Operator Product Expansion in the Gradient Flow Formalism

Robert Harlander
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Thomas Mannel Fest 2018

Oct 5, 2018

General problem

$$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

e.g.: C_n in $D = 4 - 2\epsilon$ $\langle \mathcal{O}_n \rangle$ on lattice

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flowed operators: $\mathcal{O}_n \to \tilde{\mathcal{O}}_n(t)$ finite!

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t) \mathcal{O}_m + \ldots$$

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$$\sum_n C_n \mathcal{O}_n = \sum_n \tilde{C}_n(t)\, \tilde{\mathcal{O}}_n(t)$$

Finite, and calculable in perturbation theory!

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The flow equation

fundamental QCD: $\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}$

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] \qquad D_\mu = \partial_\mu - iT^a A^a_\mu(x)$$

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flowed gauge field:

$$\frac{\partial}{\partial t}B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

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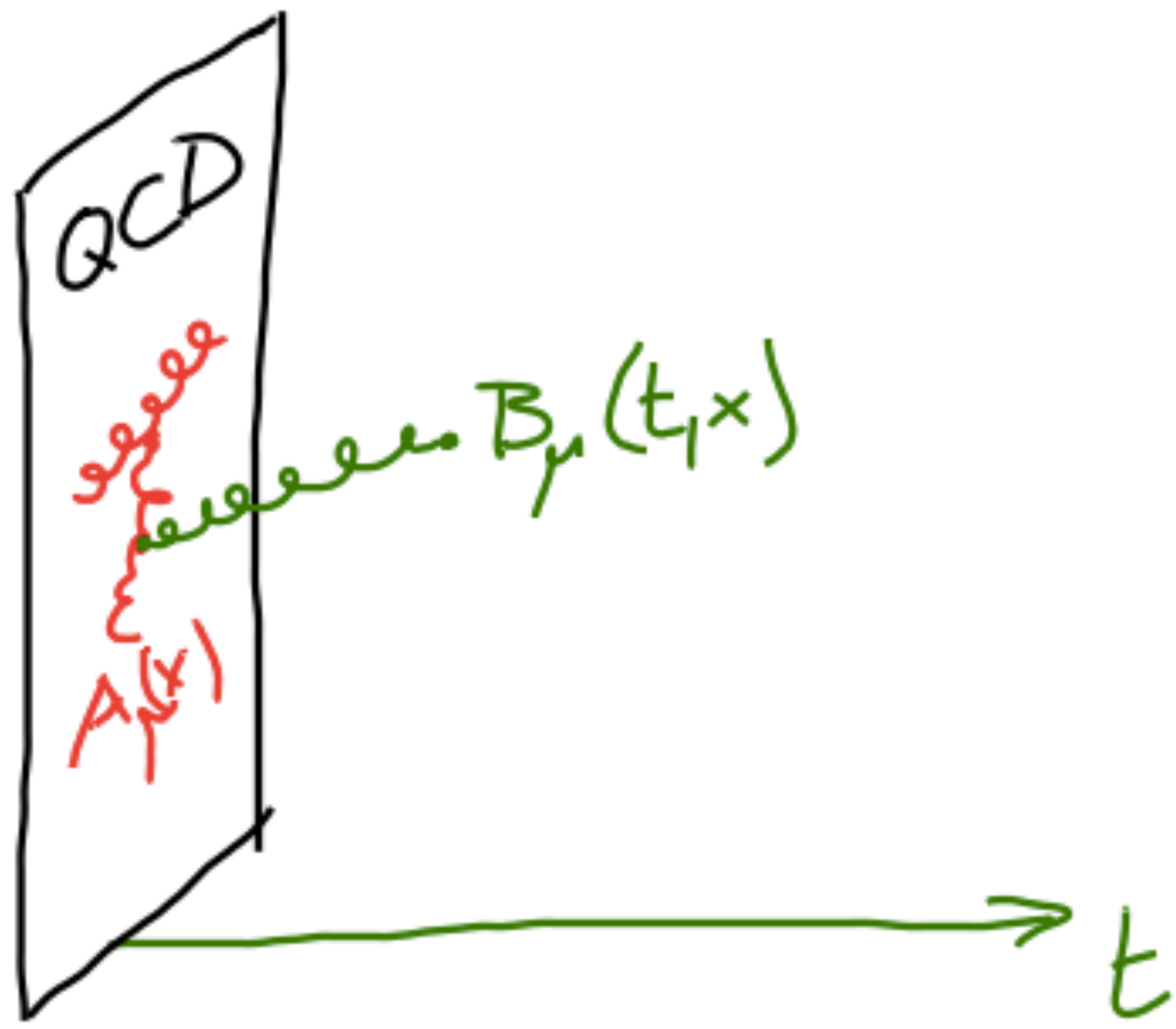
The flow field



Schematically…

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$\mathscr{D}_\mu = \partial_\mu - i T^a B^a_\mu(t,x)$$

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Schematically…

$$\frac{\partial}{\partial t}B = \mathscr{D}\ G$$

$$\mathscr{D} = \partial - B$$

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$$\frac{\partial}{\partial t} B \qquad = \mathscr{D}\ G$$

$$\mathscr{D} \ = \partial \ - \qquad B$$

$$G \sim [\mathscr{D}, \mathscr{D}] \sim \partial B + B^2$$

Schematically…

$$\frac{\partial}{\partial t} B = \mathscr{D}\, G$$

$$\mathscr{D} = \partial - B$$

$$G \sim [\mathscr{D}, \mathscr{D}] \sim \partial B + B^2$$

flow equation: $\dot{B} \sim \partial^2 B + \partial B^2 + B^3$

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Perturbative solution

flow equation: $\dot{B} \sim \partial^2 B + \partial B^2 + B^3$

$$\tilde{B}(p) = \int_x e^{ipx} B(x) \quad = g\tilde{B}_1 + g^2\tilde{B}_2 + \ldots$$

$$\mathcal{O}(g): \quad \dot{\tilde{B}}_1 = -p^2\tilde{B}_1$$

$$\mathcal{O}(g^2): \quad \dot{\tilde{B}}_2 = -p^2\tilde{B}_2 + ip\tilde{B}_1 \otimes \tilde{B}_1$$

$$\ldots$$

LO: $\dot{\tilde{B}}_1 = -p^2\tilde{B}_1 \quad \Rightarrow \quad \tilde{B}_1 = e^{-tp^2}\tilde{A}$

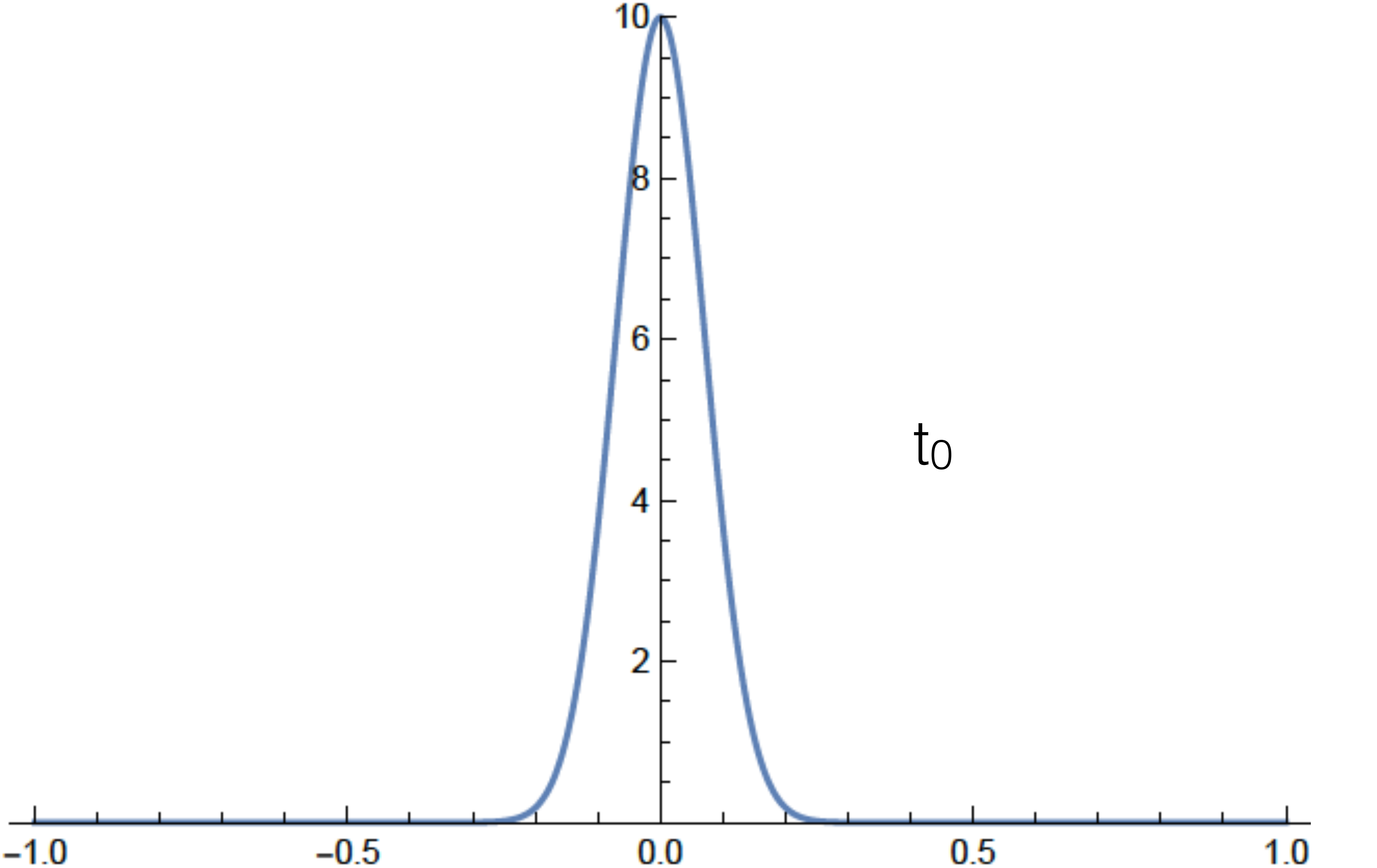
Leading order (1D)

