Recent developments in amplitude calculations

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December 21, 2017
Milan Christmas Meeting 2017
Università degli Studi di Milano

HIGH-MULTIPLICITY

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Outline

1. Introduction and motivation
2. Summary of the state of the art
   - One-loop integrand reduction and automated tools
   - Higher-loop amplitudes at high multiplicity
3. Finite fields and multivariate reconstruction
4. Applications to two-loop five-point amplitudes
5. Summary & Outlook

Introduction and motivation

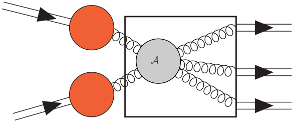
Loop amplitudes at high multiplicity

Phenomenological predictions

- Experiments at LHC
  - high-accuracy (up to % level in Run II)
  - large SM background
  - high c.o.m. energy $\Rightarrow$ multi-particle states
- We need scattering amplitudes with
  - high accuracy $\Rightarrow$ loops
  - multi-particle $\Rightarrow$ high multiplicity

Theoretical studies of amplitudes

- infer general structures in QFT and gauge theories
- exploit them in computational techniques

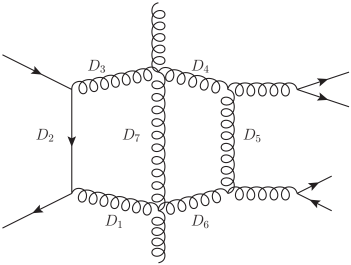


Loop amplitudes

- The integrand of a generic ℓ-loop integral:
  - is a rational function in the components of the loop momenta k_i
  - polynomial numerator $\mathcal{N}$

$$\mathcal{A}^{(\ell)} = \int d^d k_1 \cdots d^d k_\ell \;\; \mathcal{I}, \qquad \mathcal{I} \equiv \frac{\mathcal{N}}{D_1 \cdots D_n}$$

  - quadratic polynomial denominators D_i
    - they correspond to Feynman loop propagators



$$D_i = \ell_i^2 - m_i^2,$$

$$l_i^\mu = \sum_{j=1}^{\ell} \alpha_{ij} k_j^\mu + \sum_{j=1}^{n} \beta_{ij} p_j^\mu \quad (\alpha_{ij}, \beta_{ij} \in \{0, \pm 1\})$$

Summary of the state of the art:

One-loop integrand reduction and automated tools

The Integrand reduction of one-loop amplitudes

- Every one-loop integrand, can be decomposed as
  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]

$$\mathcal{I}_n = \frac{\mathcal{N}}{D_1 \cdots D_n} = \sum_{j_1 \cdots j_5} \frac{\Delta_{j_1 j_2 j_3 j_4 j_5}}{D_{j_1} D_{j_2} D_{j_3} D_{j_4} D_{j_5}} + \sum_{j_1 j_2 j_3 j_4} \frac{\Delta_{j_1 j_2 j_3 j_4}}{D_{j_1} D_{j_2} D_{j_3} D_{j_4}} + \sum_{j_1 j_2 j_3} \frac{\Delta_{j_1 j_2 j_3}}{D_{j_1} D_{j_2} D_{j_3}} + \sum_{j_1 j_2} \frac{\Delta_{j_1 j_2}}{D_{j_1} D_{j_2}} + \sum_{j_1} \frac{\Delta_{j_1}}{D_{j_1}}$$

- The residues or on-shell integrands

$$\Delta_{i_1 \cdots i_k} = \sum_i \underbrace{c_i^{(i_1 \cdots i_k)}}_{\text{process dep.}} \quad \underbrace{\mathbf{m}_i^{(i_1 \cdots i_k)}(k)}_{\substack{\text{universal basis} \\ \text{polynomials in the loop } k^\mu}}$$

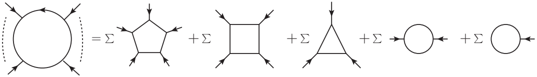
  - form a known, universal integrand basis
  - unknown, process-dependent coefficients $c_i \Rightarrow$ polynomial fit

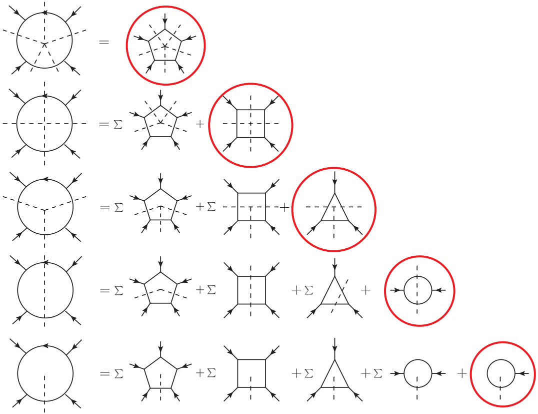
- All the integrals of the integrand basis $\mathbf{m}_i^{(i_1 \cdots i_k)}$ are known at one loop

Fit-on-the-cut at one-loop

[Ossola, Papadopoulos, Pittau (2007)]

Integrand decomposition:





Fit-on-the cut

- fit m-point residues on m-ple cuts
- Cutting a loop propagator means

$$\frac{1}{D_i} \to \delta(D_i)$$

i.e. putting it on-shell

One-loop integrand reduction: implementations

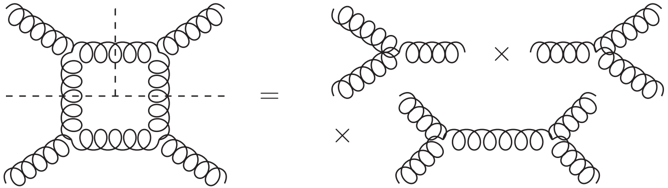
General-purpose implementations of one-loop integrand reduction:

- CUTTOOLS [Ossola, Papadopoulos, Pittau (2007)]
  - four-dimensional integrand reduction
  - extra-dimensional contributions in dim. regularization computed via process-independent (but theory-dependent) Feynman rules
- SAMURAI [Mastrolia, Ossola, Reiter, Tramontano (2010)]
  - d-dimensional integrand reduction
  - works with d dimensional integrands for any theory
- NINJA [T.P. (2014)]
  - semi-numerical integrand reduction via Laurent expansion
    Forde (2007), Badger (2008), P. Mastrolia, E. Mirabella, T.P. (2012)
  - faster and more stable integrand-reduction algorithm
  - used by GOSAM and MADLOOP (MADGRAPH5_AMC@NLO)

Generalized unitarity: loops from trees

Britto, Cachazo, Feng (2004), Giele, Kunszt, Melnikov (2008), Bern, Dixon, Kosower et al. (2008)

- Evaluating loop integrands on multiple cuts
  - the cut loop propagators are put on-shell
  - the integrand factorizes as a product of tree-level amplitudes



Loops from trees

We can compute the coefficients of loop amplitudes from products of tree-level amplitudes

- implemented in BLACKHAT, NJET and several private codes

One-loop tools

- Master Integrals
  - FF [van Oldenborg (1990)]
  - LoopTools [Hahn et al. (1998)]
  - QCDLoop [Ellis, Zanderighi (2007), Carrazza, Ellis, Zanderighi (2016)]
  - OneLOop [van Hameren (2010)]
  - ...
- Reduction
  - integrand reduction (CutTools, Samurai, Ninja)
  - tensor reduction
    - Collier [Denner, Dittmaier (since 2003), Denner, Dittmaier, Hofer (2016)]
    - Golem95 [T. Binoth, J.-P. Guillet, G. Heinrich, E. Pilon, T. Reiter (2009), J.P. Guillet, G. Heinrich, J. von Soden-Fraunhofen (2014)]
    - Iregi (part of MadLoop)
    - ...

One-loop tools (cont.)

- One-loop packages
  - HELAC-NLO: numerical recursion + OPP reduction
  - FORMCALC: analytic generation + PV or integrand reduction
  - OPENLOOPS: recursive numerical generation of tensor integrands
    - reduction via COLLIER, CUTTOOLS, SAMURAI
  - MADLOOP (MADGRAPH5_AMC@NLO) alt. OpenLoops
    - red. via NINJA, GOLEM95, IREGI, CUTTOOLS, SAMURAI, COLLIER
  - GOSAM: analytic generation (with a two-loop extension)
    - reduction via NINJA, SAMURAI, GOLEM95
  - RECOLA: recursion relations + reduction via COLLIER
  - BLACKHAT and NJET: generalized unitarity
- Montecarlo tools (Born, real+subtraction, phase-space,. . .)
  - SHERPA, AMC@NLO, MADEVENT, POWHEG, HERWIG, PYTHIA,GENEVA,. . .

Summary of the state of the art:

Higher-loop amplitudes at high multiplicity

Loop amplitudes at high multiplicity

- Loop amplitudes can be written as linear combinations of integrals

$$\mathcal{A}^{(\ell)} = \sum_i c_i \, I_i$$

- the integrals I_i are special functions of the kinematic invariants
  - at one-loop only logarithms and dilogarithms for finite part
  - at higher loops multiple polylogarithms, elliptic functions, etc. . .
- the coefficients c_i are rational functions of kinematic invariants
  - they are often a bottleneck at high multiplicity

Computing amplitudes: analytic vs numerical

QCD and SM amplitudes:

- Tree-level/One loop $\rightarrow$ mostly numerical
  - many automated codes and toolchains
  - essentially a solved problem for any process/theory/multiplicity
  - focus is on performance, stability, extension to more models, ...
- Higher loops $\rightarrow$ mostly analytic
  - more efficient/stable numerical evaluation
  - more convenient for some techniques (e.g. IBPs, diff. eqs.)
  - allows many checks/manipulations/studies (singularities, limits, ...)
  - can be used to infer general analytic/algebraic properties

  $\Rightarrow$ more control

- note that numerical algorithms (e.g. at 1 loop) often rely on good understanding of analytic/algebraic properties of the result

Techniques for loop integrals: Integration-By-Parts

Chetyrkin, Tkachov (1981), Laporta (2000)

- Amplitudes can be written as combinations of integrals of the form

$$I_T(a_1,\ldots,a_n,-b_1,\ldots,-b_m) = \int \Big(\prod_j d^d k_j\Big) \frac{S_1^{b_1}\cdots S_m^{b_m}}{D_1^{a_1}\cdots D_n^{a_n}}, \qquad a_i \lesseqgtr 0, \quad b_i \geq 0$$

  - D_i are loop denominators
  - S_i are irreducible scalar products (ISPs) depending on $\{k_i\cdot k_j,\ k_i\cdot p_j\}$

