$

$$\cdot\left[1 + \delta\Bigg(- \frac{(\tau + c\tau_0^{\frac{\kappa-3}{\kappa}}\tau^{\frac{3}{\kappa}})\partial^0 S\chi(S)\tau}{t} + \right.$$

$$+ \frac{(\tau + c\tau_0^{\frac{\kappa-3}{\kappa}}\tau^{\frac{3}{\kappa}})\chi(S)t}{Et - xp_x - yp_y - zp_z} p_\mu \partial^\mu S +$$

$$+ \frac{(Et - xp_x - yp_y - zp_z)(\mathcal{N}(S)\pi(S) - h(x,y,z,t)\nu(S))}{\tau T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{3}{\kappa}}} +$$

$$\left. + \frac{h(x,y,z,t)\nu(S)}{\mathcal{N}(S)}\Bigg)\right]$$

Single-particle distribution

$$N(p) = Nn_0 \exp\left[-\frac{E}{T_0} + \frac{p^2 R_0^2}{2T_0(2T_0 b + ER_0^2 - 2Eb)}\right] \sqrt{\frac{2\pi T_0 \tau_0^2}{E}\left(1 - \frac{T_0}{T_1}\right)}^3 \left(E - \frac{p^2}{E}\left(1 - \frac{T_0}{T_1}\right)\right) \cdot$$

$$\cdot \left[1 - \frac{\delta(\tau_0^2 + c\tau_0^2)}{r_1\sqrt{\tau_0^2 + r_1^2}} + \frac{\delta(\tau_0 + c\tau_0)}{E - \frac{p_x x_1 + p_y y_1 + p_z z_1}{\sqrt{\tau_0^2 + r_1^2}}} \left(\frac{E}{r_1} - (p_x x_1 + p_y y_1 + p_z z_1)\frac{\sqrt{\tau_0^2 + r_1^2}}{r_1^3}\right) + \right.$$

$$\left. + \frac{\delta 2bc\kappa}{(3-\kappa)R_0^2}\left(\frac{r_1}{\sqrt{\tau_0^2 + r_1^2}}\right)^3 \left(\frac{\tau_0}{r_1}\right)^4 \right] +$$

$$+ Nn_0 \exp\left[-\frac{E}{T_0} + \frac{p^2 R_0^2}{2T_0(4T_0 b + ER_0^2 - 2Eb)}\right] \sqrt{\frac{2\pi T_0 \tau_0^2}{E}\left(1 - \frac{T_0}{T_2}\right)}^3 \left(E - \frac{p^2}{E}\left(1 - \frac{T_0}{T_2}\right)\right) \cdot$$

$$\cdot \left[\frac{\delta 2bE\sqrt{\tau_0^2 + r_2^2} - p_x x_2 - p_y y_2 - p_z z_2}{R_0^2 \tau_0 T_0}\left(\frac{(\kappa+1)(\kappa-3)}{\kappa}\frac{\tau_0^2 + r_2^2}{r_2} - C\frac{\kappa}{3-\kappa}\tau_0\right)\left(\frac{r_2}{\sqrt{\tau_0^2 + r_2^2}}\right)^3 \left(\frac{\tau_0}{r_2}\right)^4 - \right.$$

$$\left. - \frac{\delta(\tau_0 + C\tau_0)}{T_0}\left(\frac{E}{r_2} - (p_x x_2 + p_y y_2 + p_z z_2)\frac{\sqrt{\tau_0^2 + r_2^2}}{r_2^3}\right)\right]$$