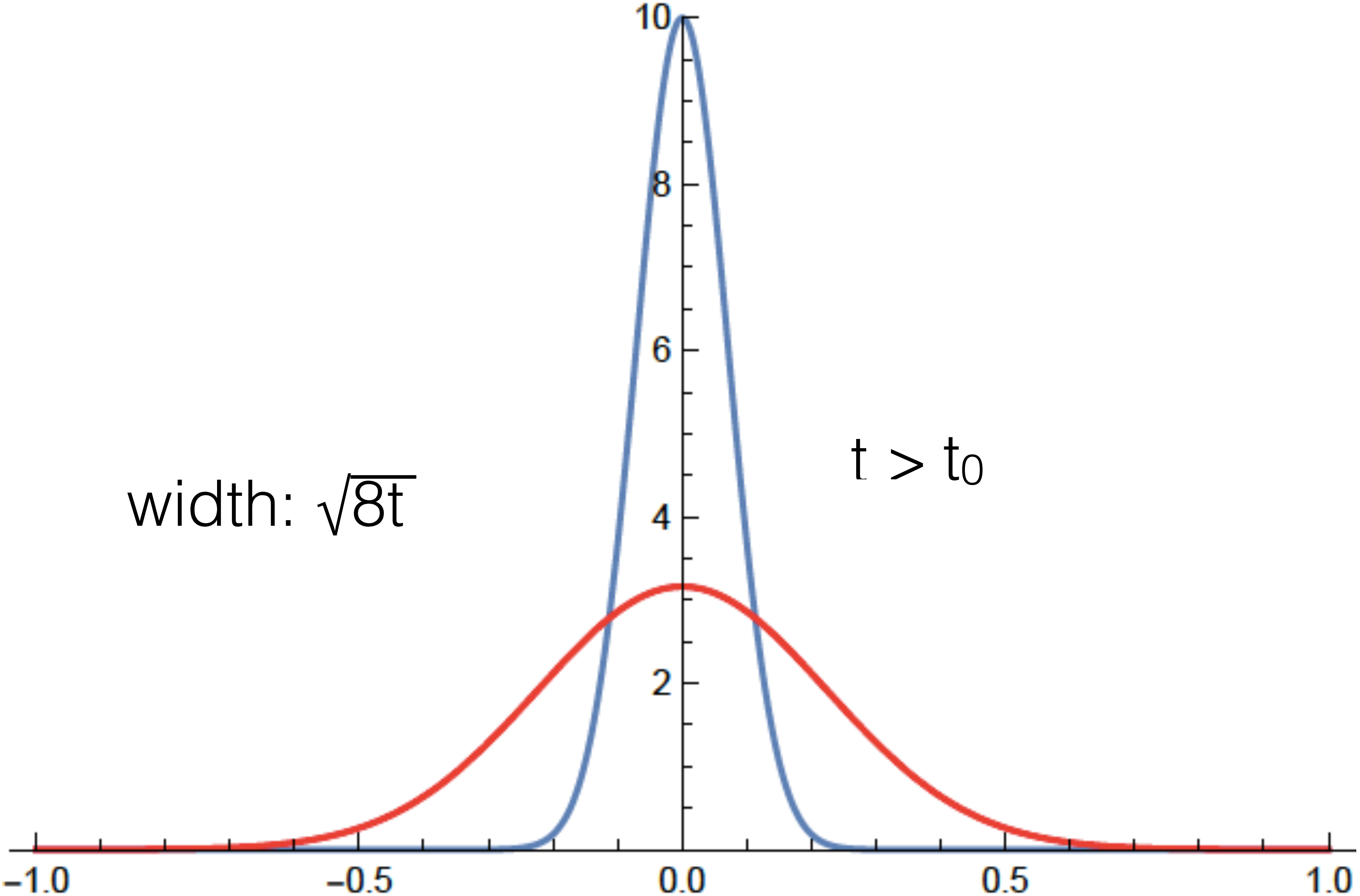
$$\dot{\tilde{B}}_1 = -p^2\tilde{B}_1 \quad \Rightarrow \quad \tilde{B}_1 = e^{-tp^2}\tilde{A}$$

example: $B_1(t=0,x) = c\,\delta(x)$

$$\tilde{B}_1(t=0,p) = \frac{c}{2\pi} \qquad \tilde{B}_1(t,p) = \frac{c}{2\pi}e^{-tp^2}$$

$$B_1(t,x) = \frac{c}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$$


10
8
6
4
2
−1.0
−0.5
0.0
0.5
1.0
t0
x



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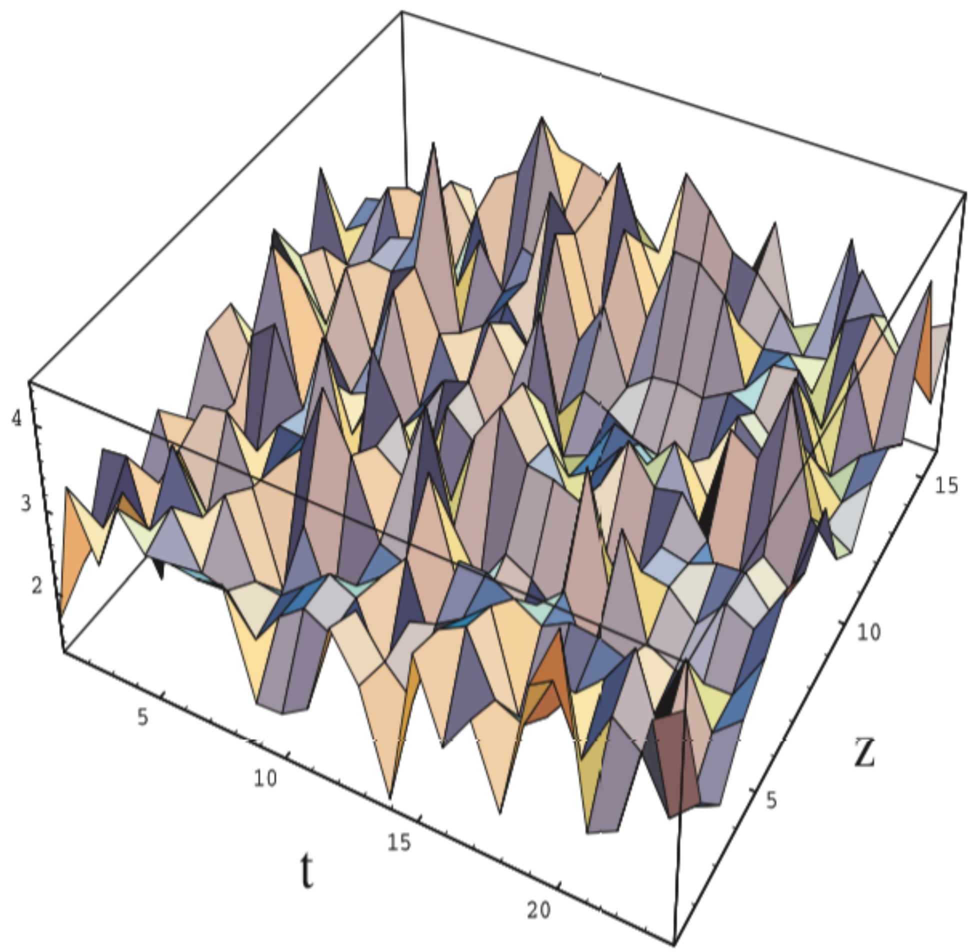
Lattice QCD

quantum fluctuations:



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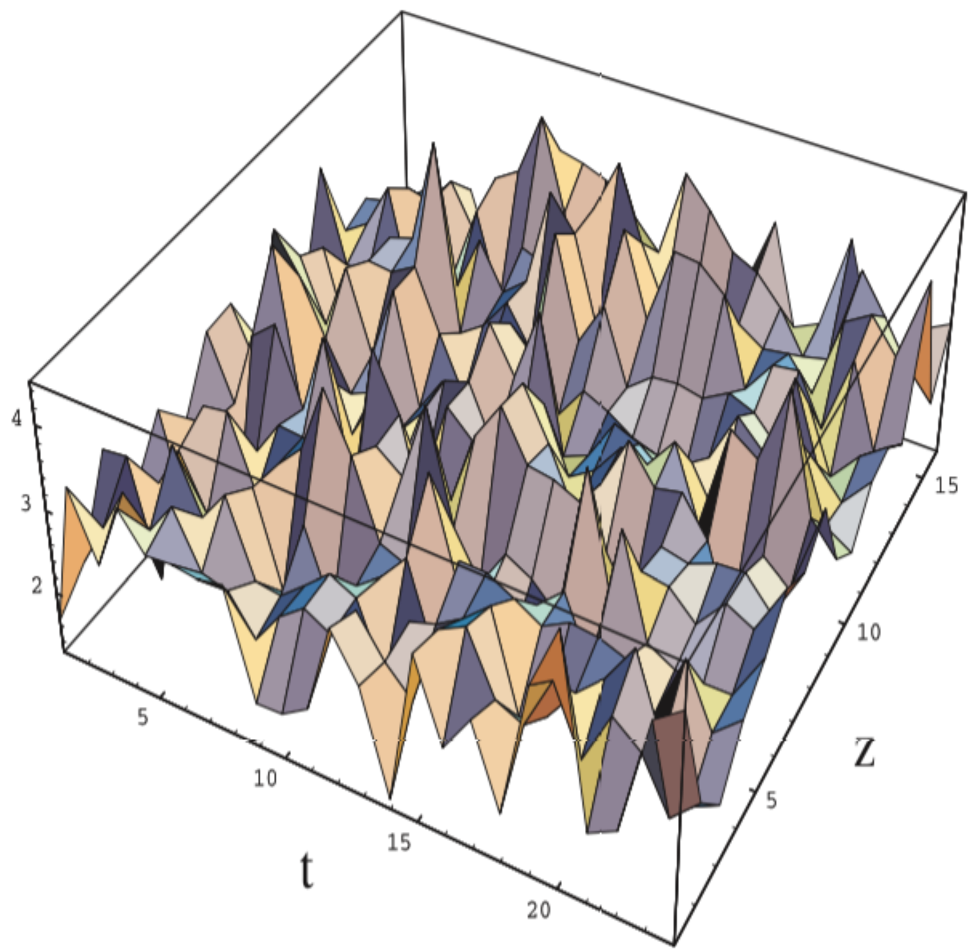
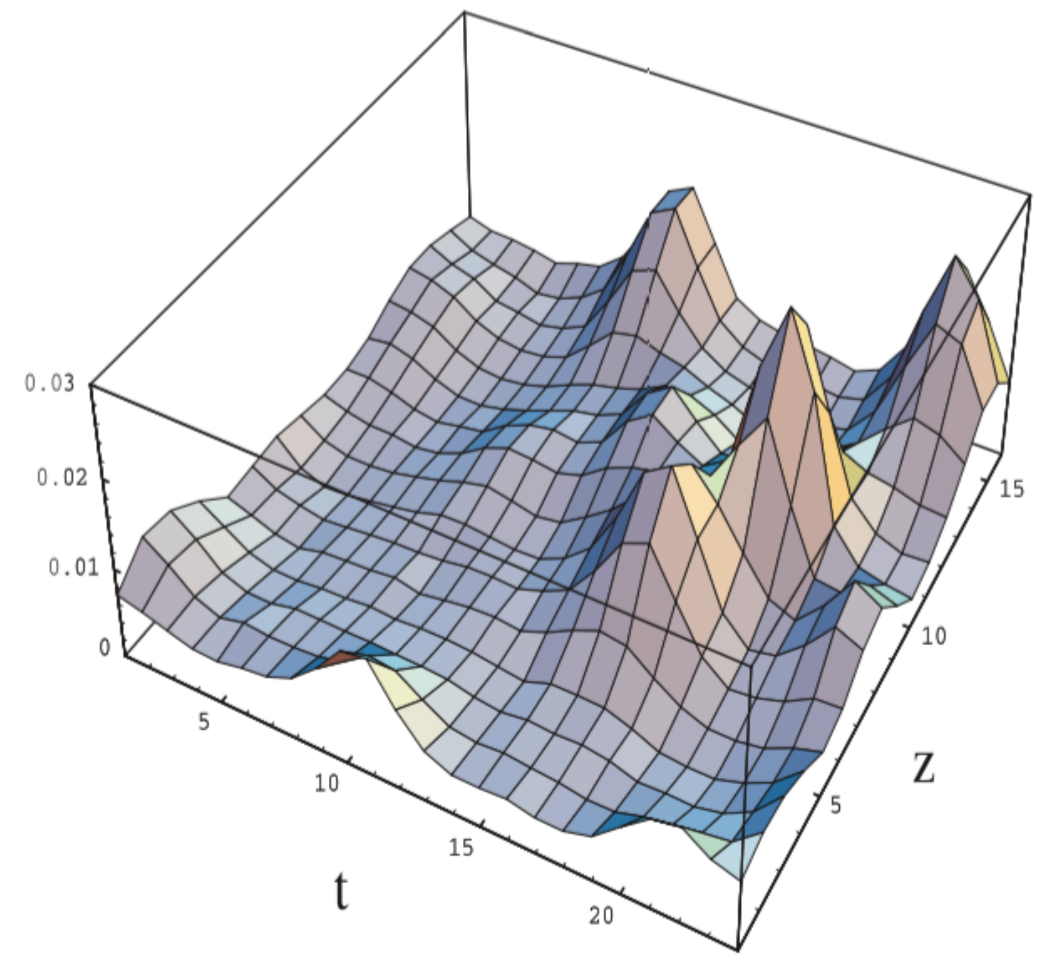


"smearing":