- These integrals can be reduced to a minimal set of Master Integrals (MIs) by solving linear relations (IBPs, LI, symmetries)
  - e.g. Integration-By-Parts (IBPs) obtained by expanding

$$\int \Big(\prod_j d^d k_j\Big) \frac{\partial}{\partial k_j^\mu}\, v^\mu\, \frac{S_1^{b_1}\cdots S_m^{b_m}}{D_1^{a_1}\cdots D_n^{a_n}} = 0, \quad v^\mu = p_i^\mu, k_i^\mu$$

Techniques for loop integrals: differential equations

Kotikov (1991), Gehrmann, Remiddi (2000)

- IBP reduction also allows to write down differential equations for MIs wrt external invariants x

$$\frac{\partial}{\partial x} I_i = \sum_j A_{ij}^{(x)}(d, x)\, I_j$$

- For special choices of the MIs (pure functions of uniform trascendentality) the system takes the form [Henn (2013)]

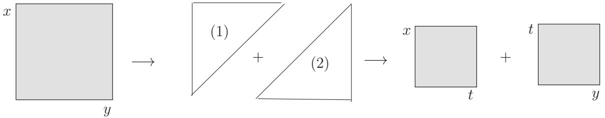
$$\frac{\partial}{\partial x} I_i = (d-4) \sum_j A_{ij}^{(x)}(x)\, I_j$$

  - much easier to solve, perturbatively in $\epsilon = (4-d)/2$
  - many recent complex calculations use this technique

Techniques for loop integrals: sector decomposition

- Numerical integration of Feynman integrals via sector decomposition
  Binoth, Heinrich (2000)
- recursively split integration region into sectors, disentangling overlapping divergences
- the main idea[1]

$$\int_0^1 dx \int_0^1 dy \, \frac{1}{(x+y)^{2+\epsilon}} = \int_0^1 dx \int_0^1 dt \, \frac{1}{x^{1+\epsilon}(1+t)^{2+\epsilon}} + \int_0^1 dy \int_0^1 dt \, \frac{1}{y^{1+\epsilon}(t+1)^{2+\epsilon}}$$



- automated in the public tools SECDEC [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke] and FIESTA [AV. Smirnov]

[1] Picture and example stolen from S. Borowka

High multiplicity at higher loops

- Higher-loop extension of integrand reduction and generalized unitarity
  S. Badger, H. Frellesvig, P. Mastrolia, E. Mirabella, G. Ossola, A. Primo, Y. Zhang, T.P. (2011—now)
- First two-loop 5-point Master Integrals via diff. eqs. in ϵ-factorized form
  Gehrmann, Henn, Lo Presti (2015), Papadopoulos, Tommasini, Wever (2015)
- All-plus two-loop 5-gluon amplitudes
  Badger, Frellesvig, Zhang (2013), Badger, Mogull, Ochirov, O'Connell (2015), Gehrmann, Henn, Lo Presti (2015)
- Two-loop 6-gluon all-plus amplitudes
  Dunbar, Perkins, Warren (2016), Badger, Mogull, T.P. (2016)
- Finite fields and functional reconstruction techniques for 2-loop generalized unitarity T.P. (2016)
- Two-loop 5-gluon amplitudes for all helicity configurations via generalized unitarity and finite-field techniques
  S. Badger, C. Brønnum-Hansen, H.B. Hartanto, T.P. (2017)
  S. Abreu, F. F. Cordero, H. Ita, B. Page, M.Zeng (2017)

Finite fields and multivariate reconstruction

Analytic calculation of scattering amplitudes

- Main bottleneck: large intermediate expressions
  - they can be orders of magnitude larger than the final result
  - not constrained by properties and symmetries of the result

The main idea

- reconstruct analytic result from "numerical" evaluations
- no large intermediate expression (just numbers!)

- Numerical evaluations over finite fields
  - using $\mathcal{Z}_p = \{0, \ldots, p-1\}$ with p prime
  - represented by machine-size integers $\Rightarrow$ fast
  - exact arithmetic operations modulo p
  - numbers and functions over $\mathcal{Q}$ can be reconstructed from their image over several fields $\mathcal{Z}_p$

Functional reconstruction over finite fields

- Finite fields
  - used under-the-hood by computer algebra systems (e.g. in polynomial factorization/GCD)
  - used for IBPs (univariate applications)
    [von Manteuffel, Schabinger (2014–2017)]

- Efficient algorithm for functional reconstruction [T.P. (2016)]
  - works on (dense) multivariate polynomials and rational functions
  - implemented in C++ code (proof of concept)
  - the input is a numerical procedure computing a function
  - the output is its analytic expression
- Applications
  - linear systems of equations and composite functions
  - spinor-helicity and tree-level recursion
  - multi-loop integrand reduction and generalized unitarity

Polynomials and rational functions

- multi-index notation: variables $z = (z_1, \ldots, z_n)$ and integer list $\alpha = (\alpha_1, \ldots, \alpha_n)$

$$z^\alpha \equiv \prod_{i=1}^{n} z_i^{\alpha_i}, \qquad |\alpha| = \sum_i \alpha_i$$

- Given a generic field $\mathcal{F}$
  - $\mathcal{F}[z]$ is the ring of polynomials in z with coefficients in $\mathcal{F}$

$$f(z) = \sum_\alpha c_\alpha z^\alpha.$$

  - $\mathcal{F}(z)$ is the field of rational functions in z with coefficients in $\mathcal{F}$

$$f(z) = \frac{p(z)}{q(z)} = \frac{\sum_\alpha n_\alpha z^\alpha}{\sum_\beta d_\beta z^\beta},$$

    - technicality: set $d_{\min\beta} = 1$ to make the representation unique.

