Using Feynman's Tree Theorem to Evaluate Loop Integrals Numerically

Tobias Kleinschmidt

27.10.09 - Radcor 2009

In collaboration with W. Kilian

Outline

Ingredients of NLO Calculations

- Tree Graphs (real corrections) with n+1 partons.
  ✓ Fully understood!
  Several generators for creation of efficient matrix elements.
  e.g. O'Mega [Ohl et.al.,'01], Alpha [Caravaglios et.al.,'95], MadGraph [Stelzer et.al.,'03], Comix [Gleisberg, Höche '08]
  Contain infrared soft and collinear divergences.
- Subtraction Terms
  Cancel divergences in real corrections locally.
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  Mainly used: [Catani, Seymour '96; Catani, Seymour, Dittmaier, Trocsanyi '02]
- Virtual Corrections to n-leg process
  Contain... Problems!

Level of Complexity rises due to:

- Length of Expressions
- Complexity of Integrals
- IR divergences, internal singularities

Calculations very time consuming! (Not only computing time!)

➠ Aim at fully automated matrix element generation and event generation.

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Major Tools and Techniques for One Loop Calculations

Feynman Diagram based methods

- Tensor reduction (PV) $\rightarrow$ set of basis integrals

✓ Scalar integrals known analytically

✗ Yields large expressions for coefficients

✗ can have delicate numerical stability

$\Rightarrow$ use modified reduction schemes, avoiding Gram determinants

GOLEM: Binoth, Guillet, Heinrich, Pilon, Reiter, ...

Denner, Dittmaier, ...

Unitarity based methods

- Decompose Amplitude into scalar integrals and coefficients

Coefficients are products of on-shell tree amplitudes, obtained by cutting techniques

✗ large expressions for coefficients, but...

✓ simpler than coefficients from PV-style reductions?

✓ P-algorithm ($\tau \propto N^9$, for N-gluon-amp)

Rocket: Ellis, Giele, Kunszt, Melnikov, Zanderighi

BlackHat: Bern, Dixon, Forde, Gleisberg, Kosower, Maitre,..

Helac-1Loop: van Hameren, Papadopoulus, Pittau, Bevilacqua, Czakon, Worek...

- Fully numerical methods: Integrate over loop momentum/Feynman parameter
  - ✓ No large expressions
  - ✗ Complicated singularity structure
    - $\Rightarrow$ Sector Decomposition, Contour Deformation

      Anastasiou, Beerli, Daleo, Krämer, Nagy, Soper, ...
  - ✗ Can again create large expressions

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Outline

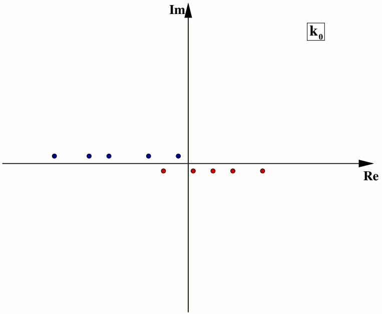
Derivation of the Feynman Tree Theorem

- Integrand $I(k)$ of a one-loop graph with loop momentum k:

$$I(k) = N(k) \prod_i F_i$$

with *Feynman* Green Functions F_i (t'Hooft-Feynman gauge):

$$F_i \equiv \frac{i}{(k+p_i)^2 - m_i^2 + i\epsilon}$$



- Partial fraction decomposition yields

$$F_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i - i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right), \qquad E_i = \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2}.$$

- Idea: Replace *Feynman* Green functions F_i by *advanced* ones A_i:

$$A_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i + i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right)$$

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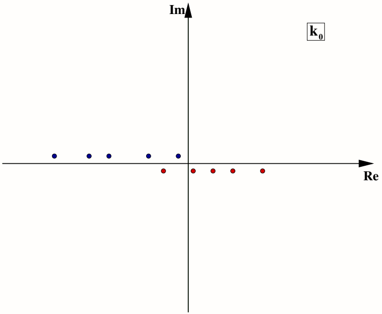