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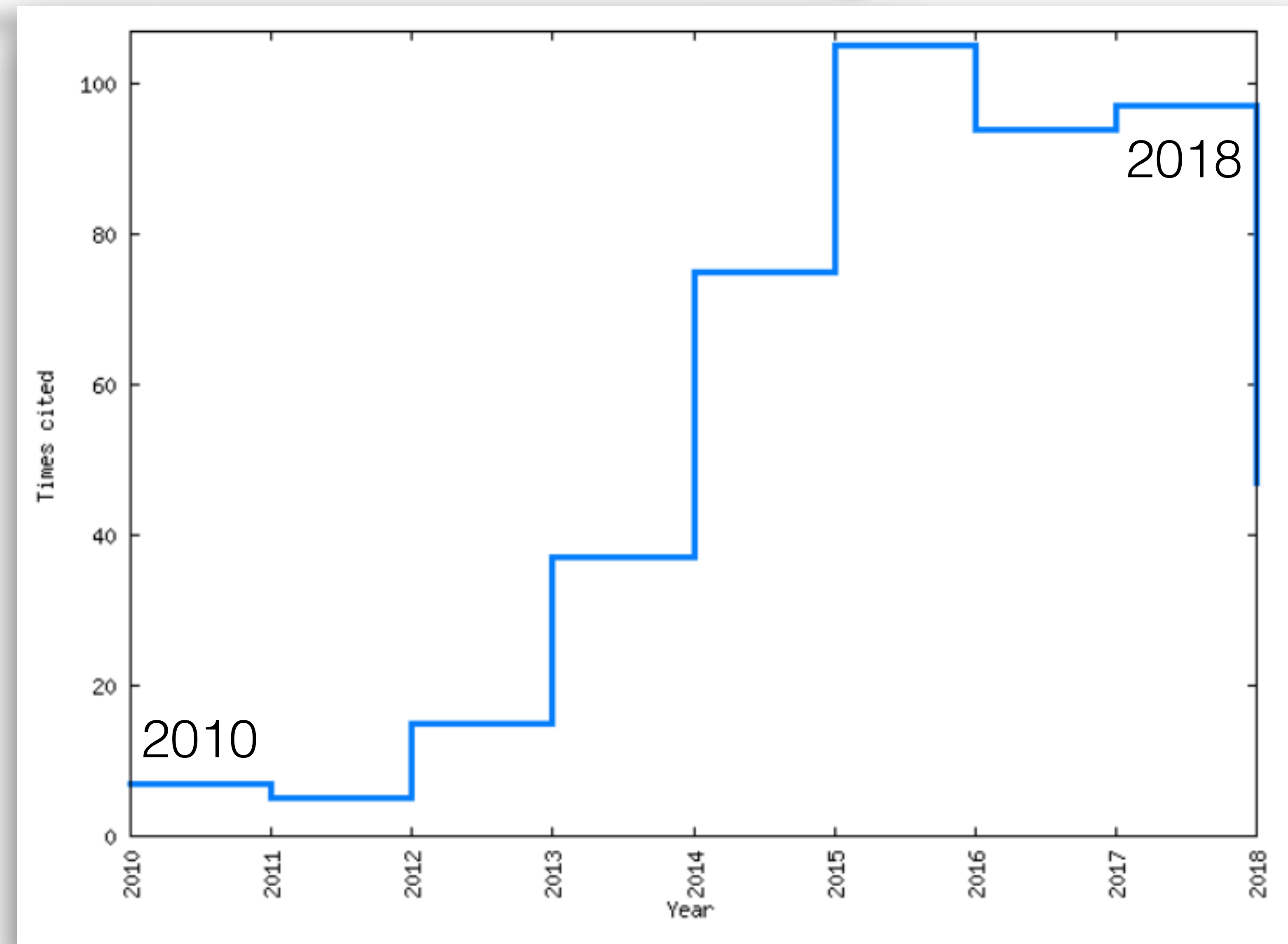
Properties and uses of the Wilson flow in lattice QCD

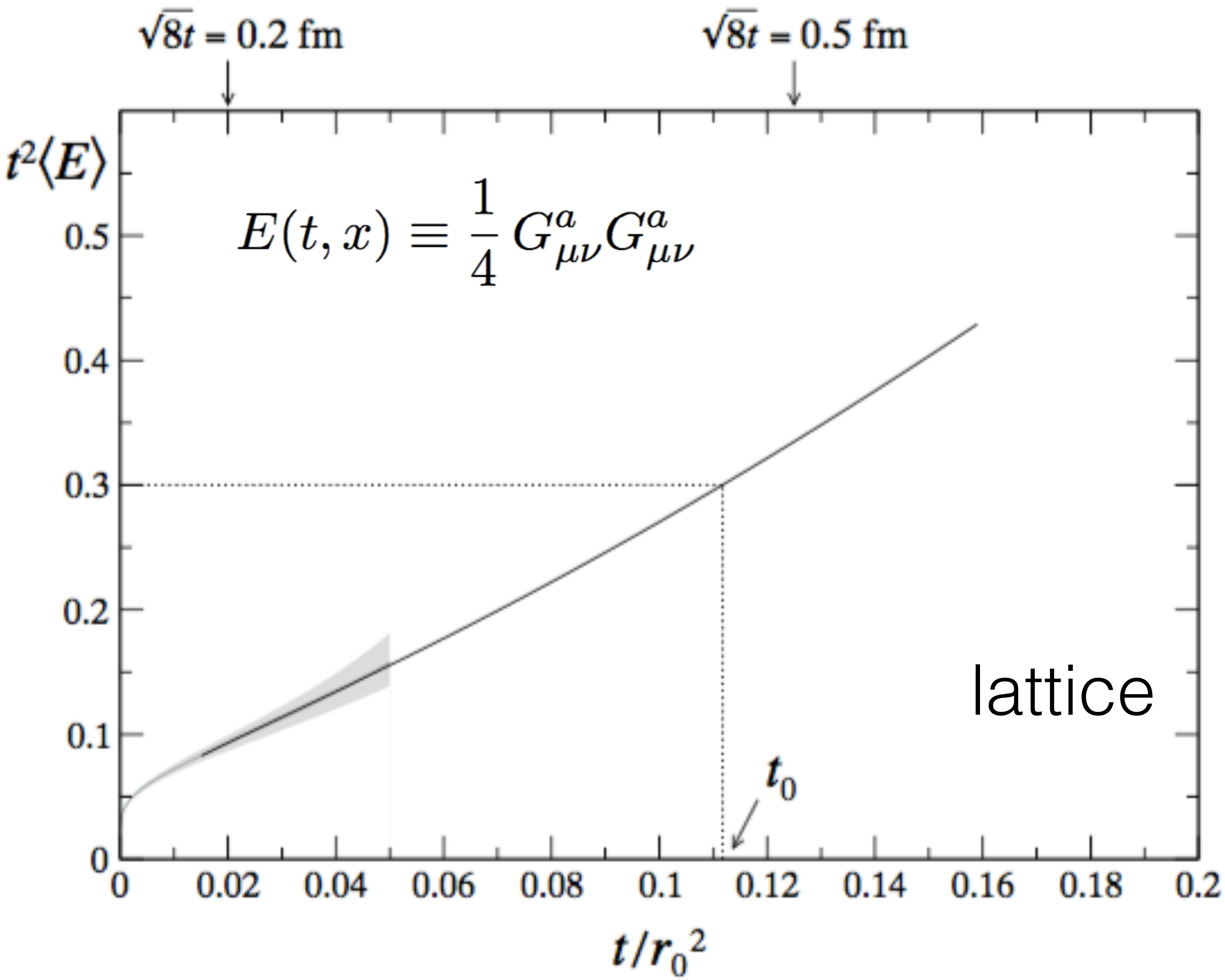
Martin Lüscher (CERN & Geneva U.)

Jun 23, 2010 - 21 pages

JHEP 1008 (2010) 071
Erratum: JHEP 1403 (2014) 092
(2010)
DOI: 10.1007/JHEP08(2010)071, 10.1007/JHEP03(2014)092
CERN-PH-TH-2010-143
e-Print: arXiv:1006.4518 [hep-lat] | PDF