The black-box interpolation problem in $\mathcal{Z}_p$

Given a polynomial or rational function f in the variables $z = (z_1, \ldots, z_n)$

- reconstruct analytic form of f, given a numerical procedure over finite fields $\mathcal{Z}_p$

$$(z, p) \longrightarrow \boxed{f} \longrightarrow f(z) \text{ mod } p.$$

- no further assumptions on f
- numbers and function over $\mathcal{Q}$ are reconstructed from images over $\mathcal{Z}_p$ using the rational reconstruction algorithm [Wang (1981)] and the Chinese remainder theorem
- numerical evaluations can be extensively parallelized

Univariate polynomials

- Newton' interpolation formula, form a sequence $\{y_0, y_1, \ldots\}$

$$f(z) = \sum_{r=0}^{R} a_r \prod_{i=0}^{r-1} (z - y_i)$$
$$= a_0 + (z - y_0)\Big(a_1 + (z - y_1)\Big(a_2 + (z - y_2)\big(\cdots + (z - y_{r-1})\, a_r\big)\Big)\Big)$$

- each coefficient a_i can be determined by evaluations $f(y_j)$ with $j \leq i$
    - good when degree is not known
- conversion into canonical form

$$f(z) = \sum_{r=0}^{R} c_r\, z^r.$$

    - addition of univariate polynomials,
    - multiplication of a univ. polynomial by a linear univ. polynomial

Univariate rational functions

- Thiele's (1838–1910) interpolation formula

$$f(z) = a_0 + \cfrac{z - y_0}{a_1 + \cfrac{z - y_1}{a_2 + \cfrac{z - y_3}{\cdots + \cfrac{z - y_{r-1}}{a_N}}}}$$

$$= a_0 + (z - y_0)\left(a_1 + (z - y_1)\left(a_2 + (z - y_2)\left(\cdots + \frac{z - y_{N-1}}{a_N}\right)^{-1}\right)^{-1}\right)^{-1},$$

- analogous to Newton's for rational functions
  - good when degrees of numerator/denominator are not known
- if degrees are known and $d_0 = 1$ (see later), just solve the system

$$f(z) = \frac{\sum_{r=0}^{R} n_r\, z^r}{\sum_{r'=0}^{R'} d_{r'}\, z^{r'}} \quad \Rightarrow \quad \sum_{r=0}^{R} n_r\, y_i^r - \sum_{r'=1}^{R'} d_{r'}\, y_i^{r'} f(y_i) = f(y_i)$$

Multivariate polynomials

- recursive Newton's formula

$$f(z_1,\ldots,z_n) = \sum_{r=0}^{R} a_r(z_2,\ldots,z_n) \prod_{i=0}^{r-1}(z_1 - y_i),$$

- like univariate with

$$f(y_j) \longrightarrow f(y_j, z_2,\ldots,z_n), \qquad a_j \longrightarrow a_j(z_2,\ldots,z_n).$$

- convert it back to canonical representation using
  - addition of multivariate polynomials,
  - multiplication of a multiv. polynomial by a linear univ. polynomial.
- very efficient, even for large polynomials

Multivariate rational functions

- dense algorithm, adapted from sparse one by A. Cuyt, W. Lee (2011)
- overall normalization
    - assume non-vanishing constant term in denominator ($d_{(0,\ldots,0)} = 1$)
    - if not the case, shift args. by appropriate vector $\boldsymbol{s}$, using $f_{\boldsymbol{s}} = f(\boldsymbol{z}+\boldsymbol{s})$
- define new function $h \in \mathcal{F}(t,\boldsymbol{z})$ as

$$h(t,\boldsymbol{z}) \equiv f(t\,\boldsymbol{z}) = f(t\,z_1,\ldots,t\,z_n) = \frac{\sum_{r=0}^{R} p_r(\boldsymbol{z})\,t^r}{1+\sum_{r'=1}^{R'} q_{r'}(\boldsymbol{z})\,t^{r'}}$$

where

$$p_r(\boldsymbol{z}) \equiv \sum_{|\alpha|=r} n_\alpha\,\boldsymbol{z}^\alpha, \qquad q_{r'}(\boldsymbol{z}) \equiv \sum_{|\beta|=r'} d_\beta\,\boldsymbol{z}^\beta.$$

$\Rightarrow$ univ. rational fun. in t with (homogeneous) multiv. polynomial coefficients

Multivariate functional reconstruction (summary)

T.P. (2016)

- Univariate polynomials
  - based on Newton's interpolation formula
- Univariate rational functions
  - based on Thiele's (1838–1910) interpolation formula
- Multivariate polynomials
  - recursive application of Newton's interpolation
- Multivariate rational functions
  - use ideas proposed for sparse interpolation [A. Cuyt, W. Lee (2011)]
  - combined with Newton and Thiele's interpolation for dense case

- Notes:
  - all implemented in C++
  - results automatically come out GCD-simplified
  - can be used from a MATHEMATICA interface

Finite-fields and functional reconstruction

- Any algorithm which can be implemented via a sequence of rational operations allows a numerical implementation over $\mathcal{Z}_p$
- Given a numerical procedure computing a rational function f over finite fields $\mathcal{Z}_p$, we can reconstruct the analytic expression of f

$\Rightarrow$ We can perform analytic calculations by implementing equivalent numerical algorithms over finite fields

