Signals processing for precision amplitude and timing applications

RD51 Open Lectures, Dec. 11th 2017

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The theory of optimum filtering in the frequency and time domain is presented.

All this should serve as food for thought …

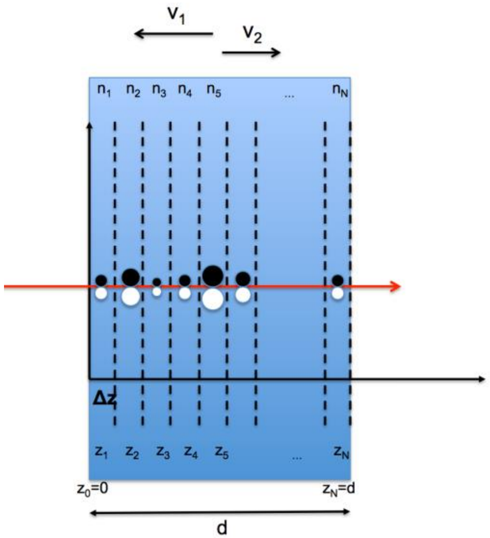
How to extract the best timing information from a signal ?

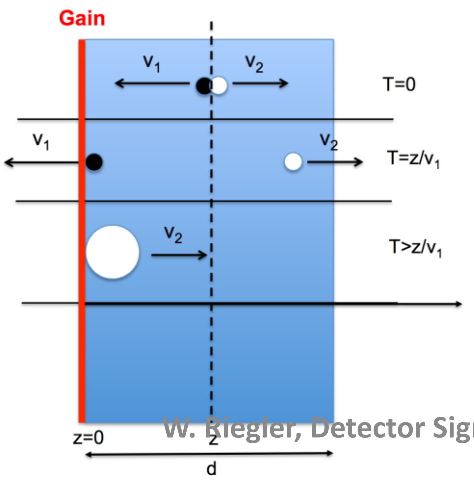
This question does significantly depend on the physics processes that determine the formation of the signal.

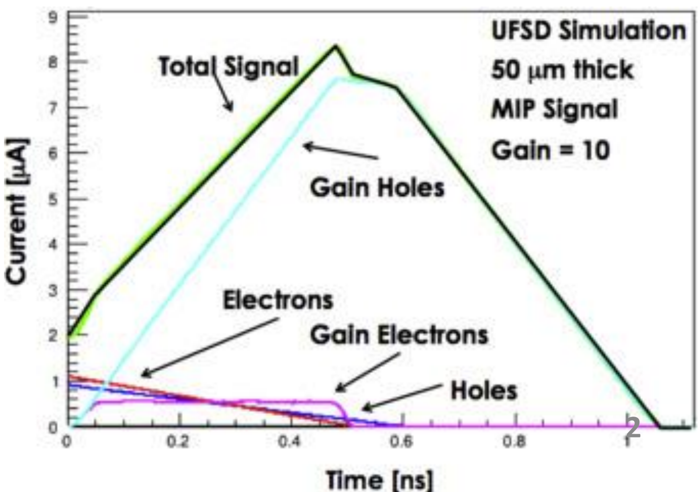
E.g. in a silicon sensor, all electrons and holes instantly start to move after they were created by the charged particle passing the sensor, so in principle the induced signal has an 'infinitely' sharp edge and is then decaying linearly (for overdepletion and velocity saturation). The time resolution is therefore mainly a question of bandwidth and signal-to-noise.

For the so called LGAD silicon sensors there is a thin gain layer on one side of the sensor in order to improve the signal to noise ratio, but now the electrons have to arrive at the gain layer before getting multiplied, so this changes the signal characteristics and the signal fluctuations are now related to the arrival time statistics of the electrons at this layer.

→ What is the best way to extract the signal for these cases ?



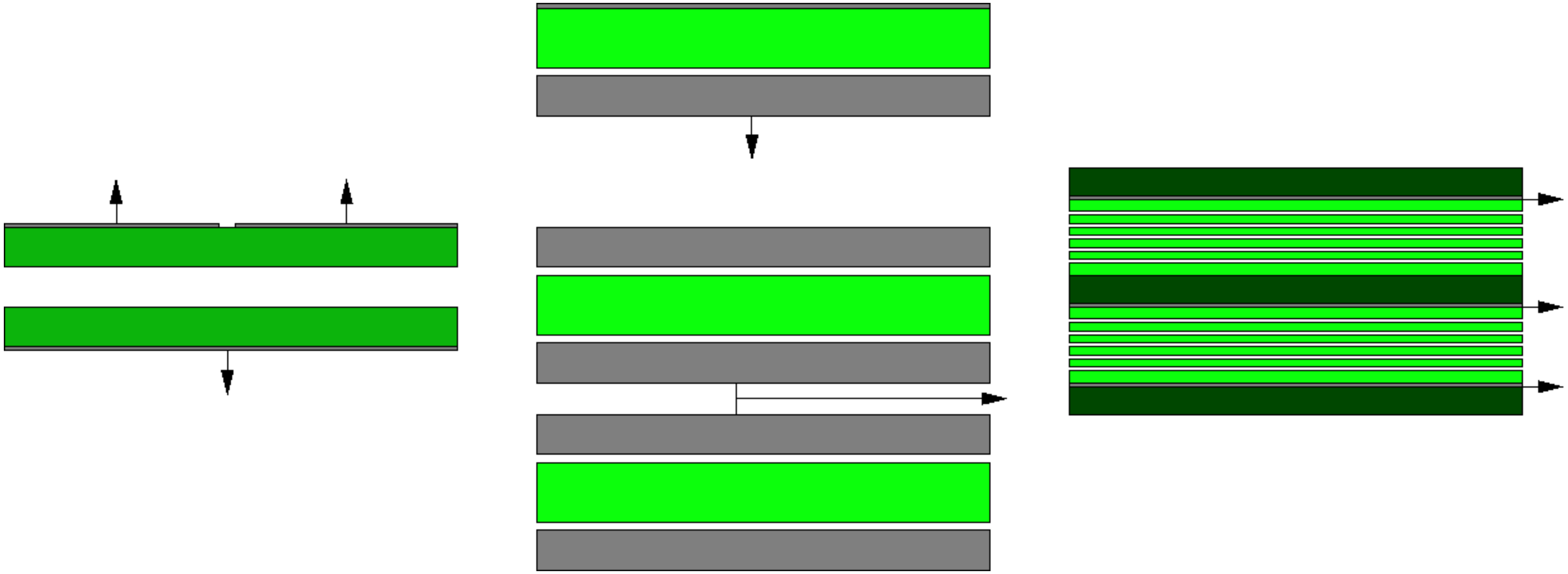




Time resolution of a Resistive Plate Chamber (RPC)

In a RPC the moving primary electrons in the gas gap produce a signal that is much to small to be detected, so a high field is applied to produce an avalanche which can be detected.

The electrons start this avalanche instantly, and the fluctuations of the avalanche will determine the time resolution of the detector.



Time resolution of a Resistive Plate Chamber (RPC)

When assuming electronics of infinite bandwidth (i.e. larger than the signal frequency content) and applying a threshold to the RPC signal, the time resolution is determined by these avalanche fluctuations.

Would there be a better way to extract more accurate time information from the signal ?

$$\begin{aligned}\frac{dP(n,x)}{dx} &= -P(n,x)n(\alpha+\eta) \\ &\quad +P(n-1,x)(n-1)\alpha \\ &\quad +P(n+1,x)(n+1)\eta\end{aligned}$$

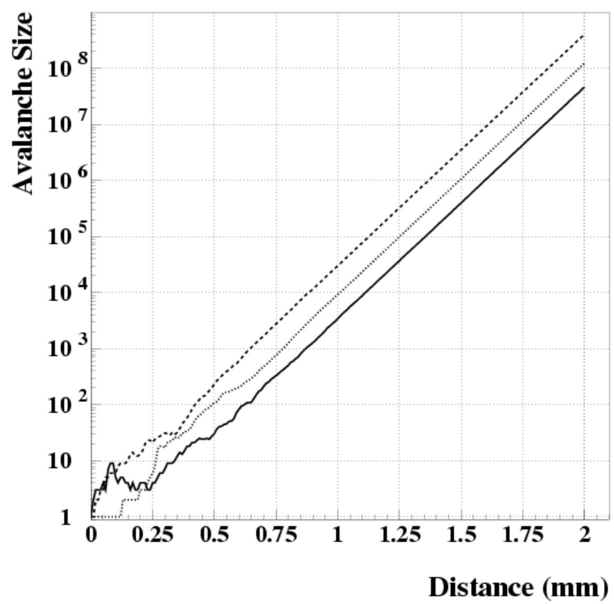
General solution:

$$\overline{n}(x) = e^{(\alpha-\eta)x} \qquad k = \frac{\eta}{\alpha}$$

$$\begin{aligned}P(n,x) &= k\frac{\overline{n}(x)-1}{\overline{n}(x)-k} & n=0 \\ &= \overline{n}(x)\left(\frac{1-k}{\overline{n}(x)-k}\right)^2\left(\frac{\overline{n}(x)-1}{\overline{n}(x)-k}\right)^{n-1} & n>0\end{aligned}$$