Scale setting



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Higher orders

$$\dot{\tilde{B}}_1 = -p^2 \tilde{B}_1 \quad \Rightarrow \quad \tilde{B}_1 = e^{-tp^2} \tilde{A}$$

$$\dot{\tilde{B}}_2 = -p^2 \tilde{B}_2 + ip\tilde{B}_1 \otimes \tilde{B}_1$$

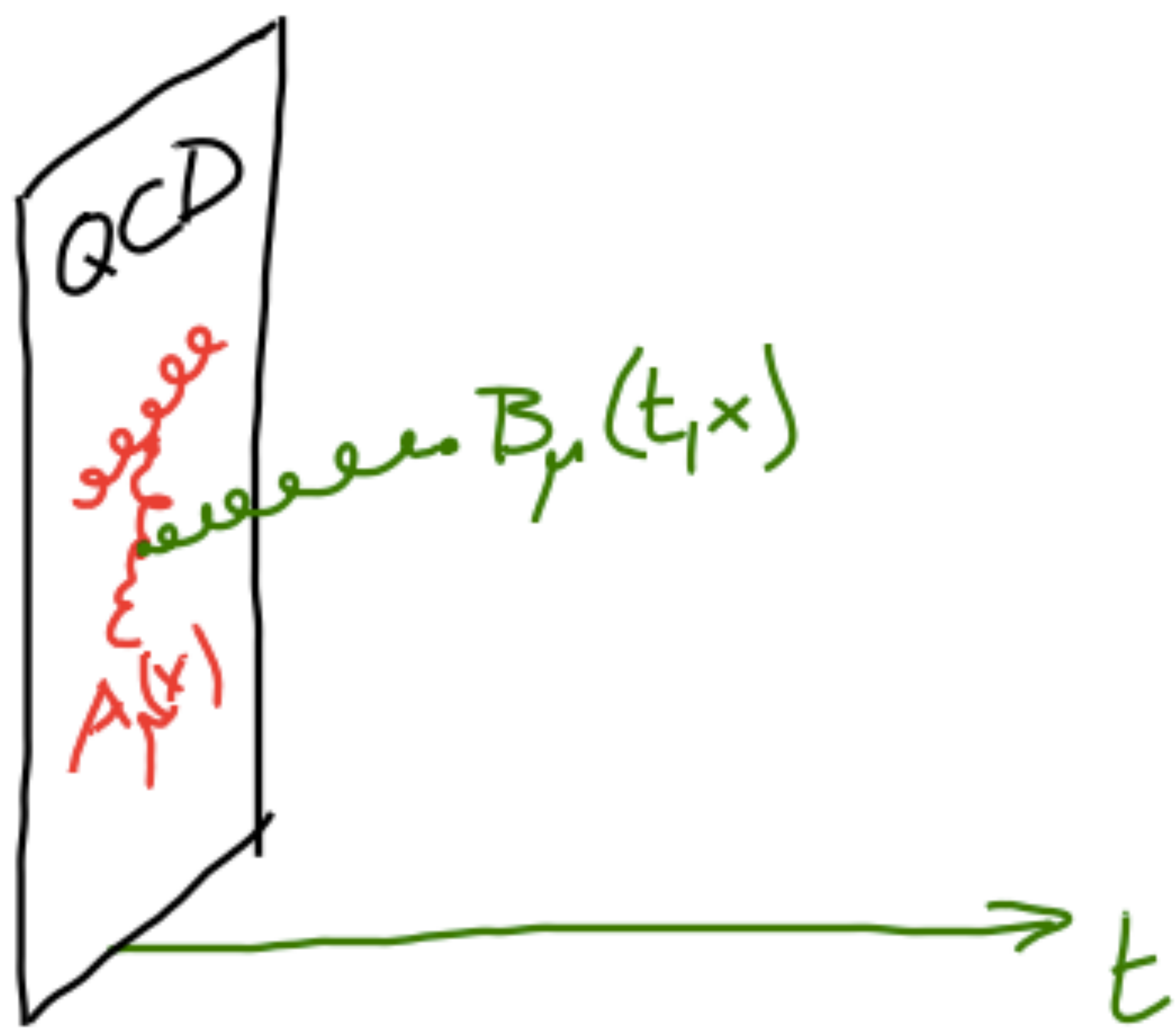
$$\Rightarrow \quad \tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

$$K(t,s,p,q) \sim \exp\left[-tp^2 - 2sq(q-p) \right]$$

etc.

Exponential damping in momentum integrals!

The flow field



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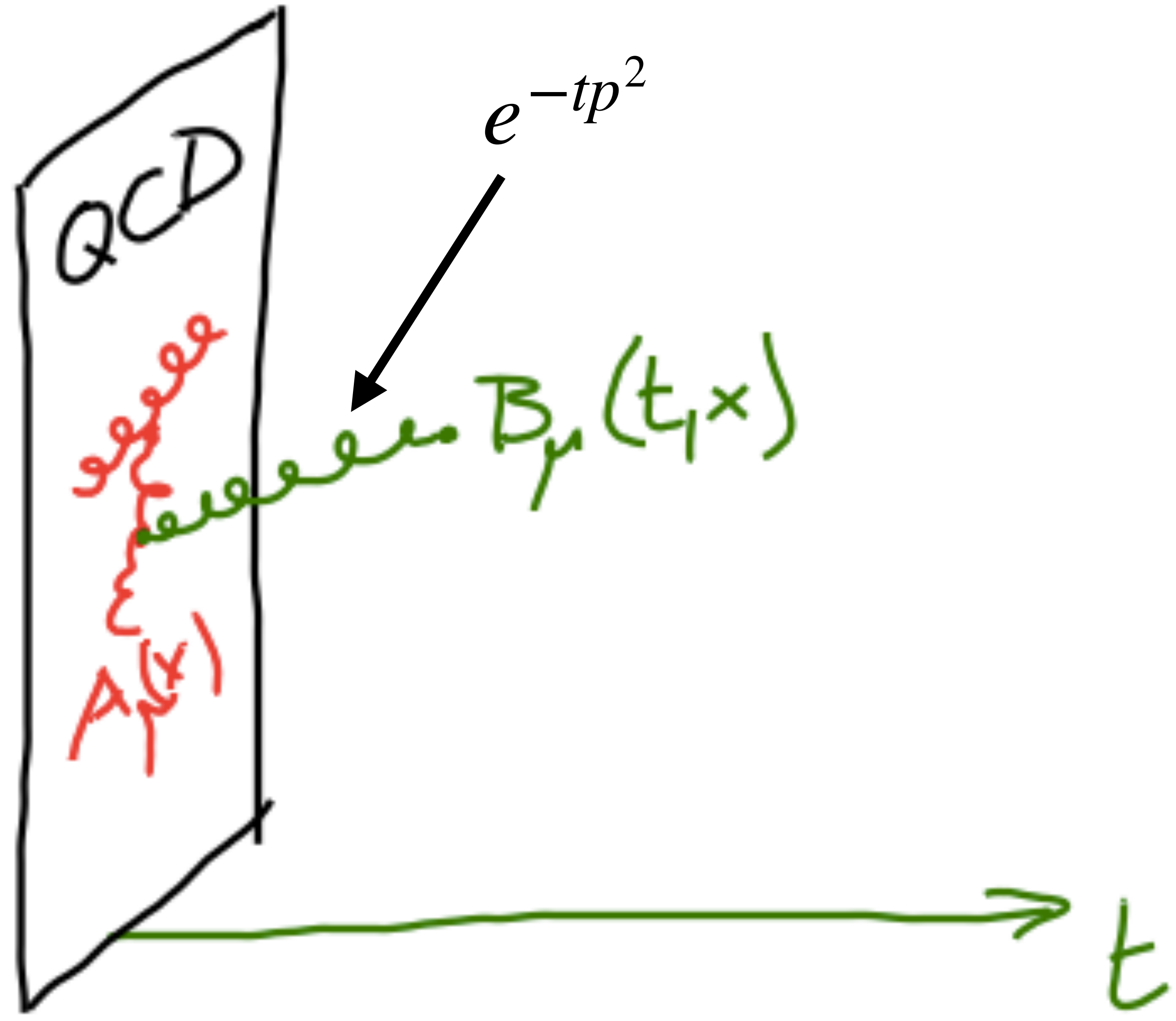
The flow field



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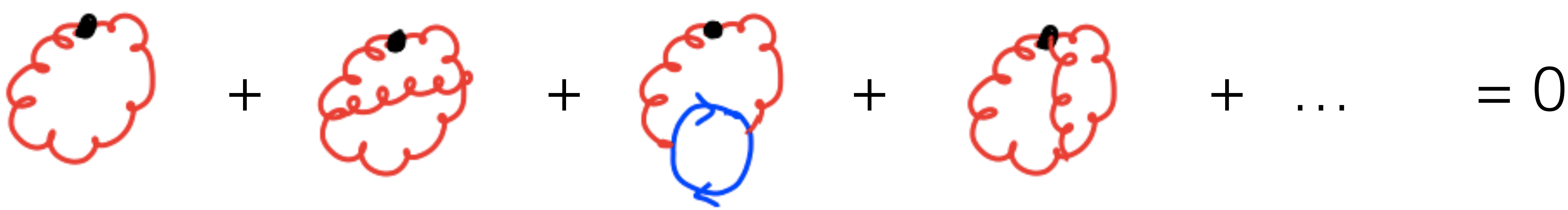
Let's calculate

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

$$\langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

at t=0 (i.e. fundamental QCD):



in dim. reg. (m_q=0)

with gradient flow: $\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$

t has dimension mass^{-2} !