Example: linear solver

- A $n \times m$ linear system with parametric rational entries

$$\sum_{j=1}^{m} A_{ij}\, x_j = b_i, \quad (j = 1, \ldots, n), \qquad A_{ij} = A_{ij}(z), \quad b_i = b_i(z)$$

- solution $\Rightarrow$ find coefficients $c_{ij} = c_{ij}(z)$ such that

$$x_i = c_{i0} + \sum_{j \in \text{indep}} c_{ij}\, x_j \qquad (i \notin \text{indep})$$

- Functional reconstruction
  - solve system numerically (over finite fields) to evaluate the coefficients $c_{ij}(z)$ of the solution
  - independent equations/variables and vanishing coefficients can be determined quickly and simplify further evaluations
- Very good efficiency compared to traditional computer algebra systems

Applications to two-loop five-point amplitudes

Choice of kinematic variables: momentum twistors

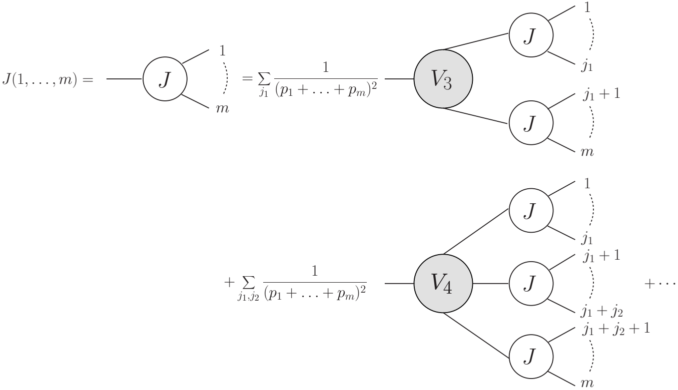
Hodges (2009), Badger, Frellesvig, Zhang (2013), Badger (2016)

- rational parametrization of the n-point phase-space and the spinor components using $3n-10$ momentum-twistor variables
- the components of spinors, external momenta and polarization vectors are rational functions of momentum twistor variables

$$|1\rangle = \binom{1}{0}, \qquad |2\rangle = \binom{0}{1}, \qquad |3\rangle = \binom{\frac{1}{x_1}}{1}, \qquad \ldots$$
$$|1] = \binom{1}{\frac{x_4-x_5}{x_4}}, \qquad |2] = \binom{0}{x_1}, \qquad |3] = \binom{x_1\, x_4}{-x_1}, \qquad \ldots$$

Both analytic and numerical calculations can be performed operating directly on the components of spinors and momenta

Tree-level amplitudes via Berends-Giele recursion



- very efficient for numerical calculations
- functional reconstruction techniques can exploit this for obtaining analytic results

Integrand reduction

Ossola, Papadopoulos, Pittau (2007)

- generic contribution to a loop amplitude

$$\int_{-\infty}^{\infty} \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{\mathcal{N}(k_i)}{\prod_j D_j(k_i)},$$

- integrand reduction (integrand as sum of irreducible contributions)

$$\frac{\mathcal{N}(k_i)}{\prod_j D_j(k_i)} = \sum_{T \in \text{topologies}} \frac{\Delta_T(k_i)}{\prod_{j \in T} D_j(k_i)}, \qquad \Delta_T(k_i) = \sum_\alpha c_{T,\alpha} \, (\boldsymbol{m}_T(k_i))^\alpha$$

- the on-shell integrands or residues Δ_T
  - $\{\boldsymbol{m}_T^\alpha\}$ forms a complete integrand basis (see below)
- fit unknown $c_{T,\alpha}$ on multiple cuts $\{D_j = 0\}_{j \in T}$
  - solutions of a linear system

Finding an integrand basis

1. use monomials in a complete set of irreducible scalar products between loop momenta k_i^μ, external momenta p_i^μ and orthogonal vectors ω_i^μ

$$\{\boldsymbol{m}_T\} = \{\boldsymbol{m}_T\}_{\text{complete}} = \{k_i \cdot k_j,\, k_i \cdot p_j\}_{\text{irreducible}} \;\cup\; \{k_i \cdot \omega_j\}_{\omega_i \perp p_j}$$

   - irreducible $\equiv$ not a combination of denominators $D_i \in T$
   - all scalar products $k_i \cdot \omega_j$ are irreducible but they can be integrated out and do not appear in the final result P. Mastrolia, A. Primo, T.P. (2016)

2. use monomials in a overcomplete set of irreducible scalar products

$$\{\boldsymbol{m}_T\} = \{\boldsymbol{m}_T\}_{\text{complete}} \;\cup\; (k_{i,[d-4]} \cdot k_{j,[d-4]}) \;\cup\; \cdots$$

   - the monomials satisfy linear relations which can be inverted (numerically over f.f.) to determine an independent basis
   - by maximizing the presence of $(k_{i,[d-4]} \cdot k_{j,[d-4]})$ we ensure a smooth $d \to 4$ limit, which yields simpler results

Other choices for an integrand basis

- Local integrands for 5- and 6-point 2-loop all-plus amplitudes
  - $\mathcal{N} = 4$ [Arkani-Hamed, Bourjaily, Cachazo, Trnka (2010)]
  - all-plus QCD [Badger, Mogull, T.P. (2016)]
  - free of spurious singularities
  - smooth soft limits to lower-point integrands
  - infrared properties manifest at the integrand level

  $\Rightarrow$ simpler results

  ✗ . . . but no general algorithm for a complete one (yet)

