Milan Christmas Meeting 2017

Five-loop renormalisation of QCD with global infrared rearrangements

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Outline

Origin of IR rearrangements

- Problem: given a diagram γ, compute the counterterm $Z(\gamma)$.

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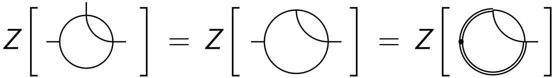
In MS scheme $Z(\gamma)$ is polynomial in the masses (Collins, 1977).

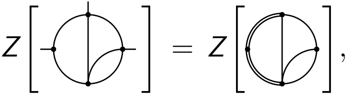
- If γ is logarithmically divergent $Z(\gamma)$ is mass-independent.
  γ can always be made log-divergent by taking derivatives.
- *Infrared rearrangements* (IRRs), acting on masses and external momenta, simplify the calculation of $Z(\gamma)$ (Vladimirov, 1980).



Double lines are massive propagators, dotted lines are squared propagators.

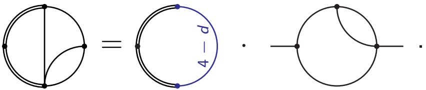
One-mass tadpoles

- IRR to one-mass tadpoles reduces the complexity of **1 loop**.
  Ex: begin with **three-loop** vertex



(1.1)

Factorisation in **one-loop tadpole** and **two-loop propagator**



(1.2)

Dealing with IR singularities

$= \Pi(p)$

$$\Pi(p) = \int d^d k_{1,2} \frac{1}{(p+k_1)^2} \frac{1}{(k_1+k_2)^2 \left[k_2^2\right]^2} \tag{1.3}$$

- IR regulating mass (Chetyrkin, Misiak, Munz; van Ritbergen, Larin, Vermaseren)
- IR counterterms: apply the R^* operation (Chetyrkin, Smirnov, Tkachov)

$$R^*(\Gamma) = \widetilde{R} \circ R(\Gamma) \tag{1.4}$$

$\widetilde{R}$ recursively subtracts IR divergences from each diagram.

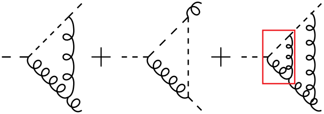
Global approach

- R^* subtracts the IR poles generated in IRRs. No restriction on the choice of rearrangement.
- IR counterterms are computed diagram-by-diagram, many IR counterterms per diagram $\rightarrow$ **Bottleneck** at high orders.

Global R^* (Chetyrkin 1991) avoids this problem my using a single IR counterterm for the whole process.

The ghost-gluon vertex

We determine the ghost-gluon vertex counterterm with global R^*

$\delta\Gamma(g, p) =$  $+ \cdots$

The UV vertex divergence is renormalised by Z_1

$g \to g_0$ renormalises all the remaining subdivergences.

$$\delta\Gamma(g_0, p) + Z_1 + \delta Z_1 \cdot \delta\Gamma(g_0, p) = \text{finite} \tag{2.5}$$

The ghost-gluon vertex

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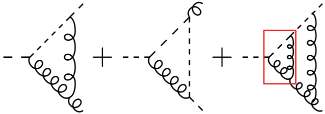


$$+ \cdots$$

The UV vertex divergence is renormalised by Z_1
$g \to g_0$ renormalises all the remaining subdivergences.

$$\delta Z_1 = -K_\epsilon \Big[Z_1 \cdot \delta\Gamma(g_0, p) \Big] \tag{2.5}$$

Operator K_ϵ extracts the poles in ϵ.

Global IR subtraction

- $\delta\Gamma(g_0, p)$ is log-divergent
  - Z_1 is **independent** on the momenta p_1, p_2, p_3.
- IR poles arise only at $p_i = 0$ and are subtracted with $\widetilde{R}$

$$\delta Z_1 = -K_\epsilon\Big[Z_1 \cdot \widetilde{R}\big[\delta\Gamma(p=0)\big]\Big] \tag{2.6}$$

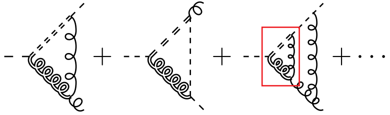
- Determine the **globally subtracted** vertex

$$\boxed{\widetilde{R}\big[\delta\Gamma(p=0)\big] = -\frac{\delta Z_1}{Z_1}} \tag{2.7}$$

Global infrared rearrangement

We rearrange all the Feynman diagrams into **factorised tadpoles**.