Let's calculate

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$$\langle E \rangle \sim \langle \partial B \, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

LO:  $\sim \int d^D p \, e^{-2tp^2} \sim t^{-D/2} \to t^{-2} \quad \neq 0$

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explicitly: $\langle E(t) \rangle = \frac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$

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→ measure α_s on the lattice?

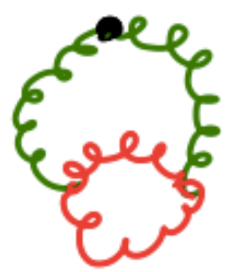
Higher orders

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \qquad \langle E \rangle \sim \langle \partial B\, \partial B \rangle + \langle B^2 \partial B \rangle + \langle B^4 \rangle$$

$$\tilde{B}_1(t,p) = e^{-tp^2} \tilde{A}(p)$$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

$+$



$+ \ldots$

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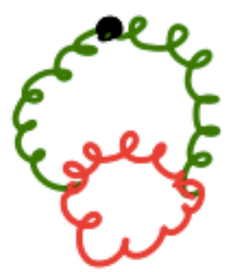

+ …

- one more momentum integration

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 +  + …

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- additional integration over flow-time parameter *s*

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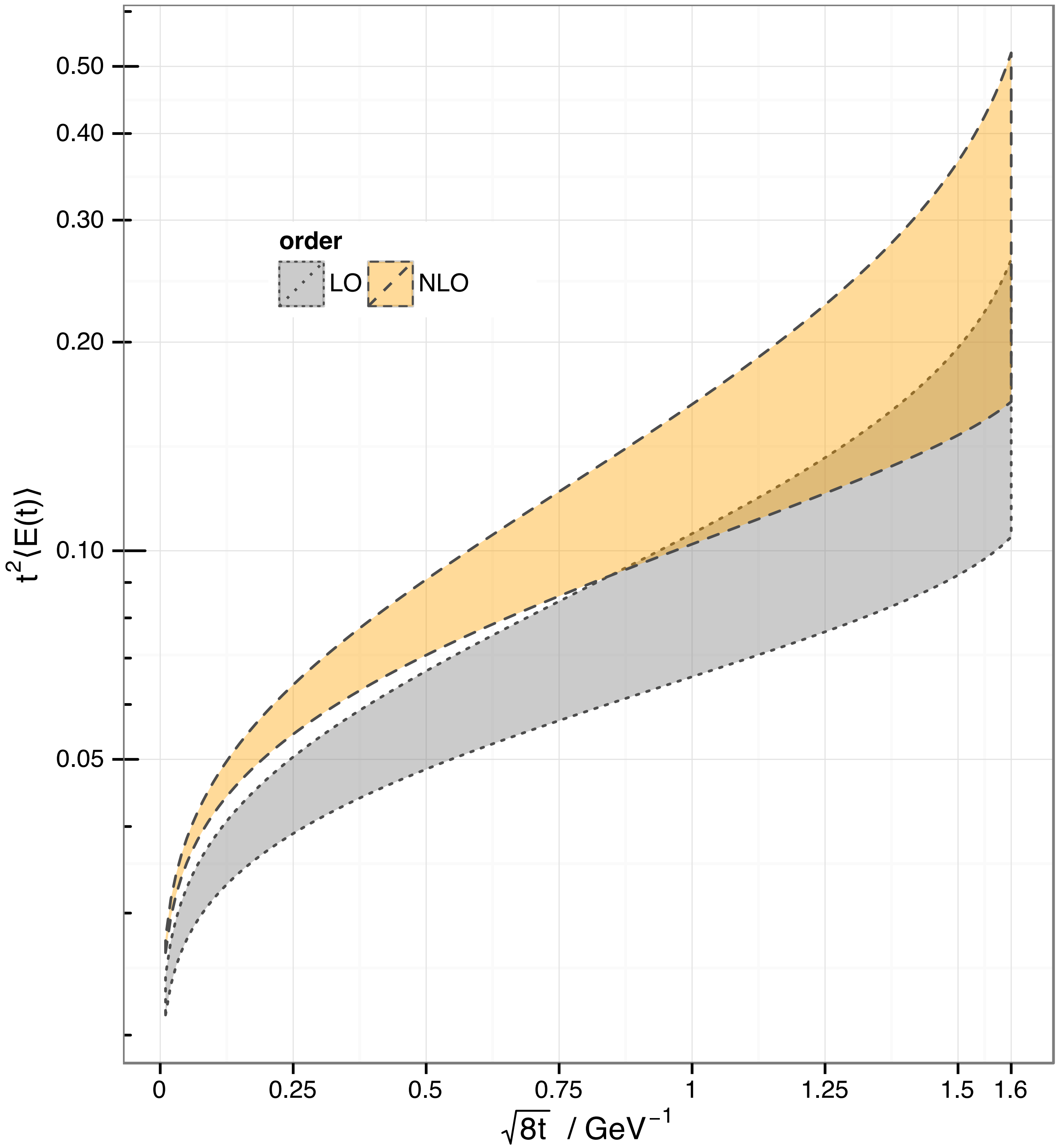
- one more momentum integration
- additional integration over flow-time parameter *s*
- renormalization: same as fundamental QCD!