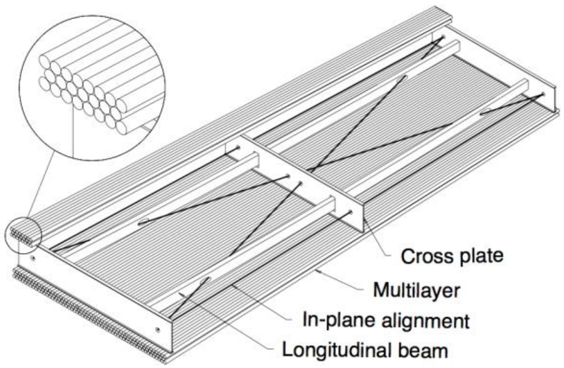
Variance:

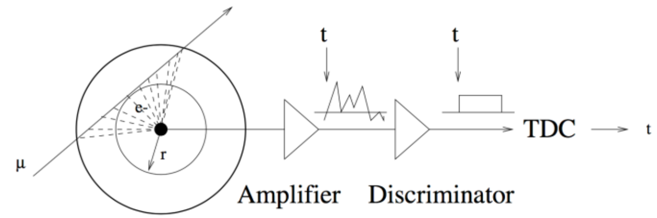
$$\sigma^2(x) = \left(\frac{1+k}{1-k}\right)\overline{n}(x)\left(\overline{n}(x)-1\right)$$

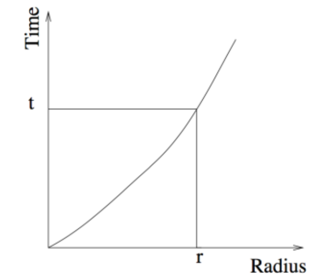


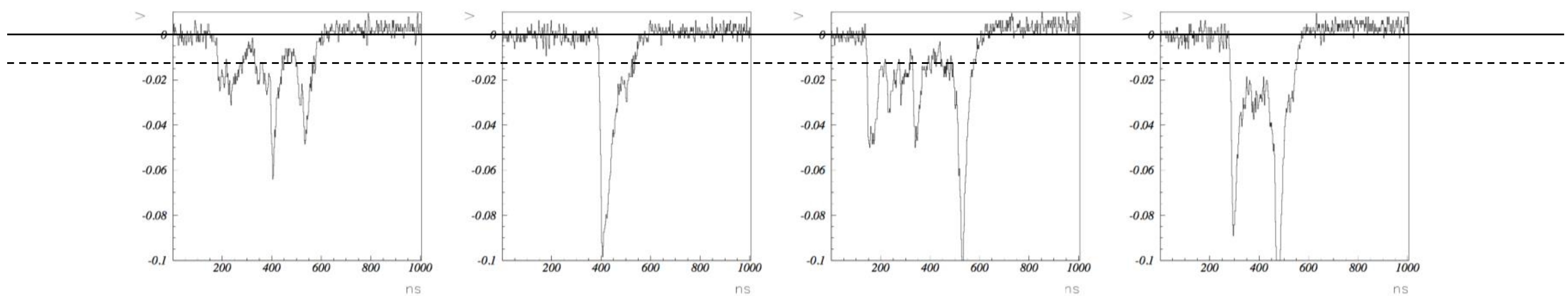
$$\begin{aligned}i(t) &= Ae^{(\alpha-\eta)vt} := thr \\ t(A) &= \frac{1}{(\alpha-\eta)v}\ln\frac{thr}{A} \\ P(A) &= \frac{1}{A_{av}}e^{\frac{-A}{A_{av}}} \\ \sigma_t &= \sqrt{\overline{t^2}-\overline{t}^2} \\ &= \frac{1.28255}{(\alpha-\eta)v}\end{aligned}$$

Drift Tubes (Example ATLAS Muon Spectrometer)

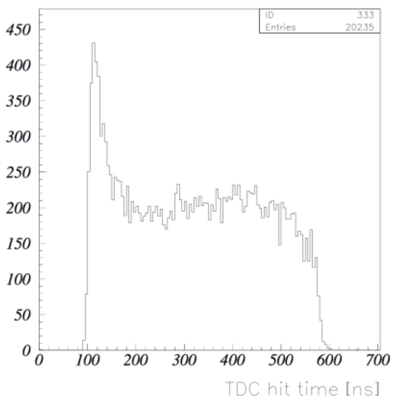








Leading edge time distribution

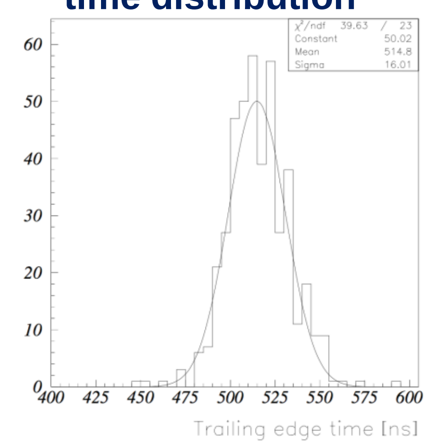


The leading edge is not a good measure of the time the particle passed:

Leading edge:
$500/\sqrt{12}$=144ns

Trainling edge:
20ns

Trailing edge time distribution



How to extract the best timing information from a signal ?

This depends on the specific sensor.

Leading edge discrimination, constant fraction discrimination, trailing edge discrimination, signal centroid, √(trailing edgex41)/(Leading edge)^2… ?

For a given sample of simulated (induced current) signals, the problem to solve is:

Assume a sample of simulated detector signals, for a particle passing the detector at time tau, sampled at times t_1, t_2, … , t_N giving amplitudes A_1, A_2, … , A_N. How to extract the time from these amplitudes that gives the best estimate of tau ?

Construct $P(\tau, A_1, A_2, \ldots, A_N)$ and apply maximum likelihood formalism … !
Or send it though a neural net … !

Once this is known one can apply electronics and noise and different specific algorithms to see how close one can get to the intrinsic optimum timing information.

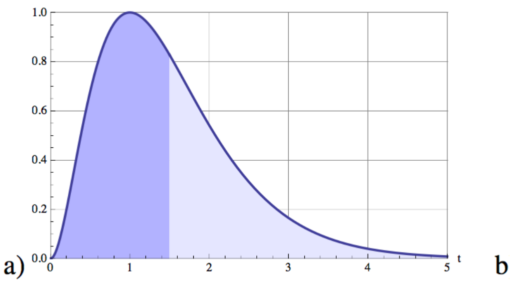
The same can be done for measured signals from the testbeam, but this will already contain the electronics and noise contribution.

→ Homework !!

Centriod Time of a Signal

A specific way to extract the timing information from a signal is measuring the centroid time of a signal i(t): $I(s) = \mathcal{L}[i(t)]$

$$\tau_{cur} = \frac{\int_0^\infty t\, i(t)dt}{\int_0^\infty i(t)dt} = \frac{\int_0^\infty t\, i(t)dt}{q} = -\frac{I'(0)}{I(0)}$$



If this signal is sent through an amplifier with delta response f(t)

$$v(t) = \int_0^t f(t-t')i(t')dt' \qquad V(s) = F(s)I(s)$$

The centroid time becomes

$$\tau_v = -\lim_{s\to 0}\frac{V'(s)}{V(s)} = -\frac{F'(0)I(0)+F(0)I'(0)}{F(0)I(0)} = -\frac{F'(0)}{F(0)} - \frac{I'(0)}{I(0)} = \tau_{amp} + \tau_{cur}$$

And since the amplifier delta response f(t) does not fluctuate, the fluctuation of the centroid time of the amplifier output signal is equal to the fluctuation of the centroid time of the input signal, independent of the amplifier.