- Other properties worth looking for in the future
  - correspondence with uniform-weight integrals for easier integration (cfr. J. Henn (2013))

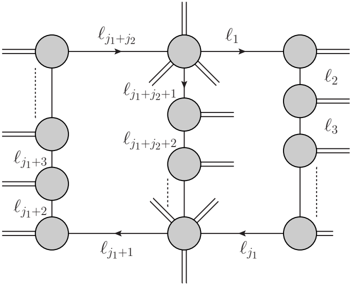
- Looking for a good choice using functional reconstruction
  - the functional reconstruction algorithm allows to quickly compute the degree of multivariate functions without a full reconstruction
  - the degree can be used to assess the complexity of the result

Integrand reduction and generalized unitarity

Britto, Cachazo, Feng (2004), Giele, Kunszt, Melnikov (2008), Bern, Dixon, Kosower et al. (2008)

- Generalized unitarity
  - build irreducible integrands from multiple cuts
  - multiple-cuts ⇒ loop propagators go on-shell, $\ell_i^2 = 0$
  - integrand factorizes as product of trees (summed over internal helicities)
  - multiple cuts ⇒ unitarity cuts

- # unitarity cuts ≪ # diagrams
  - lower complexity
- Every intermediate step is gauge invariant
  - no ghosts
  - more compact expressions



Two-loop unitarity cuts in d dimensions

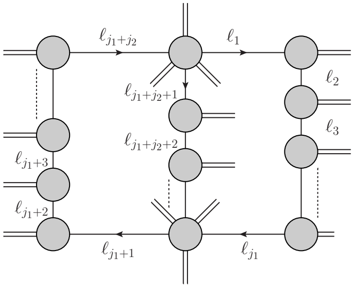
Bern, Carrasco, Dennen, Huang, Ita (2010), Davies (2011), Badger, Frellesvig, Zhang (2013)

- d-dim. dependence of loops $k_i^\mu \Rightarrow$ embed k_i^μ in $\mathcal{D}$ dimensions $(\mathcal{D} > 4)$
- unitarity cuts $\ell_i^2 = 0 \Rightarrow$ explicit $\mathcal{D}$-dim. representation of loop components
- describe internal on-shell states with $\mathcal{D}$-dim. spinor-helicity formalism
  see e.g. six-dim. formalism by Cheung, O'Connell (2009)
- additional gluon states as $d_s - \mathcal{D}$ scalars ($d_s = 4, d$ in FDH, tHV)

$\mathcal{D} = 6$ sufficient up to two loops:

$$\mathcal{A}^{(2)} = \sum_{i=0}^{2} (d_s - 2)^i \, \mathcal{A}_i^{(2)}$$

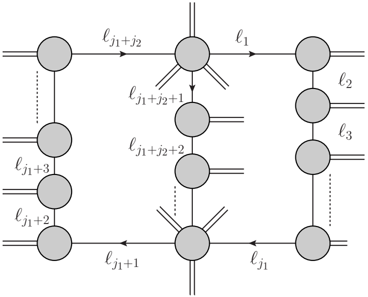
numerical evaluation over finite fields using an explicit (rational) representation of internal states



Generalized unitarity over finite fields

T.P. (2016)

- Amplitudes over finite fields
  - momentum-twistor variables
  - loop states: embed in 6-dim.
  - spinor-helicity in 4 and 6 dim.
  - tree-level recursion
  - two-loop d-dim. unitarity cuts



Finite-field implementation

- explicit six-dim. representation of loop states
- efficient numerical techniques for analytic calculations
- two-loop unitarity cuts by sewing Berends-Giele currents
  - sum over helicities only for 2 internal lines
  - the others replaced by contraction of currents

Two-loop five-gluon helicity amplitudes

S. Badger, C. Brønnum-Hansen, H.B. Hartanto, T.P. (2017)

Two-loop leading-colour (planar) five-gluon helicity amplitudes

$$\mathcal{A}^{(2)}(1,2,3,4,5) = g_s^3 \sum_{\sigma \in S_5/Z_5} \mathrm{tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(5)}}\right) A^{(2)}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right)$$

helicity	flavour	non-zero coefficients	non-spurious coefficients	contributions @ $\mathcal{O}(\epsilon^0)$
$+++++$	$(d_s-2)^0$	50	50	0
	$(d_s-2)^1$	175	165	50
	$(d_s-2)^2$	320	90	60
$-++++$	$(d_s-2)^0$	1153	761	405
	$(d_s-2)^1$	8745	4020	3436
	$(d_s-2)^2$	1037	100	68
$--+++$	$(d_s-2)^0$	2234	1267	976
	$(d_s-2)^1$	11844	5342	4659
	$(d_s-2)^2$	1641	71	48
$-+-++$	$(d_s-2)^0$	3137	1732	1335
	$(d_s-2)^1$	15282	6654	5734
	$(d_s-2)^2$	3639	47	32

Two-loop five-gluon helicity amplitudes

S. Badger, C. Brønnum-Hansen, H.B. Hartanto, T.P. (2017)

- complete parametrization of a generic (massless) 5-point integrand
- numerical integrand reduction over finite fields
- partial reduction of integrals via IBPs
- numerical calculation of some integrals via sector decomposition techniques
- functional reconstruction of kinematic dependence of most of the integrands (the remaining ones will be available soon)
- first numerical benchmark points for a 2-loop 5-point amplitude for a complete set of set of helicity configurations