$$\langle t^2 E(t)\rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher ‘10

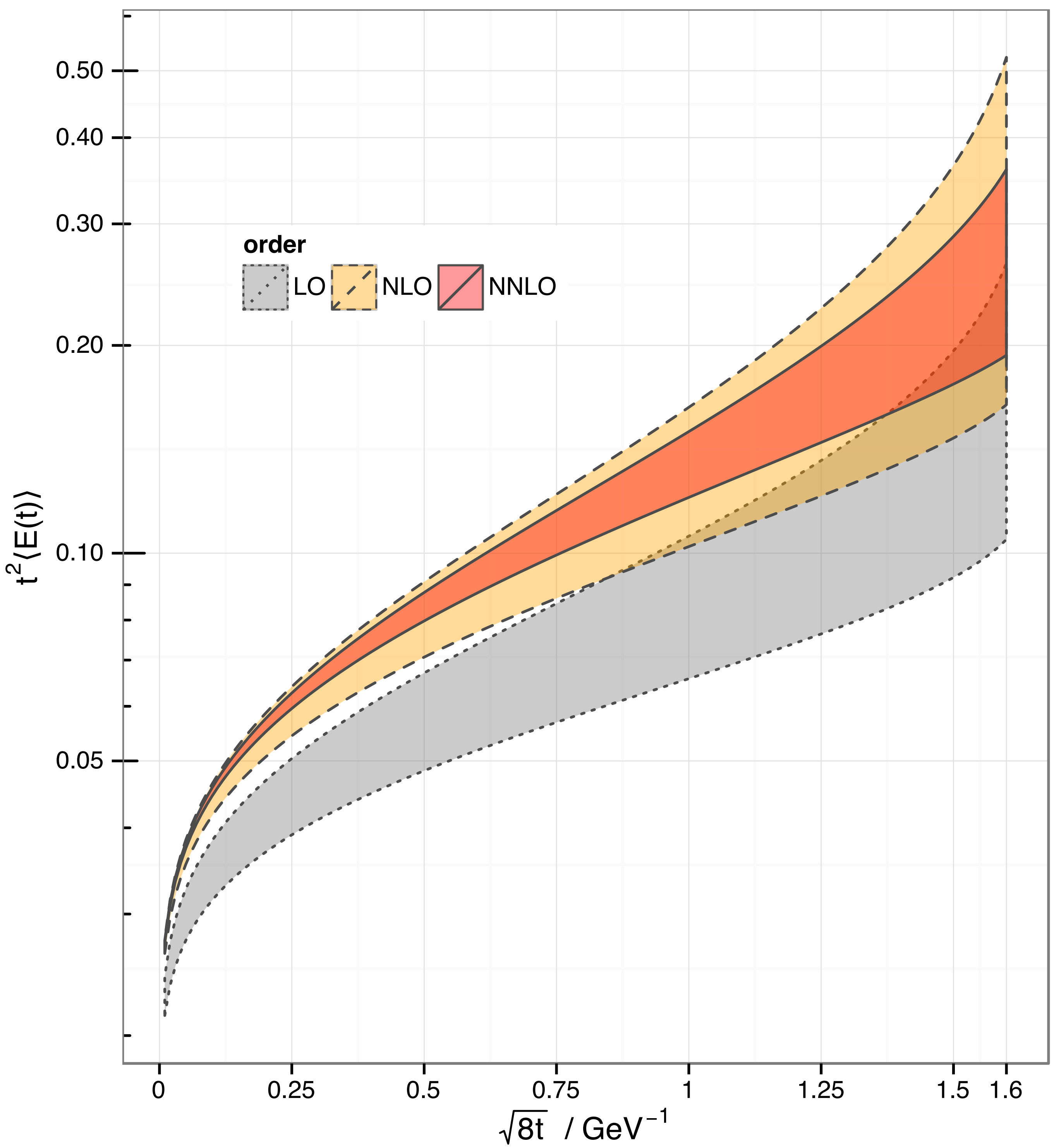


resulting perturbative accuracy on α_S: ± 3-5%

PDG: ± 1%

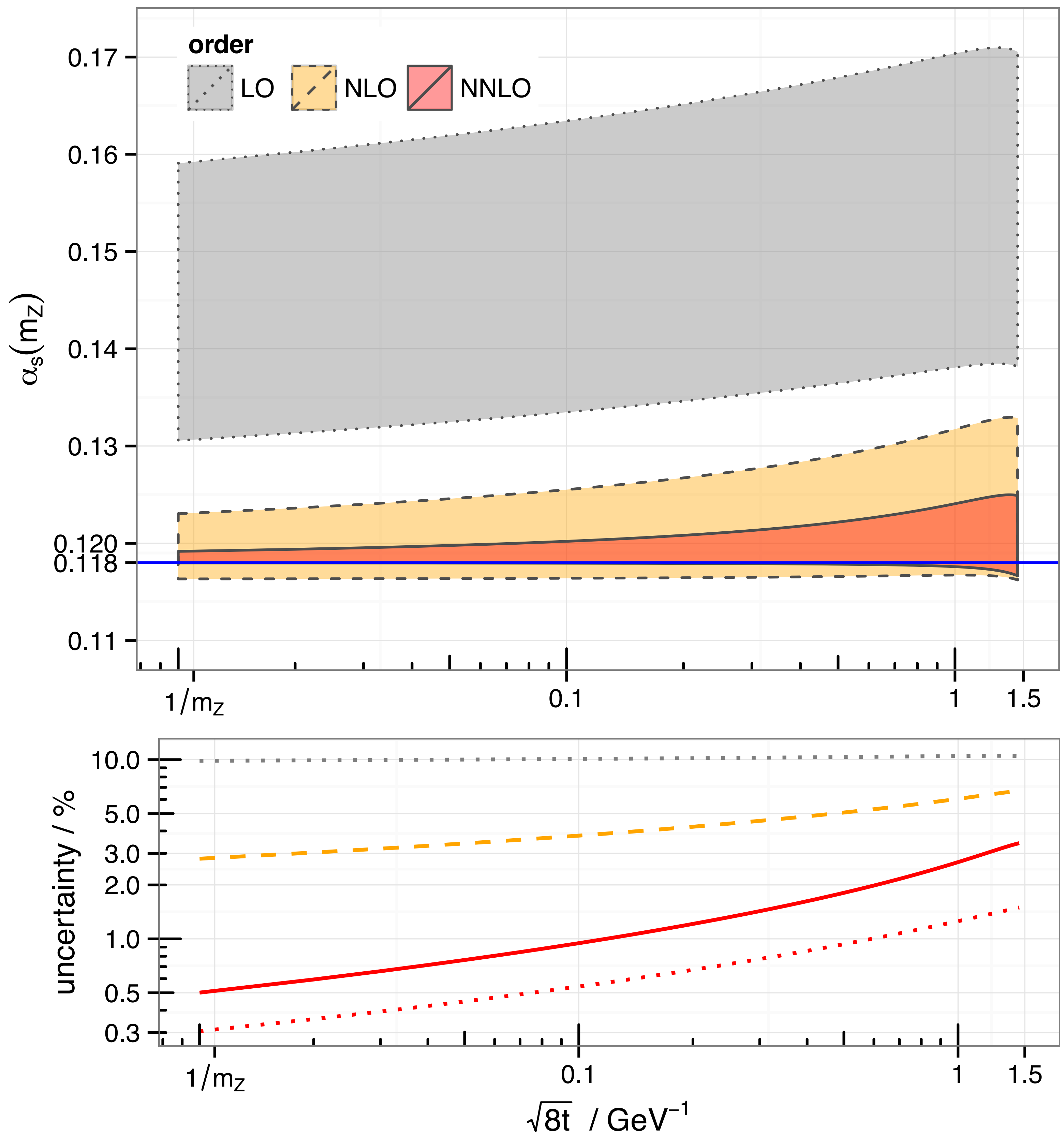
$$\langle t^2 E(t) \rangle = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann ‘16



resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%



RH, Neumann '16

Flowed operators

$$\tilde{\mathcal{O}}_n(t) = \sum_m \zeta_{nm}(t)\mathcal{O}_m + \ldots$$

finite because of e^{-tp^2}

$$P_n[\mathcal{O}_k] = \delta_{nk}$$

$$\zeta_{nm}(t) = P_m[\tilde{\mathcal{O}}_n(t)]$$

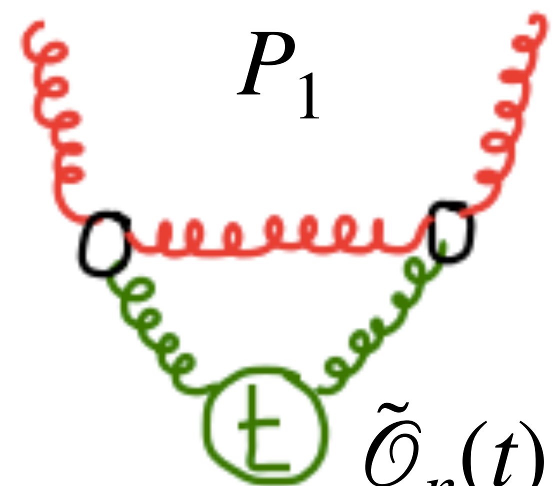
$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

P_1

$\mathcal{O}_1$

$\rightarrow \sum_n$

P_1



$\tilde{\mathcal{O}}_n(t)$

Energy-momentum tensor

in QCD:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2,\mu\nu}(x)\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu}(x)$$

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho}$$

$$\mathcal{O}_{2,\mu\nu} = \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}$$

$$\mathcal{O}_{3,\mu\nu} = \overline{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi$$

$T_{\mu\nu}(x)$ is finite!

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EMT in Gradient Flow

$$T_{\mu\nu}(x) = \sum_{i=1}^{4} c_i(t) \mathcal{Q}_{i,\mu\nu}(t,x)$$

$$c_i(t) = \frac{1}{{g_0}^2}\left[\zeta_{1i}^{-1}(t) - \frac{1}{4}\zeta_{2i}^{-1}(t)\right] + \frac{1}{4}\zeta_{3i}^{-1}(t)$$

idea and NLO result: H. Suzuki '14

NNLO result

$$c_1(t) = \frac{1}{g^2}\Bigg\{1 + \frac{g^2}{(4\pi)^2}\left[-\frac{7}{3}C_A + \frac{3}{2}T_F - \beta_0\, L(\mu,t)\right]$$
$$+ \frac{g^4}{(4\pi)^4}\Bigg[-\beta_1\, L(\mu,t) + C_A^2\left(-\frac{14482}{405} - \frac{16546}{135}\ln 2 + \frac{1187}{10}\ln 3\right)$$
$$+ C_A T_F\left(\frac{59}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{10873}{810} + \frac{73}{54}\pi^2 - \frac{2773}{135}\ln 2 + \frac{302}{45}\ln 3\right)$$
$$+ C_F T_F\left(-\frac{256}{9}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{2587}{108} - \frac{7}{9}\pi^2 - \frac{106}{9}\ln 2 - \frac{161}{18}\ln 3\right)\Bigg]$$
$$+ \mathcal{O}(g^6)\Bigg\},$$