1 Introduction of the mass

$\delta\Gamma_M(g, p, M) =$ 

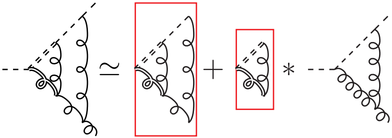
- Vertex divergence (mass independent) renormalises with Z_1
- The **massive vertex** subdivergence changes

$$Z_1 + \delta\Gamma_M(g_0, p, M) + \delta Z_1 \cdot \delta\Gamma(g_0, p) = \text{finite} \qquad (2.8)$$

Zero momentum limit (I)

$$\delta Z_1 = -K_\epsilon \Big[\delta\Gamma_M(g_0, p, M) + \delta Z_1 \cdot \delta\Gamma(g_0, p) \Big] \qquad (2.9)$$

2 Leading term at $p \to 0$ comes from the hard mass expansion $M \gg p$ (Smirnov 1996)



$$\delta\Gamma_M(g_0, p, M) \simeq \boxed{\delta\Gamma_M(g_0, 0, M)} + \boxed{\delta\Gamma_M(g_0, 0, M)} \cdot \delta\Gamma(g_0, p, 0)$$

Zero momentum limit (II)

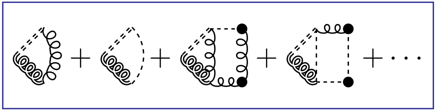
$$\delta Z_1 = -K_\epsilon \left[\underbrace{\delta\Gamma_M(g_0, 0, M)}_{\text{tadpole}} + \Big(\delta\Gamma_M(g_0, 0, M) + \delta Z_1\Big) \cdot \delta\Gamma(g_0, p) \right]$$

3 **Global subtraction** of the IR poles of $\delta\Gamma$ at $p = 0$.

$$\boxed{\delta Z_1 = -K_\epsilon \left[\frac{\delta\Gamma_M(g_0, 0, M)}{Z_1} - \frac{(\delta Z_1)^2}{Z_1} \right]} \qquad (2.10)$$

Towards 5-loop renormalisation

$$\delta Z_1 = -K_\epsilon\left[\frac{\boxed{\delta\Gamma_M(g_0,0,M)}}{Z_1} - \boxed{\frac{(\delta Z_1)^2}{Z_1}}\right] \quad (3.11)$$



Counterterms from lower orders.

- We reduced the calculations to factorized L-loop tadpoles.
- We automated the computation of factorized tadpoles using `Forcer` (Ruijl, Ueda, Vermaseren 2017) and determined Z_1 **to 5 loops.**

Highly efficient approach

Renormalisation of QCD

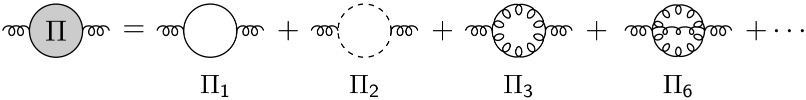
- The ghost-gluon vertex renormalisation constant Z_1 is the first necessary ingredient to renormalise QCD.
- One of the possible choices includes
  - The ghost field renormalisation Z_3^c
  - The gluon field renormalisation Z_3
  - The quark field renormalisation Z_2
- Ward identities fix the remaining constants

$$Z_g = \frac{Z_1}{Z_3^c \sqrt{Z_3}} = \frac{Z_1^{qqg}}{Z_2 \sqrt{Z_3}} = \frac{Z_1^{3g}}{(Z_3)^{\frac{3}{2}}} \tag{3.12}$$

The gluon field renormalisation

- The gluon renormalisation Z_3 is complicate: first time undertaken in 2016 for $SU(3)$ (Baikov, Chetyrkin, Kühn).