Centriod Time

Let's assume the amplifier to be much 'slower' than the signal, i.e. the total signal duration T of the signal i(t) to be much smaller than the amplifier peaking time. The amplifier output signal for t>T is then:

$$
\begin{aligned}
v(t) &= \int_0^T f(t-t')i(t')dt' \approx \int_0^T \left[f(t) - f'(t)t'\right] i(t')dt' \\
&= q\left[f(t) - f'(t)\frac{\int_0^T t'i(t')dt'}{q}\right] = q\left[f(t) - f'(t)\tau_{\mathrm{cur}}\right] \\
&\approx q\, f(t-\tau_{\mathrm{cur}})
\end{aligned}
$$

so the output signal is simply equal to the amplifier delta response, scaled by the total signal charge and displaced by the centroid time of the input signal !

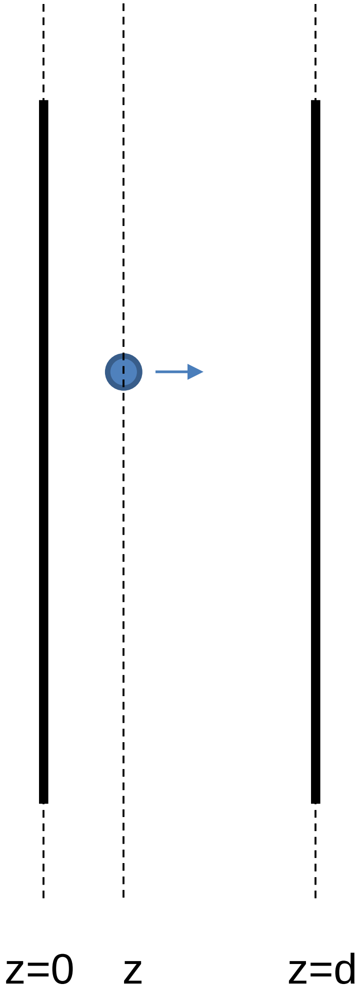
→ The easiest way to measure the centroid time of a signal is to simply send it process it by a 'slow' amplifier and determine the time of the pulse of known shape f(t).

→ Whether this centroid time is a good measure of the time depends on the sensor.

Example of the RPC

Assuming a single primary electron starting an avalanche at t=0, and recording the time when the avalanche crosses a threshold of n_{thr} electrons:

$$n(t) = Ae^{\alpha v t} \qquad p_1(A) = e^{-A}$$



Applying a threshold n_{thr} to the signal the threshold crossing time becomes

$$t_{thr}(A) = \frac{1}{\alpha v} \ln \frac{n_{thr}}{A}$$

$$\overline{t_{thr}} = \int e^{-A} t_{thr}(A) dA = \frac{1}{\alpha v}(\gamma + \ln n_{thr})$$

$$\overline{t_{thr}^2} = \int e^{-A} t_{thr}(A)^2 dA = \frac{1}{\alpha^2 v^2}(\pi^2/6 + \gamma^2 + 2\gamma \ln n_{thr} + (\ln n_{thr})^2)$$

So the standard deviation σ is

$$\sigma^2 = \overline{t_{thr}^2} - \overline{t_{thr}}^2 = \frac{1}{\alpha^2 v^2} \frac{\pi^2}{6}$$

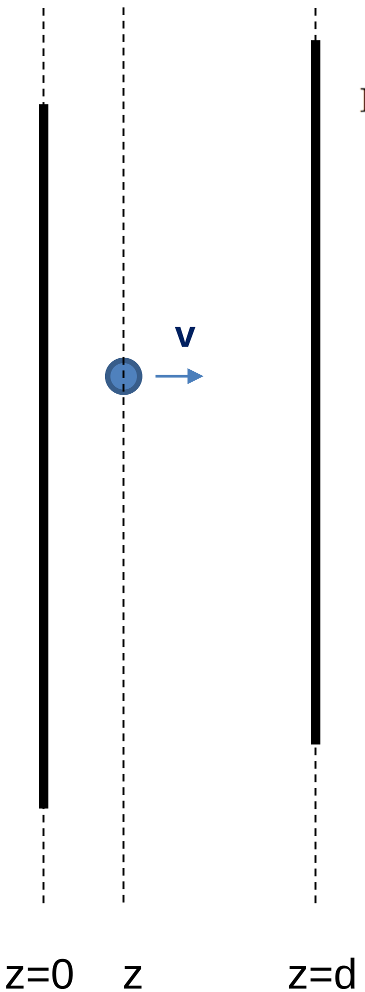
The time resolution is therefore independent of the threshold and independent of the position of the primary electron (as long as the avalanche is large enough to cross the threshold).

This is the situation when we apply the threshold to the induced signal directly, i.e. when we have electronics with a bandwidth larger than the intrinsic bandwidth of the signal.

What happens of we use much slower electronics and measure the centroid time of the signal ?

Example of the RPC

Average position of the primary electron is z=λ

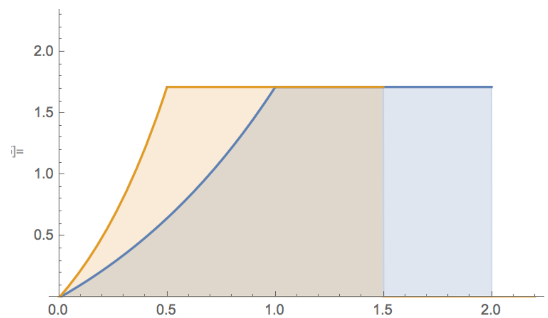


We know that the avalanches in RPC saturate at a given number of electrons N_{sat}

If a single primary electron is produced at position z, the avalanche will first grow until is has reached a size of N_{sat} and these electrons will then move to the end of the gas gap and induce a constant signal.

For a given position z of the primary electron and a given avalanche size A the signal is given by

$$s(t) = Ae^{\alpha v t} \quad \text{for} \quad 0 < t < t_s \qquad s(t) = N_s \quad \text{for} \quad t_s < t < \frac{d-z}{v}$$



$$t_{cent}(z, A) = \frac{\int t\, s(t) dt}{\int s(t) dt} \approx \frac{d-z}{2v} + \frac{1}{2\alpha v} \ln \frac{N_s}{A} \qquad p_1(A) = e^{-A} \qquad p_2(z) = \frac{1}{\lambda} e^{-z/\lambda}$$

$$\bar{t}_{cent} = \int_0^\infty \int_0^\infty t_{cent}(z, A)\, p_1(A) p_2(z) dA dz \qquad \overline{t^2_{cent}} = \int_0^\infty \int_0^\infty t_{cent}(z, A)^2\, p_1(A) p_2(z) dA dz \tag{42}$$

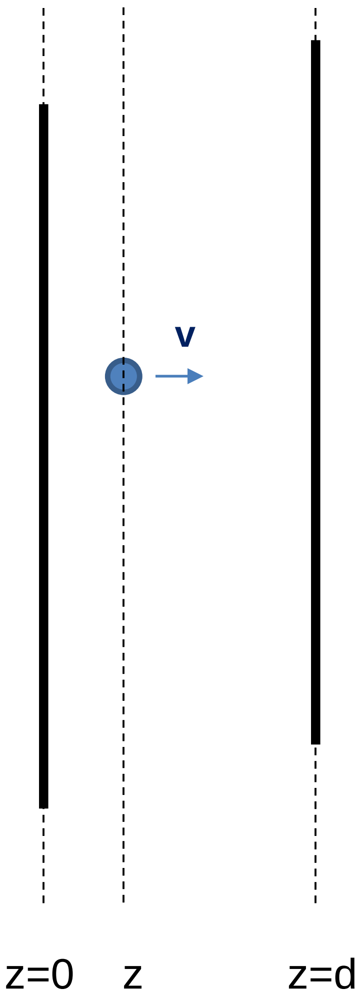
$$\sigma^2 = \overline{t^2_{cent}} - \bar{t}^2_{cent} = \frac{1}{4v^2} \left(\lambda^2 + \frac{1}{\alpha^2} \frac{\pi^2}{6} \right) \tag{43}$$

Since $\lambda \approx 0.1\,\mathrm{mm}$ and $1/\alpha \approx 0.01 mm$ the variation of the centroid time is now dominated by the position of the primary electron and the standard deviation of the centroid time is about a factor 10 worse compared to the standard deviation of the threshold crossing time. The total charge of the signal