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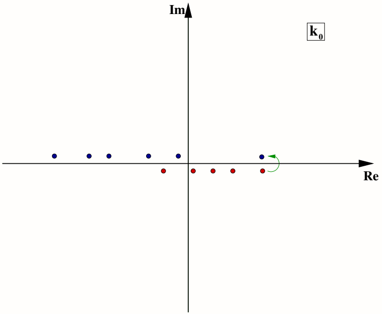

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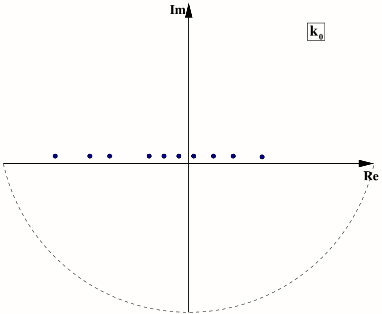

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- Difference of *Feynman* and *advanced* Green function:

$$\Delta_i^l \equiv F_i - A_i = \frac{i}{2E_i}\left(\frac{1}{k^0-(-p_i^0+E_i)+i\epsilon} - \frac{1}{k^0-(-p_i^0+E_i)-i\epsilon}\right)$$

$$\stackrel{\epsilon\to 0}{=} \frac{2\pi}{2E_i}\delta(k^0-(-p_i^0+E_i)).$$

$\Rightarrow \Delta_i^l$ sets momentum $k+p_i$ on-shell with positive energy component E_i.

$$0 = \int N(k)\prod_i^n A_i$$

- Replacing A_i with $F_i - \Delta_i^l$ yields:

Feynman Tree Theorem (FTT)

$$0 = \int N(k)\left[F\cdots F - \sum \Delta^l F\cdots + \sum \Delta^l\Delta^l F\cdots - \ldots + (-1)^n \sum \Delta^l\cdots\Delta^l\right]$$

Acta. Phys. Polon. **24** (1963) 697

- Recent interest: Brandhuber ea.[hep-th/0510253], Catani ea.[0804.3170]
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- Make use of identity:

$$\frac{1}{x-a\pm i\epsilon} = \mathcal{P}\frac{1}{x-a} \mp i\pi\delta(x-a)$$

- Rewrite Feynman Green function F_i:

$$F_i = P_i + \frac{1}{2}\Delta_i^l + \frac{1}{2}\Delta_i^u$$

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- Replace any F_i in subleading terms of FTT:

Feynman Tree Theorem - Improved Version

$$\int I(k) = \int N(k)[\Delta_1^l P_2\cdots P_n + P_1\Delta_2^l P_3\cdots P_n + \ldots + P_1\cdots P_{n-1}\Delta_n^l]$$

$$+\int N(k)\sum_{\substack{perm.\\ U+L\geq 2}} C_{LUP}\,\Delta^{l^L}\Delta^{u^U}P^P,$$

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Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1}\Delta_n^l\right]$$

- Performing k_0 integration, Δ_i^l act as *opening* or *cutting* the loop:
  - Momentum $k + p_i$ is set on-shell
  - Numerator of cut propagator is product of wave functions, summed over all internal states
  - Loop integral is replaced by phase space integral

$$(\not k + \not p_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \to \sum_\sigma \epsilon^*_\mu(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$

Loop corrections for a $2 \to n$ process can be computed by considering all possible $2+1 \to n+1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

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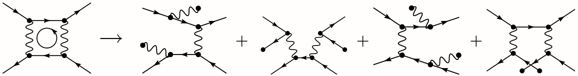

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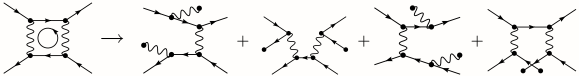

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- Phase space integration over additional particles can be performed simultaneously with integrations over external particle momenta.