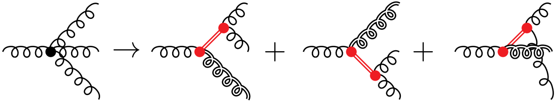
$$K_\epsilon \left[Z_3 \left(1 + \sum_{i=1,2,3,6} \Pi_i(g_0, q) \right) \right] = 0, \qquad (3.13)$$



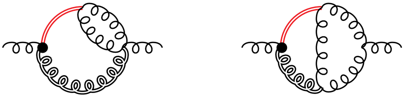
(a) Several interactions modified with the mass insertion.
(b) New rearrangements of 4-point vertices.

Some problems (b)

- Mass insertion in the 4-gluon vertex via an auxiliary field



- New "special", self-energy-like, subdivergences

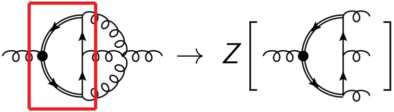


"Special" "Non-special"

Some problems (a)

Highly non-trivial renormalisation of the modified vertices:

- vertices mix among each other. *E.g.*

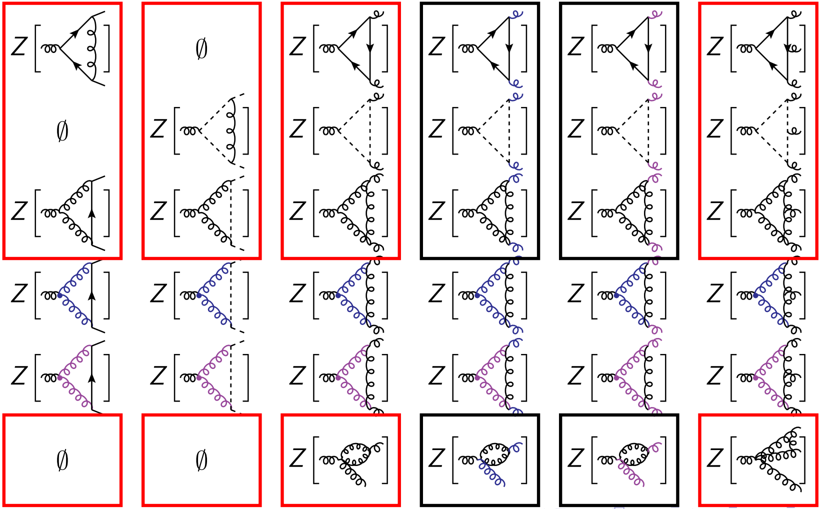


counterterm mixing quark and 4-gluon operators.

- Mixing into new operators (not in the QCD Lagrangian).
  We analysed the operators appearing for general gauge group
  - Two new 3-gluon vertices

  - New 4-gluon vertices appear at each loop order: we selected a basis of six new vertices $O_7, .., O_{12}$ needed to this order. Note that each of them is splitted with the auxiliary field.

A 12x12 mixing matrix



Solution

Only a small subset of the matrix elements z_{ij} contributes and

$$\sum_{i=1,2,3,6} z_{ij} = \begin{cases} 0, & \text{if } j \in \{4,5,7,..12\} \text{ is a gauge variant operator} \\ Z_1^j, & \text{if } j \in \{1,2,3,6\} \text{ is a QCD operator} \end{cases}$$

where $Z_1^j \in \{Z_1^{ccg}, Z_1^{qqg}, Z_1^{ggg}, Z_1^{gggg}\}$ are QCD vertex renormalisation constant.