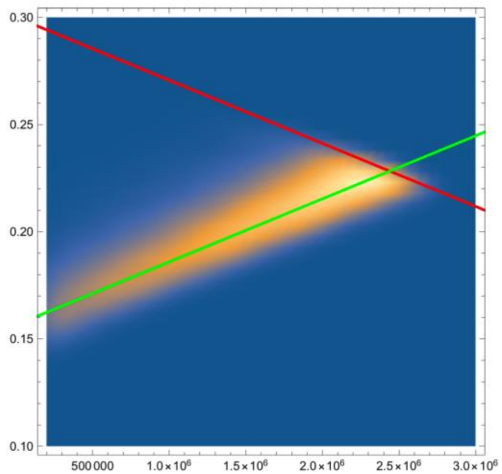
Measuring the centroid time of this signal gives a 10x worse resolution compared to the threshold crossing time

Example of the RPC

Average position of the primary electron is z=λ



We can however still exploit a possible correlation between the total charge and the centroid time:



For this 'toy' example we indeed find a correlation between the time and the total charge:

The charge for a given position and amplitudes is

$$q(z, A) \approx \frac{N_s}{\alpha v}\left[\alpha(d-z) - \ln\frac{N_s}{A}\right]$$

Expressing A and z from this equation gives

$$A(q,z) = N_s \exp\left[-\alpha\left(d - z - \frac{qv}{N_s}\right)\right] \qquad z(q,A) = d - \frac{qv}{N_s} - \frac{1}{\alpha}\log\frac{N_s}{A}$$

The probability to find the amplitude A under the condition that the total charge is q is

$$p_q(A)dA = \frac{e^{-A}\left(\frac{1}{A}\right)^{\frac{1}{\alpha\lambda}}}{\Gamma(1-\frac{1}{\alpha\lambda})}dA$$

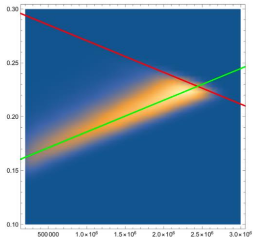
Inserting z from the above expression in Eq. 40 we have

$$t_{cent}(q,A) = \frac{q}{2N_s} + \frac{1}{\alpha v}\log\frac{N_s}{A}$$

So the average time for a given q is given by

$$\bar{t}_{cent}(q) = \int_0^\infty t_{cent}(q,A)p_q(A)dA = \frac{q}{2N_s} + c_1(\alpha, \lambda, v, N_s)$$

Example of the RPC



We use this correlation to correct the measured centroid time with the charge:

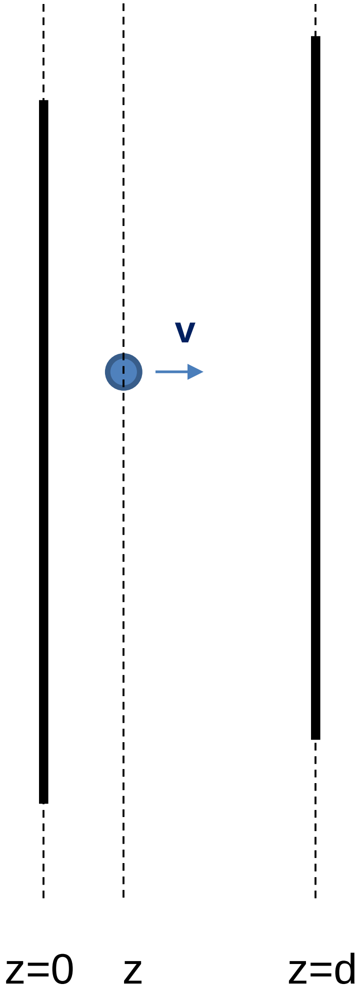
Using this correlation to correct the centre of gravity time with the measured charge (we can neglect the constant offset) we have the time $t_{correct}$ given by

$$t_{corr} = t_{cent} - \frac{q}{2N_s} \tag{50}$$

To find the variance of the time t_{corr} we insert the expressions

$$t_{corr}(z,A) = t_{cent}(z,A) - \frac{q(z,A)}{2N_s} = \frac{1}{\alpha v}\log\frac{N_s}{A} \tag{51}$$

The variance of the amplitude corrected time is therefore again the same as the one where we just apply a threshold.



The expression for t_{corr} is the same as the one when applying a threshold directly to the induced signal, so we conclude:

Applying a threshold directly to the induced current signal, or measuring the signal centroid and correcting with the total signal charge gives the same time resolution.

If this toy model applies to a real RPC detector is means that one is not forced to use fast electronics for RPCs but one can use electronics with modest bandwidth

(if noise does not start to play an important role)

Key Point

The strategy for obtaining the optimum time resolution from a detector signal depends on the specific detector type and the detector physics processes leading to the detector signal.

For some detectors, the centroid time of the signal, with some correction related to the total charge, can give results that are close to the optimum achievable resolution.

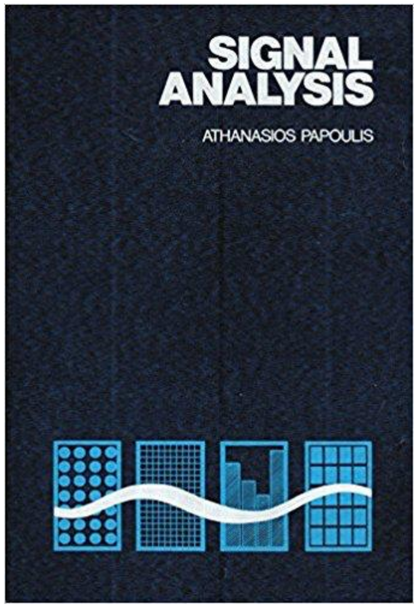
Since processing the signal with an amplifier that is 'slower' than the signal duration leads to an amplifier output signal identical to the amplifier delta response, scaled by the charge of the signal and displaced by the centroid time, this is a very practical way to perform the measurement.

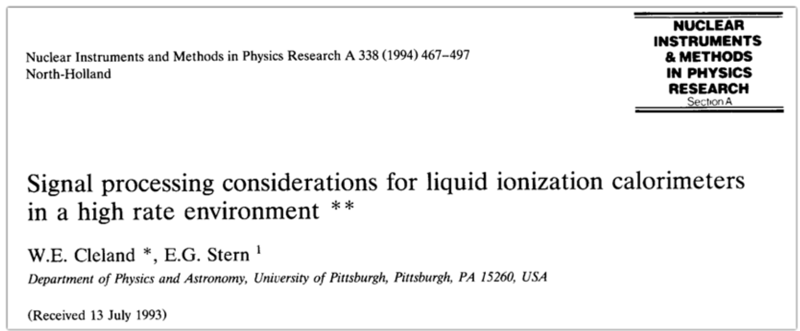
In this case the time resolution is given by the fluctuation of the centroid time, added in square with the error in finding the signal time due to electronics noise.

The problem has a precise mathematical formulation:

Given a signal of shape f(t) with superimposed noise of noise power spectrum S(f), what is the optimum way of extracting the signal amplitude A and the signal time tau from the measurement.

Optimum Filtering

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NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH
Section A

Signal processing considerations for liquid ionization calorimeters in a high rate environment **

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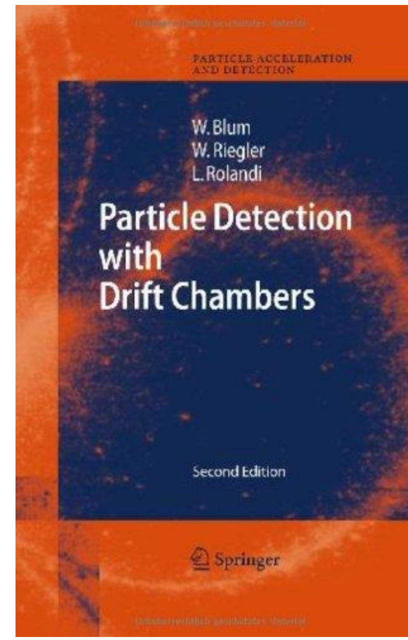
(Received 13 July 1993)



Fourier Transform

1. Fourier transform

The Fourier transform $F(i\omega)$ of the function $f(x)$ and the inverse Fourier transform are defined by

$$F(i\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega x} d\omega \tag{1}$$

Proof:

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x') e^{-i\omega x'} dx' \right] e^{i\omega x} d\omega && (2) \\
&= \int_{-\infty}^{\infty} f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega (x-x')} d\omega \right] dx' && (3) \\
&= \int_{-\infty}^{\infty} f(x') \delta(x-x') dx' && (4) \\
&= f(x) && (5)
\end{aligned}$$

Convolution

2. Convolution

For the convolution of two functions $f(t)$ and $g(t)$ it holds that

$$\int_{-\infty}^{\infty} f(x-y)g(y)dy = h(x) \qquad F(i\omega)G(i\omega) = H(i\omega) \tag{6}$$

Proof:

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} g(y)f(x-y)dy &(7)\\ &= \int_{-\infty}^{\infty} g(y)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega(x-y)}d\omega\right]dy. &(8)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)\left[\int_{-\infty}^{\infty} g(y)e^{-i\omega y}dy\right]e^{i\omega x}d\omega &(9)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)G(i\omega)e^{i\omega x}d\omega &(10) \end{aligned}$$

And therefore

$$F(i\omega)G(i\omega) = H(i\omega) \tag{11}$$

Parseval's Theorem

3. Parseval's Theorem

$$\int_{-\infty}^{\infty} f(t)\overline{f}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)\overline{F}(i\omega)d\omega \qquad (12)$$

or

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(i\omega)|^2 d\omega \qquad (13)$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\overline{f}(t)dt &= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}d\omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{F}(i\omega')e^{-i\omega' t}d\omega'\right]dt && (14)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(i\omega)\overline{F}(i\omega')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt\right]d\omega' d\omega && (15)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(i\omega)\overline{F}(i\omega')\delta(\omega-\omega')d\omega' d\omega && (16)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)\overline{F}(i\omega)d\omega && (17)
\end{aligned}$$

Noise Power Spectrum

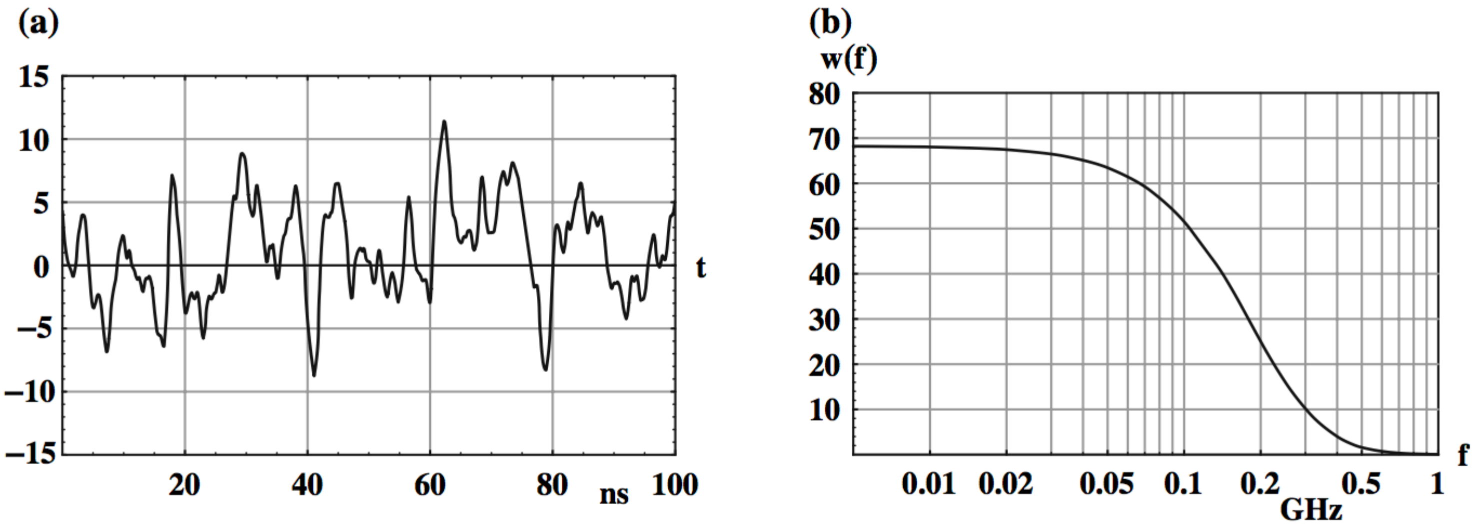


Fig. 6.28 (**a**) Random noise signal $n(t)$. (**b**) Power spectral noise density $w(f)$ of the noise signal, which characterizes the noise signal in the frequency domain

Noise Power Spectrum

4. Noise power spectrum

Assuming a stationary noise process leading to a noise signal $n(t)$, and assuming the average to be zero, we can define the variance σ^2 of this noise signal in the time range $-T, T$ as

$$\sigma^2 = \frac{1}{2T}\int_{-T}^{T} n(t)^2 dt \tag{18}$$

If $N(i\omega)$ is the Fourier transform is this noise signal on this interval

$$N(i\omega, T) = \int_{-T}^{T} n(t)e^{-i\omega t}dt \tag{19}$$

we can use Parseval's theorem and write

$$\frac{1}{2T}\int_{-T}^{T} n(t)^2 dt = \frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty} |N(i\omega, T)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2T}|N(i\omega, T)|^2\right] d\omega \tag{20}$$

To include all frequencies we take the limit of $T \to \infty$ and write the results as

$$\sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega \qquad S(\omega) = \lim_{T\to\infty}\frac{1}{2T}|N(i\omega, T)|^2 \tag{21}$$

where $S(\omega)$ is called the power spectral density of the noise. Since we have

$$\sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega = \int_{-\infty}^{\infty} S(2\pi f)df \tag{22}$$

$S(2\pi f)$ is contribution to the variance in frequency range between f and $f + df$.

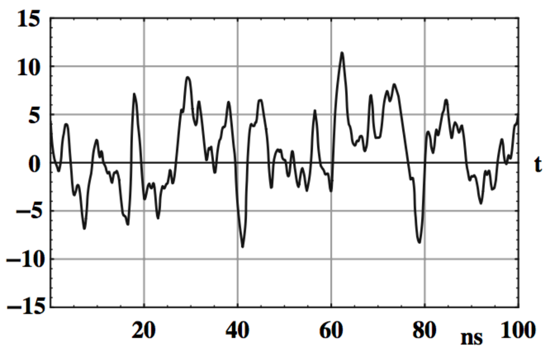
Autocorrelation Function

5. Autocorrelation function

The Autocorrelation function $\psi(t)$ of a noise signal in the interval $-T, T$ is defined as

$$\psi(t) = \frac{1}{2T}\int_{-T}^{T} n(t+\tau)n(\tau)d\tau \tag{23}$$

and is a measure of the correlation between different points of the noise as a function of dime difference t. We can express this function by the Fourier transform $N(i\omega)$ in the following way:



$$\begin{aligned}
\psi(t) &= \frac{1}{2T}\int_{-T}^{T} n(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} N(i\omega,T)e^{i\omega(t+\tau)}d\omega\right]d\tau && (24)\\
&= \frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty} N(i\omega,T)\left[\int_{-T}^{T} n(\tau)e^{i\omega\tau)}d\tau\right]e^{i\omega t}d\omega && (25)\\
&= \frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty} N(i\omega,T)\overline{N}(i\omega,T)e^{i\omega t}d\omega && (26)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2T}|N(i\omega,T)|^2\right]e^{i\omega t}d\omega && (27)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega t}d\omega && (28)\\
& && (29)
\end{aligned}$$

so by extending the interval T to infinity we find that the autocorrelation function $\psi(t)$ is just the inverse Fourier transform of the noise power spectrum

$$\psi(t) = \mathcal{F}^{-1}[S(\omega)] \tag{30}$$

which is called theWiener-Khinchin theorem theorem. We have the relation

$$\psi(0) = \frac{1}{2\pi}\int_{0}^{\infty} S(\omega)d\omega = \sigma^2 \tag{31}$$

We therefore define $u(t) = \psi(t)/\sigma^2$ and write the autocorrelation function $\psi(t)$ as

$$\psi(t) = \sigma^2 u(t) \qquad u(0) = 1 \tag{32}$$

Signal and noise through a linear network

7. Filtering of signal and noise

Considering a signal $Af(t)$ with a superimposed noise of power spectrum $S(\omega)$ being processed by a linear network with transfer function $H(i\omega)$ the output signal and noise power spectrum are given by

$$F(i\omega) \to F(i\omega)\,H(i\omega) \qquad S(\omega) \to S(\omega)|H(i\omega)|^2 \tag{39}$$