Two-loop five-gluon helicity amplitudes

S. Abreu, F. F. Cordero, H. Ita, B. Page, M.Zeng (2017)

A similar result published a few days later:

- similar calculation, using numerical generalized unitarity over finite fields
- a few key differences
  - IBPs embedded at the integrand level [H. Ita (2015)]
  - reduction to MIs known analytically
- numerical benchmark point
- in agreement with our calculation

Summary & Outlook

Summary

- Novel developments for high-multiplicity two-loop calculations
  - multi-loop integr. reduction via gen. unitarity over finite fields
  - recent methods for computing integrals via IBPs and DE
- Finite-fields and functional reconstruction techniques
  - can be use to solve complex algebraic problems
  - any function which can be implemented as a sequence of rational operations is suited for these algorithms

Outlook

- analytic integral representation of five-point two-loop amplitudes and stable evaluation in the Minkowski region
- apply finite fields reconstruction algorithms to other techniques (e.g. diagrammatic techniques, tensor reduction, IBPs,. . .)

THANKS!

BACKUP SLIDES

Finite fields

- In this talk we consider finite fields $\mathcal{Z}_p$, with p prime
- We define

$$\mathcal{Z}_n = \{0, \ldots, n-1\}$$

  - addition, subtraction, and multiplication via modular arithmetic

$$4 + 5\Big|_{\mathcal{Z}_7} = (4+5) \text{ mod } 7 = 2$$

  - if $a \in \mathcal{Z}_n$ and a, n are coprime, we can define an inverse

$$b = a^{-1} \in \mathcal{Z}_n, \qquad a \times b \text{ mod } n = 1$$

  - if $n = p$ prime, an inverse exists for every $a \in \mathcal{Z}_p \Rightarrow \mathcal{Z}_p$ is a field
  - every rational operation is well defined in $\mathcal{Z}_p$

Rational reconstruction

Functional reconstruction

Reconstruct the monomials z^α and their coefficients from numerical evaluations of the function (over finite fields)

- from $\mathcal{Q}$ to $\mathcal{Z}_p$

$$q = a/b \in \mathcal{Q} \quad \longrightarrow \quad q \bmod p \equiv a \times (b^{-1} \bmod p) \bmod p$$

- how to go back from $\mathcal{Z}_p$ to $\mathcal{Q}$?
- rational reconstruction algorithm: given $c \in \mathcal{Z}_n$ find its pre-image $q = a/b \in \mathcal{Q}$ with "small" a, b [Wang (1981)]
  - it's correct when $a, b \lesssim \sqrt{n}$
- make n large enough using Chinese reminder theorem
  - solution in $\mathcal{Z}_{p_1}, \mathcal{Z}_{p_2} \ldots \Rightarrow$ solution in $\mathcal{Z}_{p_1 p_2 \ldots}$

Extended euclidean algorithm

- given integers a, b, find s, t such that

$$a\, s + b\, t = \gcd(a,b).$$

- algorithm: generate sequences of integers $\{r_i\}$, $\{s_i\}$, $\{t_i\}$ and the integer quotients $\{q_i\}$ as follows

$$\begin{aligned} r_0 &= a & \cdots &= \cdots \\ s_0 &= 1 & q_i &= \lfloor r_{i-2}/r_{i-1} \rfloor \\ t_0 &= 0 & r_i &= r_{i-2} - q_i\, r_{i-1} \\ r_1 &= b & s_i &= s_{i-2} - q_i\, s_{i-1} \\ s_1 &= 0 & t_i &= t_{i-2} - q_i\, t_{i-1} \\ t_1 &= 1 & & \end{aligned}$$

- stop when $r_k = 1 \Rightarrow t = t_{k-1}$, $s = s_{k-1}$, $\gcd(a,b) = r_{k-1}$
- multiplicative inverse: if $b = n$ and $\gcd(a,n) = 1 \Rightarrow s = a^{-1}$.

Chinese reminder theorem

- given $a_1 \in \mathcal{Z}_{n_1}$, $a_2 \in \mathcal{Z}_{n_2}$ (n_1, n_2 co-prime) find $a \in \mathcal{Z}_{n_1 n_2}$ such that

$$a \text{ mod } n_1 = a_1, \qquad a \text{ mod } n_2 = a_2.$$

- rational reconstruction over $\mathcal{Q}$
  - reconstruct a function f over several finite fields $\mathcal{Z}_{p_1}, \mathcal{Z}_{p_2}, \ldots$
  - recursively combine the result in $\mathcal{Z}_{p_1 p_2 \cdots}$ using the Chinese reminder
  - use the rational reconstruction algorithm on the combined result over $\mathcal{Z}_{p_1 p_2 \cdots}$ to obtain a guess over $\mathcal{Q}$
  - when $\prod_i p_i$ is large enough the reconstruction is successful
  - the termination criterion is consistency over several finite fields
- we can choose the primes p_i small enough to use machine-size integers
- multi-precision arithmetic only required for Chinese reminder
- 1, 2 or 3 primes are often sufficient

Rational reconstruction: example

- Reconstruct $q = -611520/341$ from its images over finite fields
- $\mathcal{Z}_{p_1}$, with $p_1 = 897473$

$$a_1 = q \text{ mod } p_1 = 13998,$$

$$\text{first guess: } a_1 \xrightarrow{\text{rational rec. over } \mathcal{Z}_{p_1}} g_1 = -411/577$$