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Exploiting these features we write the gluon renormalisation constant in terms of factorized tadpoles $\Pi_i(M)$ and lower order counterterms

$$\begin{aligned} \delta Z_3 \quad &= \quad -K_\epsilon \Bigg\{ \sqrt{Z_3} \sum_{i=1,2,3,6} \Bigg\{ \sum_{j,k} \Big[z_{ij}^{sp} \mathbf{Z_j} \Big(\Pi_j(M) + \delta\Gamma_{jk}^{ns} \Pi_k(q) \Big) \Big] \\ &+ \quad (\sqrt{Z_3} - \mathbf{Z}_i)\Pi_i(q) - \sum_j \delta z_{ij}^{sp} \mathbf{Z}_j \Pi_j(q) \Bigg\} \Bigg\}. \end{aligned}$$

Results

- Z_1^{ccg}, Z_1^{qqg}, Z_2, Z_3^c to 5 loops with all the powers of the gauge parameter ξ.
  - Verified $Z_1^{ccg} \propto (1-\xi)$.
  - Verified consistency with the Ward identities.
- Z_3 to 5 loops, retaining linear terms in ξ.
- We computed the coupling renormalisation

$$Z_\alpha = \frac{(Z_1^{ccg})^2}{Z_3\,(Z_3^c)^2}, \tag{4.14}$$

  - Verified independence on ξ to first order.

Complete renormalisation of QCD to 5 loops in covariant gauges for general gauge group.

Landau gauge quark anomalous dimension

$$
\begin{aligned}
(\gamma_2)_4 \quad &= C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{1985}{24} + \frac{781753}{192}\zeta_7 - \frac{1458845}{384}\zeta_5 + \frac{135731}{192}\zeta_3 + \frac{3577}{64}\zeta_3^2 \right] \\
&+ T_R\, n_f \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{6200}{9} - \frac{1425}{4}\zeta_6 + \frac{27377}{6}\zeta_7 + \frac{1113}{4}\zeta_4 - \frac{9915}{2}\zeta_5 \right. \\
&\left. - \frac{2468}{3}\zeta_3 + \frac{91}{2}\zeta_3^2 \right] + T_R\, n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{7360}{9} + 640\,\zeta_5 + \frac{704}{3}\zeta_3 \right] \\
&+ C_F\, n_f^4\, T_R^4 \left[\frac{1328}{243} - \frac{256}{27}\zeta_3 \right] + C_F \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{113}{6} - \frac{125447}{8}\zeta_7 \right. \\
&\left. + 1015\,\zeta_5 + 17554\,\zeta_3 - 4884\,\zeta_3^2 \right] + C_F\, n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{5984}{3} - 8680\,\zeta_7 \right. \\
&\left. + 18080\,\zeta_5 - 12096\,\zeta_3 + 3648\,\zeta_3^2 \right] + C_F^2\, n_f^3\, T_R^3 \left[-\frac{2636}{243} - 64\,\zeta_4 + \frac{832}{9}\zeta_3 \right] \\
&+ C_F^3\, n_f^2\, T_R^2 \left[-\frac{2497}{27} - 128\,\zeta_4 + 320\,\zeta_5 + \frac{400}{9}\zeta_3 \right] + C_F^4\, n_f\, T_R \left[\frac{29209}{36} \right. \\
&\left. + \frac{6400}{3}\zeta_6 - 800\,\zeta_4 - \frac{46880}{9}\zeta_5 + \frac{22496}{9}\zeta_3 + \frac{1024}{3}\zeta_3^2 \right] + C_F^5 \left[\frac{4977}{8} \right. \\
&\left. - 47628\,\zeta_7 + 22600\,\zeta_5 + 16000\,\zeta_3 + 2496\,\zeta_3^2 \right]
\end{aligned}
$$