Optimum Filter

8. Optimum filtering

Let us consider the following question: A signal $A \times f(t)$ with $f(t_p) = 1$ and a superimposed noise of spectral noise power density of $w(i\omega)$ is processed by a linear network of transfer function $H(i\omega)$. How can one determine the transfer function $H(i\omega)$ that maximises the signal to noise ratio ? The output filter output signal $g(t)$ and the noise at the output are given by

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} AF(i\omega)H(i\omega)e^{i\omega t} d\omega \qquad \sigma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)|H(i\omega)|^2 d\omega \tag{40}$$

The signal $g(t)$ has a maximum at some time t_m where the signal to noise ratio is

$$\left(\frac{S}{N}\right)^2 = \left(\frac{g(t_m)}{\sigma}\right)^2 = \frac{A^2}{2\pi} \frac{|\int_{-\infty}^{\infty} F(i\omega)H(i\omega)e^{i\omega t_m} d\omega|^2}{\int_{-\infty}^{\infty} S(\omega)|H(i\omega)|^2 d\omega} \tag{41}$$

This expression has an upper limit given by the Schwarz inequality, which states the following relation for two complex-valued functions $\psi(x)$ and $\phi(x)$:

$$|\int_a^b \overline{\psi}(x)\phi(x)dx|^2 \leq \int_a^b |\psi(x)|^2 dx \int_a^b |\phi(x)|^2 dx \tag{42}$$

where ther equal sign applies if $\psi(x) = c_1\phi(x)$. If we insert

$$\psi(\omega) = \frac{\overline{F}(i\omega)}{\sqrt{S(\omega)}} e^{-i\omega t_m} \qquad \phi(\omega) = \sqrt{S(\omega)}H(i\omega) \tag{43}$$

and have $a = -\infty$ and $b = \infty$ the inequality reads as

$$|\int_{-\infty}^{\infty} F(i\omega)H(i\omega)e^{i\omega t_m} d\omega|^2 \leq \int_{-\infty}^{\infty} \frac{|F(i\omega)|^2}{S(\omega)} d\omega \int_{-\infty}^{\infty} S(\omega)|H(i\omega)|^2 d\omega \tag{44}$$

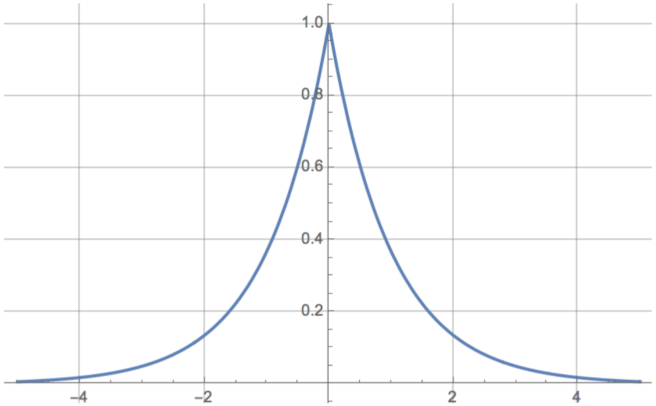
and therefore

$$\left(\frac{S}{N}\right)^2 \leq \frac{A^2}{2\pi} \int_{-\infty}^{\infty} \frac{|F(i\omega)|^2}{S(\omega)} d\omega \tag{45}$$

Optimum Filter

So processing a signal f(t) with superimposed noise S(omega) with a linear network, the signal to noise ratio will always be smaller than

$$\left(\frac{S}{N}\right)^2 \leq \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|F(i\omega)|^2}{S(\omega)}d\omega$$



The filter that maximizes this signal to noise ratio is

$$H(i\omega) = \frac{A}{c_1}\frac{\overline{F}(i\omega)}{S(\omega)}e^{-i\omega t_m}$$

The output signal becomes

$$G(i\omega) = \frac{A^2}{c_1}\frac{F(i\omega)^2}{S(\omega)} \qquad g(t) = \mathcal{F}^{-1}[G(i\omega)] \tag{47}$$

As an example we assume a delta signa $f(t) = Q\delta(t)$ and a parralel and series noise at the input of the amplifier according to

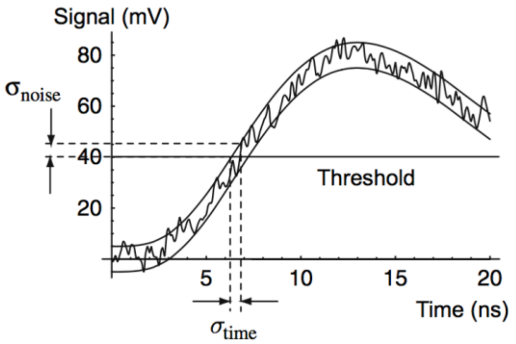
$$S(\omega) = \frac{1}{2}\left(\frac{4kT}{R_p} + 4kTR_sC_D^2\omega^2\right) = \frac{1}{2}(a^2 + b^2\omega^2) \tag{49}$$

which gives

$$\left(\frac{S}{N}\right)^2 \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Q^2}{a^2+b^2\omega^2}d\omega = \frac{Q^2}{ab} \tag{50}$$

$$H(i\omega) = \frac{1}{c_1}\frac{1}{a^2+b^2\omega^2} \qquad h(t) = \mathcal{F}^{-1}[H(i\omega)] = c_2e^{-|t|/\tau_c} \qquad \tau_c = \frac{b}{a} \tag{51}$$

where τ_c is called the *noise corner constant*. The delta response is called the *infinite cusp function*.



Optimum Filter for best Timing

If we want to find the time of the pulses by applying a threshold, the time resolution σ_t is related to the noise σ and the slope of the signal $g'(t)$ by

$$\sigma_t = \frac{\sigma}{g'(t)} \tag{52}$$

so if we want to minimize the time resolution σ_t we have to maximize the expression

$$\left(\frac{1}{\sigma_t}\right)^2 = \left(\frac{g'(t_m)}{\sigma}\right)^2 = \frac{A^2}{2\pi}\frac{|\int_{-\infty}^{\infty} i\omega F(i\omega)H(i\omega)e^{i\omega t_m}d\omega|^2}{\int_{-\infty}^{\infty} S(\omega)|H(i\omega)|^2 d\omega} \tag{53}$$

Proceeding as before using Schwarz' inequality we find

$$\left(\frac{1}{\sigma_t}\right)^2 \leq \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|\omega F(i\omega)|^2}{S(\omega)}d\omega \tag{54}$$

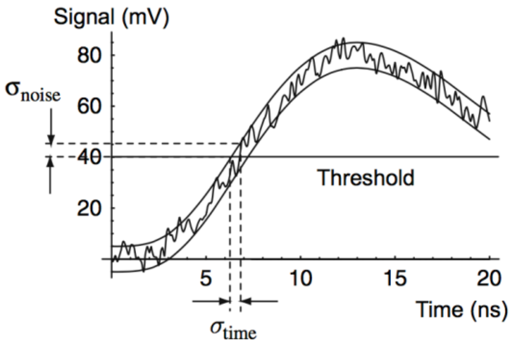
and this minimum is achieved by

$$H(i\omega) = i\omega\frac{A}{c_1}\frac{\overline{F}(i\omega)}{S(\omega)}e^{-i\omega t_m} \tag{55}$$

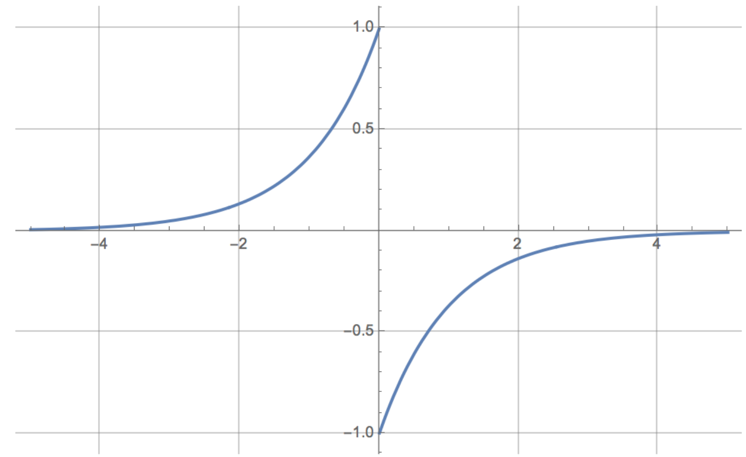
The output signal becomes

$$G(i\omega) = i\omega\frac{A^2}{c_1}\frac{F(i\omega)^2}{S(\omega)} \qquad g(t) = \mathcal{F}^{-1}[G(i\omega)] \tag{56}$$

These signals differ in the frequency domain only by a factor $i\omega$ from the signals optimizing the signal to noise ratio, so we learn that the amplifier delta response that maximizes the time resolution is just the derivative of the delta response that maximises the signal to noise ratio !
Since the delta response maximizing the signal to noise ratio is symmetric with respect to $t = 0$, the delta response maximizing the time resolution is antisummetric with respect to zero and the time of the maximum slop is at $t = 0$ i.e. the zero crossing of this signal gives the best time resolution.