- $\mathcal{Z}_{p_2}$, with $p_2 = 909683$

$$a_2 = q \text{ mod } p_2 = 835862$$

$$g_1 \text{ mod } p_2 = 807205 \quad \Rightarrow \quad \text{guess } g_1 \text{ is wrong}$$

- Chinese reminder: $a_1, a_2 \longrightarrow a_{12} \in \mathcal{Z}_{p_1 p_2}$, with $p_1 p_2 = 816415931059$

$$a_{12} \equiv q \text{ mod } p_1 p_2 = 629669763217 \xrightarrow{\text{rational rec. over } \mathcal{Z}_{p_1 p_2}} g_2 = -611520/341$$

- calculation over other fields $\mathcal{Z}_{p_3}, \ldots$ confirm the guess g_2

Choice of variables: spinor-helicity formalism

Mangano, Parke

- tree-level amplitudes and coefficients of loop integrals are rational functions of spinor variables $|p\rangle$ and $|p]$
- satisfying the Dirac equation (in Weyl components)

$$p^\mu \, \sigma_\mu |p\rangle = p^\mu \, \sigma_\mu |p] = 0$$

- momenta and polarization vectors

$$p^\mu = \frac{1}{2} \langle p| \, \sigma^\mu \, |p], \quad \epsilon_+^\mu(p) = \frac{\langle \eta|\sigma^\mu|p]}{\sqrt{2}\,\langle \eta\, p\rangle}, \quad \epsilon_-^\mu(p) = \frac{\langle p|\sigma^\mu|\eta]}{\sqrt{2}\,[p\,\eta]}$$

- helicity amplitudes are combinations of spinor products, e.g.

$$\mathcal{A}_{5g}(1^+, 2^-, 3^+, 4^-, 5^+) = i\, g_s^3 \, \frac{\langle 2\,4\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,1\rangle}$$

- redundancy: spinor components are not all independent

A brief digression on spinor phases

- under a little group tranformation (complex redefinition of phase)

$$|i\rangle \to t_i\,|i\rangle, \qquad |i] \to \frac{1}{t_i}\,|i],$$

an n-point amplitude $\mathcal{A}(1,\ldots,n)$ transforms as

$$\mathcal{A}(1,\ldots,n) \to \left(\prod_{i=1}^{n} t_i^{-2\,h_i}\right)\mathcal{A}(1,\ldots,n),$$

where h_i is the helicity of the i-th particle (e.g. $\pm 1/2$ for fermions and ± 1 for gluons)

- extract from the amplitude an overall factor $\mathcal{A}^{(\mathrm{phase})}(1,\ldots,n)$ which transform as the amplitude
- consider $\tilde{A}$ such that

$$\mathcal{A} = \underbrace{\mathcal{A}^{(\mathrm{phase})}}_{\text{only depends on helicities}} \times \underbrace{\tilde{\mathcal{A}}(x_i)}_{\text{phase-free} \to \text{mom. twist.}}$$

A brief digression on spinor phases

Examples (loop independent):

- possible choices for 5-gluon amplitudes

$$\mathcal{A}^{(\text{phase})}(1^+,2^+,3^+,4^+,5^+) = \frac{1}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}$$

$$\mathcal{A}^{(\text{phase})}(1^-,2^+,3^+,4^+,5^+) = \frac{(\langle 1\,2\rangle\,[23]\langle 3\,1\rangle])^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}$$

$$\mathcal{A}^{(\text{phase})}(1^-,2^-,3^+,4^+,5^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle},$$

- a choice n-gluon amplitudes S. Badger (2016)

$$\mathcal{A}^{(\text{phase})}(1^{h_1},\ldots,n^{h_n}) = \left(\frac{\langle 3\,2\,1]}{\langle 3\,1\rangle}\right)^{(h_1-\sum_{i=2}^{n} h_i)} \prod_{i=2}^{n} \langle i\,1\rangle^{-2h_i}$$

Choice of kinematic variables (phase-free part)

Hodges (2009), Badger, Frellesvig, Zhang (2013), Badger (2016)

- $3n - 10$ momentum-twistor variables
- 5-point example → 5 variables $\{x_1, \ldots, x_5\}$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1] = \begin{pmatrix} 1 \\ \frac{x_4 - x_5}{x_4} \end{pmatrix}, \qquad x_k = x_k(s_{ij}, \mathrm{tr}(\sigma_5\, 1\, 2\, 3\, 4))$$

$$|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |2] = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \qquad p_i^\mu = \frac{\langle i|\, \sigma^\mu \,| i]}{2}$$

$$|3\rangle = \begin{pmatrix} \frac{1}{x_1} \\ 1 \end{pmatrix}, \qquad |3] = \begin{pmatrix} x_1\, x_4 \\ -x_1 \end{pmatrix},$$

$$|4\rangle = \begin{pmatrix} \frac{1}{x_1} + \frac{1}{x_1\, x_2} \\ 1 \end{pmatrix}, \qquad |4] = \begin{pmatrix} x_1 (x_2\, x_3 - x_3\, x_4 - x_4) \\ -\frac{x_1\, x_2\, x_3\, x_5}{x_4} \end{pmatrix},$$

$$|5\rangle = \begin{pmatrix} \frac{1}{x_1} + \frac{1}{x_1\, x_2} + \frac{1}{x_1\, x_2\, x_3} \\ 1 \end{pmatrix}, \qquad |5] = \begin{pmatrix} x_1\, x_3 (x_4 - x_2) \\ \frac{x_1\, x_2\, x_3\, x_5}{x_4} \end{pmatrix}.$$