$$+C_A \frac{d_R^{abcd} d_A^{abcd}}{N_R} \Big[-\frac{173959}{144} + \frac{15675}{16}\zeta_6 + \frac{3016307}{256}\zeta_7 - \frac{12243}{16}\zeta_4$$
$$+\frac{609425}{96}\zeta_5 - \frac{574393}{32}\zeta_3 + \frac{16935}{4}\zeta_3^2\Big]$$
$$+C_A n_f \frac{d_R^{abcd} d_R^{abcd}}{N_R} \Big[\frac{33464}{9} + \frac{23632}{3}\zeta_7 - \frac{48640}{3}\zeta_5 + 8992\zeta_3 - 2320\zeta_3^2\Big]$$
$$+C_A C_F n_f^3 T_R^3 \Big[-\frac{3566}{243} + 64\zeta_4 - \frac{1984}{27}\zeta_3\Big] + C_A C_F^2 n_f^2 T_R^2 \Big[\frac{101485}{162}$$
$$+\frac{1600}{3}\zeta_6 + 176\zeta_4 - \frac{3712}{3}\zeta_5 - \frac{6160}{9}\zeta_3 + \frac{256}{3}\zeta_3^2\Big] + C_A C_F^3 n_f T_R \Big[-\frac{167263}{108}$$
$$-4800\zeta_6 - 13944\zeta_7 + 2120\zeta_4 + \frac{58720}{3}\zeta_5 - \frac{25804}{9}\zeta_3 - 64\zeta_3^2\Big]$$
$$+C_A C_F^4 \Big[-\frac{835739}{144} - \frac{17600}{3}\zeta_6 + 123977\zeta_7 + 2200\zeta_4 - \frac{248960}{9}\zeta_5 - \frac{530884}{9}\zeta_3$$
$$-\frac{24632}{3}\zeta_3^2\Big] + C_A^2 C_F n_f^2 T_R^2 \Big[\frac{120037}{162} - \frac{800}{3}\zeta_6 - \frac{441}{2}\zeta_7 - 179\zeta_4 + \frac{3584}{9}\zeta_5$$
$$+\frac{3140}{3}\zeta_3 - \frac{128}{3}\zeta_3^2\Big] + C_A^2 C_F^2 n_f T_R \Big[\frac{717409}{432} + 1150\zeta_6 + \frac{42203}{3}\zeta_7 - \frac{1411}{4}\zeta_4$$
$$-\frac{95792}{9}\zeta_5 - \frac{14287}{24}\zeta_3 - 1214\zeta_3^2\Big] + C_A^2 C_F^3 \Big[\frac{827215}{72} + 13200\zeta_6 - \frac{1789067}{16}\zeta_7$$
$$-4664\zeta_4 - \frac{188795}{12}\zeta_5 + \frac{1365227}{18}\zeta_3 + \frac{18097}{2}\zeta_3^2\Big] + C_A^3 C_F n_f T_R \Big[-\frac{31919039}{7776}$$
$$+\frac{4825}{16}\zeta_6 - \frac{440419}{144}\zeta_7 - \frac{8705}{32}\zeta_4 + \frac{28721}{18}\zeta_5 - \frac{144377}{864}\zeta_3 + \frac{4067}{6}\zeta_3^2\Big]$$

$$
\begin{aligned}
&+C_A^3\, C_F^2 \Big[-\frac{42214139}{3888} - \frac{43175}{6}\zeta_6 + \frac{9074513}{192}\zeta_7 + \frac{3815}{4}\zeta_4 + \frac{5957573}{288}\zeta_5 \\
&-\frac{5503507}{144}\zeta_3 - \frac{78041}{24}\zeta_3^2 \Big] + C_A^4\, C_F \Big[\frac{368712343}{62208} + \frac{227975}{192}\zeta_6 \\
&-\frac{312820991}{36864}\zeta_7 + \frac{87067}{128}\zeta_4 - \frac{16237513}{3072}\zeta_5 + \frac{46196783}{6912}\zeta_3 + \frac{23555}{128}\zeta_3^2 \Big] .
\end{aligned}
$$

- More complicated structure compared to the beta-function:
  - zeta-values up to weight 7.
- The leading and next-to-leading n_f terms agree with the all-order results of (Ciuchini et al. '99).

Outlook

- Global R^* allows highly efficient computations.
- It requires careful analysis of subdivergences

  $\longrightarrow$ it's difficult to automate.
- Interesting for renormalisation problems in different context
  - Operators in EFT
  - Splitting functions
- Any suggestion?

Thank you