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1. Renormalization and Regularization

- Born Level ⇒ Loop Level: Relation bare ⇔ physical parameters broken
  - Add renormalization constants to Lagrangian (also absorb UV divergences)

    ⇒ Additional degrees of freedom

  - Fix by renormalization conditions
- Use *on-shell* renormalization scheme [Ross and Taylor, '73]:

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$$\sigma_v^{(1)} = \Phi \int d\Pi_n \, 2\,\mathrm{Re}(\mathcal{M}^{\text{Born}}(\mathcal{M}_n^{\text{loop}} + \mathcal{M}_{n,\text{CT}}^{\text{loop}})^*)$$
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3-dim integral UV convergent. ✓

2. Infrared divergent terms in $\mathcal{M}_{n+1}^{\text{Tree}}$ *and* $\mathcal{M}_{n+1,\text{CT}}^{\text{Tree}}$. Compensated by addition of real emission graphs [Kinoshita, '63; Lee, Nauenberg, '64]

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Contains implicit δ-function conserving overall momentum.
$\Rightarrow$ Need approximation for soft real emission diagrams.

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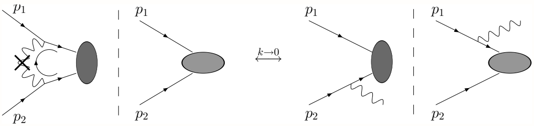

- Virtual IR-divergence arises solely from cut of massless particle.
- In limit $k \to 0$, expressions for cut loop and real emission compensate each other.
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$$\mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}} \to \mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}} \theta(|\vec{k}| - E_s), \qquad E_s\text{: soft cut}$$

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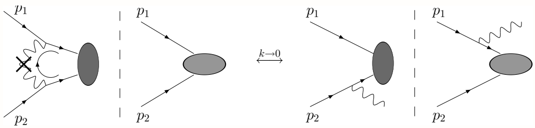

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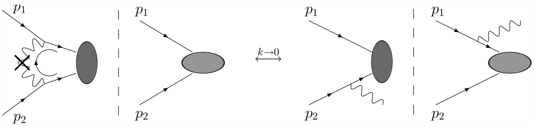

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3. Threshold Singularities

- Propagators of tree graphs can become singular in parts of integration region.

$$P_j = \frac{i}{(k+p_j)^2 - m_j^2} = \frac{i}{\left(k^0 - (-p_j^0 + E_j)\right)\left(k^0 - (-p_j^0 - E_j)\right)}$$

- After cutting propagator P_i, one of the two factors in P_j can get zero:

$$(p_j^0 - p_i^0) + (E_i \mp E_j) = 0$$

  - Vanishing of first factor corresponds to coincidence of original poles in lower half plane. $\Rightarrow$ Singularities cancel in the sum of tree graphs
  - Vanishing of second factor corresponds to coincidence of poles in lower and upper half plane. $\Rightarrow$ Singularity not canceled in the sum of tree graphs.

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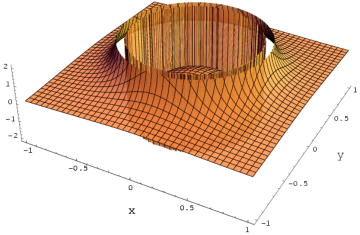
Construction of Fix Functions

- In rest frame of p_{ji}, peak of threshold singularity is spherical:

$$I(\mathbf{k}') \propto \frac{1}{\mathbf{k}' - \mathbf{k}_s}, \qquad \mathbf{k}_s = \frac{\lambda^{\frac{1}{2}}({p^0_{ji}}^2, m_i^2, m_j^2)}{2|p^0_{ji}|}$$



- Problematic for integration algorithms.
- Idea: Subtract zero from integrand:

$$\frac{\text{Res}(k_s')}{\mathbf{k}' - \mathbf{k}_s}$$

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$$\text{Fix}(\mathbf{k}', k_s') \equiv \frac{\mathbf{k}_s R(\Lambda^{-1} k_s' - p_i)}{4p^0_{ji}} \left(\frac{1}{\mathbf{k}' - \mathbf{k}_s} - 2\frac{\mathbf{k}' - \mathbf{k}_s}{c^2} + \frac{(\mathbf{k}' - \mathbf{k}_s)^3}{c^4} \right) \theta(\mathbf{k}', k_s', c)$$