Optimum Filter for best Timing



For the above example we have

$$\left(\frac{1}{\sigma_t^2}\right)^2 \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Q^2\omega^2}{a^2+b^2\omega^2}d\omega = \infty \tag{57}$$

so we find the curious result that for the optimum filter that has finite bandwidth the time jitter at the threshold crossing can be made negligible. We still have to consider that this filter is not causal i.e. it does not have $h(t) = 0$ for $t < t_0$, so any true relaization of a filter will just be an approximation to this filter with a resuklting finite time resolution.

Signal Sampling

Instead of sending a signal through an optimum filter to get the best amplitude measurement or the best time measurement we can also use a non-optimum filter and sample the signal at a finite number of points and then find the best fit to the known signal shape.

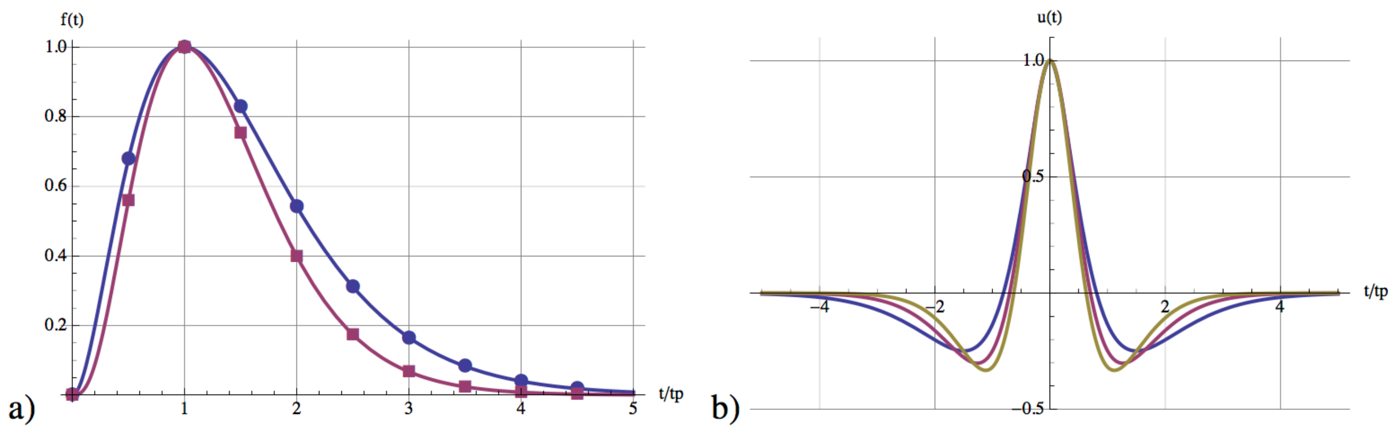
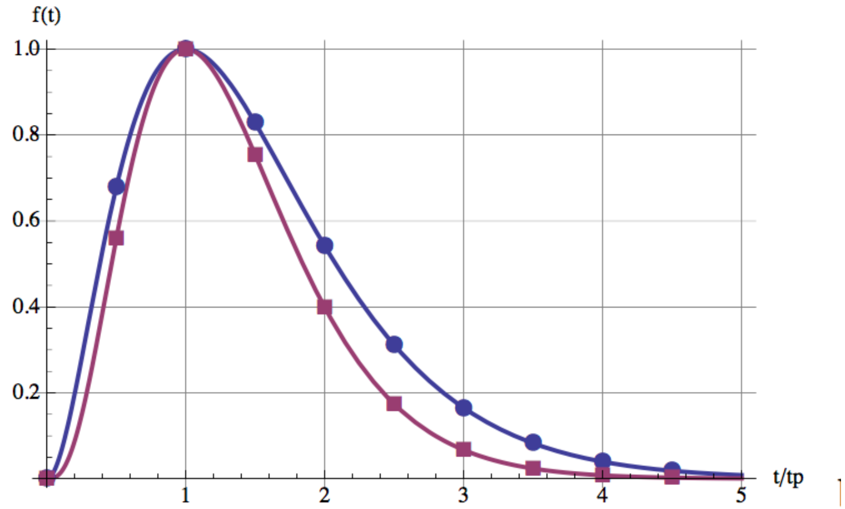


Figure 8. a) Sampling the signal at constant frequency. b) Autocorrelation function of $f'(t)$ for $n = 2, 3, 4$. For times smaller than $0.5\,t_p$ the samples become highly correlated.

The problem to solve consists in finding the best possible 'fit' of the known signal shape to the data samples. Since the samples are correlated the autocorrelation function of the noise has to be taken into account.

Signal Sampling



We assume a signal of known shape $f(t)$ but unknown amplitude A and time of appearance τ with superimposed noise of autocorrelation function $\psi(t)$. This signal is sampled at a finite number of times i.e. we have measurements S_n at times t_n. The problem to solve consists in finding the best estimate for A and τ from this samples, as well as the variance of A and τ. Linearizing this expression for small values of τ we have

$$A\,f(t-\tau) \approx A\,f(t) - A\,f'(t)\tau = \alpha_1\,f(t) - \alpha_2\,f'(t) \qquad \alpha_1 = A \qquad \alpha_2 = A\tau \tag{58}$$

Finding the best estimate of α_1, α_2 for a signal signal $S_1, S_2, ..., S_N$ sampled at times $t_1, t_2, ..., t_N$ leads to the familiar problem of linear regression. We proceed as outlined in [**?**] where the problem is stated

Signal Sampling

The problem is solved by calculating the matrix v_{ij} which is the inverse of the nosie autocorrelation function $\psi_{ij} = \psi(t_i - t_j)$

$$\sum_j \psi_{ij} v_{jk} = \delta_{ik} \tag{60}$$

and the calculating the quantities

$$Q_1 = \sum_i \sum_j f(t_i) v_{ij} f(t_j) \tag{61}$$

$$Q_2 = \sum_i \sum_j f'(t_i) v_{ij} f'(t_j) \tag{62}$$

$$Q_3 = \sum_i \sum_j f'(t_i) v_{ij} f(t_j) \tag{63}$$

$$Q_4 = \sum_i \sum_j f(t_i) v_{ij} S_j \tag{64}$$

$$Q_5 = \sum_i \sum_j f'(t_i) v_{ij} S_j \tag{65}$$

The best estimate for A and τ is then given by

$$A = \frac{Q_2 Q_4 - Q_3 Q_5}{Q_1 Q_2 - Q_3^2} \qquad A\tau = \frac{Q_1 Q_5 - Q_3 Q_4}{Q_1 Q_2 - Q_3^2} \tag{66}$$

The variance of these quantities is

$$\sigma_A^2 = \frac{Q_2}{Q_1 Q_2 - Q_3^2} \qquad A^2 \sigma_\tau^2 = \frac{Q_1}{Q_1 Q_2 - Q_3^2} \tag{67}$$