$$L(\mu,t) \equiv \ln\left(2\mu^2 t\right) + \gamma_{\mathrm{E}}$$

RH, Kluth, Lange '18

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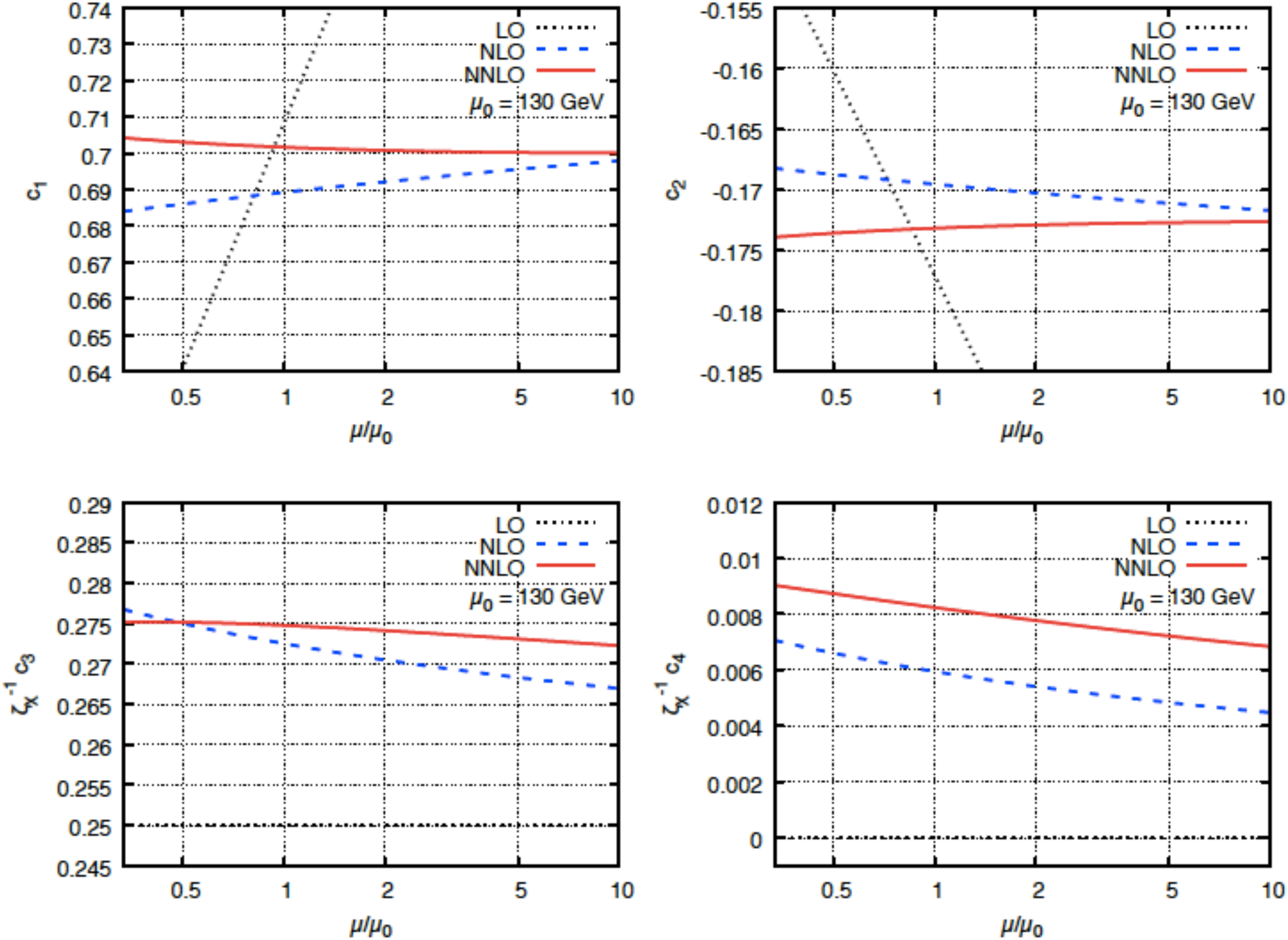
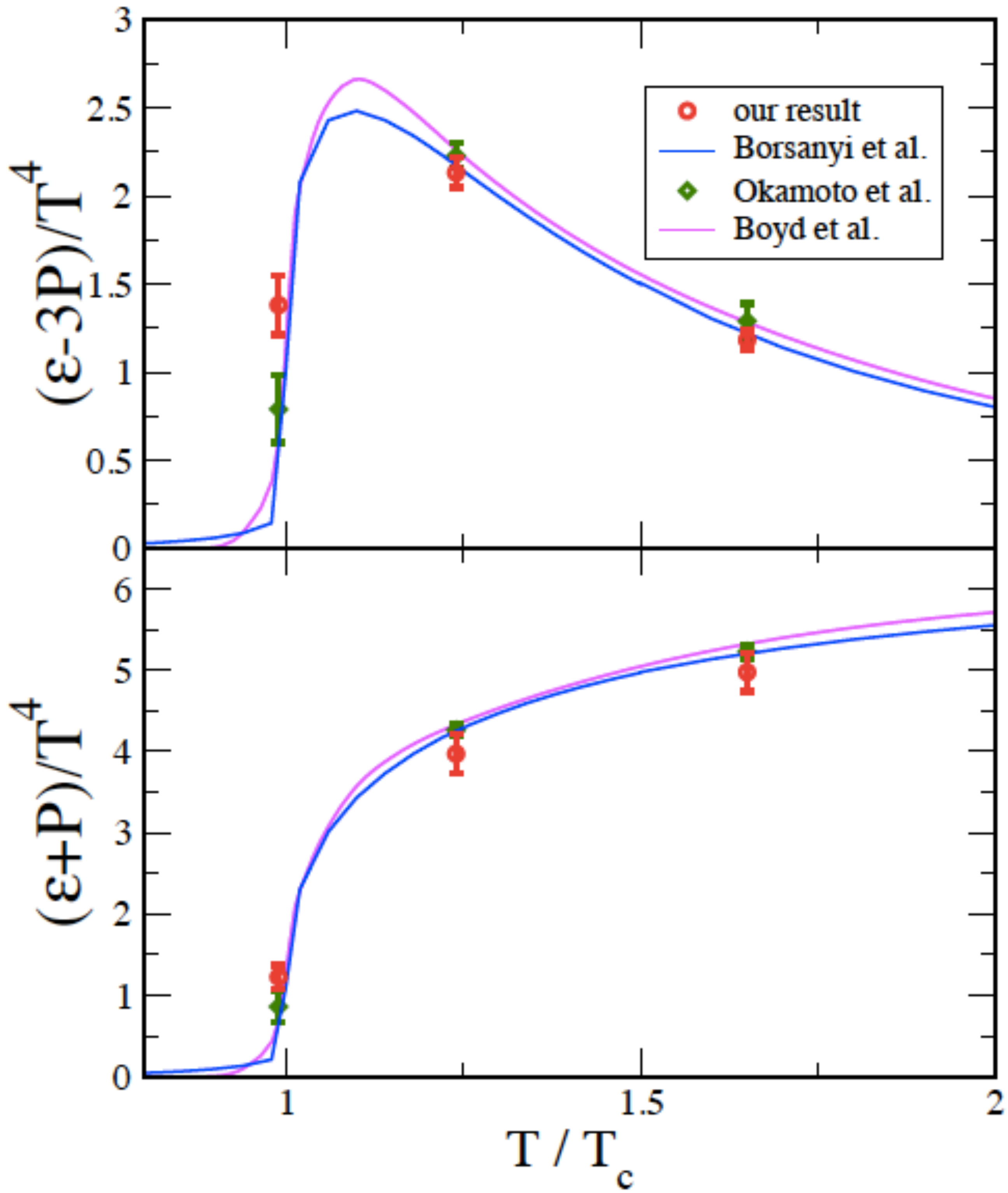

LO
NLO
NNLO
$\mu_0 = 130$ GeV
c_1
c_2
$\zeta_\chi^{-1} c_3$
$\zeta_\chi^{-1} c_4$
μ/μ_0
0.5
1
2
5
10



O(10%) uncertainty from lattice

FlowQCD coll. ‘15

Summary

- Gradient Flow is a (relatively) new tool
- Extremely successful in lattice QCD
- Perturbative approach not yet fully explored
- Here:
  - potential measurement of α_s on the lattice
  - lattice definition of energy-momentum tensor

Outlook:

- Applications in other fields? (Flavor?)
- Renormalization group vs. flow time?

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"Strong enough coffee will kill all germs."

T. Mannel

Happy Birthday!

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