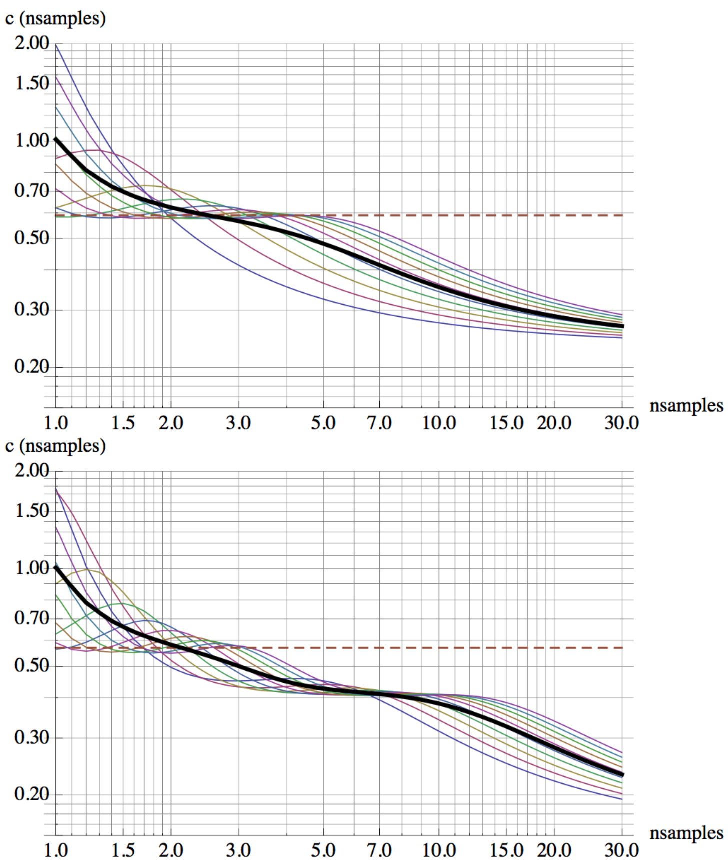
Signal Sampling

$$f(t) = \left(\frac{t}{t_p}\right)^n e^{n(1-t/t_p)}\,\Theta(t)$$

Improvement of the time resolution as a function of the number of samples taken within the peaking time.

The top plot refers to n=2 and the bottom plot to n=3.

$$\frac{\sigma_\tau}{t_p} = \frac{\sigma_{\text{noise}}}{A}\sqrt{\frac{Q_1(n_s)}{Q_1(n_s)Q_2(n_s) - Q_3(n_s)^2}} = \frac{\sigma_{\text{noise}}}{A}\,c(n_s)$$



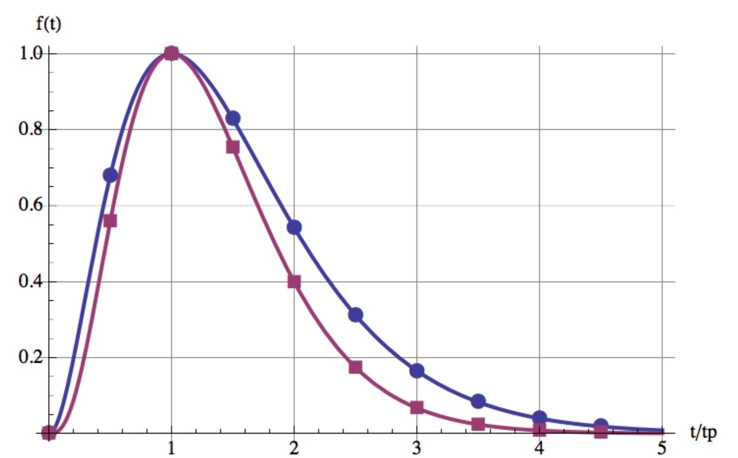


Figure 9. The function $c(n_s)$ for an amplifier with $n = 2$ (top) and $n = 3$ (bottom). The horizontal line is the result for constant fraction discrimination at the maximum slope from eq. (4.20).

Infinite Sampling Limit

What is the best amplitude and time resolution that can be achieved in the limit of infinite sampling frequency for a signal f(t) superimposed by noise of power density S(omega) and related autocorrelation function psi(t) ?

$$
\begin{aligned}
Q_1 &= \sum_i \sum_j f(t_i) v_{ij} f(t_j) \\
Q_2 &= \sum_i \sum_j f'(t_i) v_{ij} f'(t_j) \\
Q_3 &= \sum_i \sum_j f'(t_i) v_{ij} f(t_j)
\end{aligned}
\longrightarrow
\begin{aligned}
Q_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) v(x,y) f(y) dx dy \\
Q_2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f'(x) v(x,y) f'(y) dx dy \\
Q_3 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) v(x,y) f'(y) dx dy
\end{aligned}
$$

$$
\sum_j \psi_{ij} v_{jk} = \delta_{ik} \longrightarrow \int_{-\infty}^{\infty} \psi(x-y) v(y,z) dy = \delta(x-z)
$$

The expression represents a convolution, and since the Fourier Transform of the autocorrelation function $\psi(t)$ is equal to the noise power density $S(\omega)$ the relation reads in the frequency domain as

$$
S(\omega) V(\omega, z) = e^{-i\omega z} \quad \to \quad V(\omega, z) = \frac{e^{-i\omega z}}{S(\omega)} \tag{72}
$$

We can evaluate the expressions the following way:

$$
\begin{aligned}
Q_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v(x,y) f(x) f(y) dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega y}}{S(\omega)} e^{i\omega x} d\omega \right] f(x) f(y) dx dy & (73) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{S(\omega)} \left[\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right] \left[\int_{-\infty}^{\infty} f(y) e^{i\omega y} dy \right] d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|F(i\omega)|^2}{S(\omega)} d\omega & (74)
\end{aligned}
$$

Infinite Sampling Limit

and for Q_2 and Q_3 we have

$$Q_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{S(\omega)}\left[\int_{-\infty}^{\infty} f'(x)e^{i\omega x}dx\right]\left[\int_{-\infty}^{\infty} f'(y)e^{-i\omega y}dy\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|i\omega F(i\omega)|^2}{S(\omega)}d\omega. \quad (76)$$

$$Q_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{S(\omega)}\left[\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx\right]\left[\int_{-\infty}^{\infty} f(y)e^{i\omega y}dy\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\omega|F(i\omega)|^2}{S(\omega)}d\omega \quad (77)$$

Since $S(\omega)$ and $|F(i\omega)|^2$ are symmetric functions, Q_3 becomes zero and we have

$$\frac{A^2}{\sigma_A^2} = A^2 Q_1 = \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|F(i\omega)|^2}{S(\omega)}d\omega \quad (78)$$

and for the time resolution we have

$$\frac{1}{\sigma_\tau^2} = A^2 Q_2 = \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|i\omega F(i\omega)|^2}{S(\omega)}d\omega \quad (79)$$

→ The same expression as the ones when using optimum filters !!!

Infinite Sampling Limit

11. Infinite sampling limit with arbitrary transfer function

If we assume the signal $f(t)$ and the noise $n(t)$ to be processed by an arbitrary filter $K(i\omega)$ and the resulting signal is then sampled in order to derive τ and A, the infinite sampling limit of this precedure will again result in the same

$$\frac{A^2}{\sigma_A^2} = A^2 Q_1 = \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|F(i\omega)K(i\omega)|^2}{S(\omega)|K(i\omega)|^2}d\omega = \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|F(i\omega)|^2}{S(\omega)}d\omega \qquad (80)$$

$$\frac{1}{\sigma_\tau^2} = A^2 Q_2 = \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|i\omega F(i\omega)K(i\omega)|^2}{S(\omega)|K(i\omega)|^2}d\omega = \frac{A^2}{2\pi}\int_{-\infty}^{\infty}\frac{|i\omega F(i\omega)|^2}{S(\omega)}d\omega \qquad (81)$$

So even if we use for $K(i\omega)$ the optimum filter $H(i\omega)$ and we sample the signal infinitely often we will not improve beyond a single measurement on the peak of the optimally filtered signal.

Whatever transfer function one uses to process the signal, the infinite sampling limit of this processed signal always yields the same result.

Infinite Sampling Limit

In order to extract the best amplitude and time information from a signal of known shape f(t) with superimposed noise of power spectral density S(omega) one can

→ **Use an optimum amplitude filter and find the peak of the filter output and use an optimum timing filter (which is the derivative of the optimum amplitude filter) and record the zero crossing.**

→ **Use an arbitrary filter, sample the signal many (infinite) times and find the best fit of the samples to the know shape f(t) by linear regression taking into account the autocorrelation of the noise**

Both methods yield exactly the same result.

The fact that the optimum filters are not causal and that we cannot ample infinitely often will pose the practical limits on these methods.

Conclusion

The best way to extract the time information from a detector signal is very specific to the physics processes in this detector.

Systematic attempts to find these intrinsic limits should be pursued (Likelihood, Neural network).

In case the signal centroid in combination with some amplitude correction gives a close to optimum time resolution, the time measurement can be obtained by an amplifier that is 'slow' compared to the signal time.

Optimum filter methods, either a linear filter or multiple sampling are well defined methods with the same theoretical optima and probably different practical limits of implementation.

Homework:
Work all of this out for the specific RD51 ps timing ideas !!