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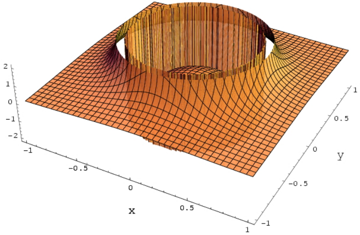

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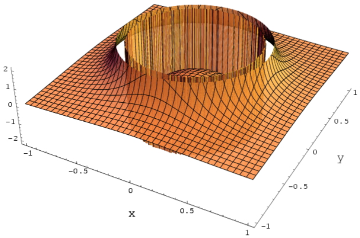

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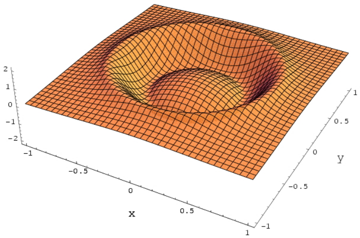

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Overlapping Peaks

- Possible for 6-pt functions with on-shell external particles or 3-pt functions with off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to r' in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

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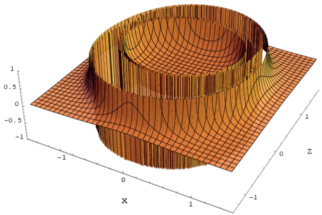
- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



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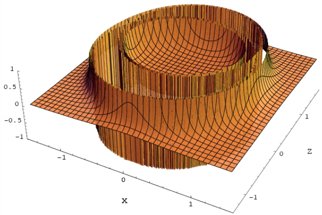

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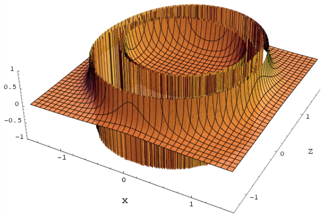

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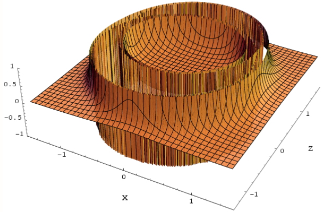

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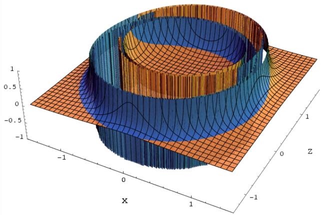

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  $\Rightarrow$ Trade-off between accuracy and efficiency!

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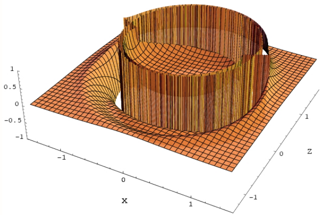

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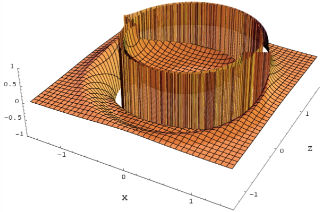

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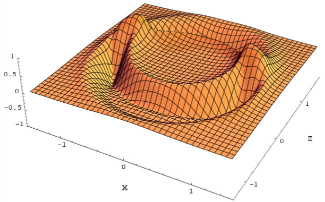

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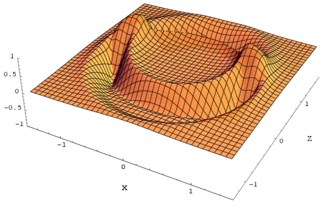

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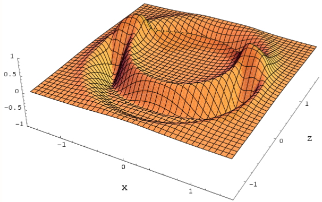

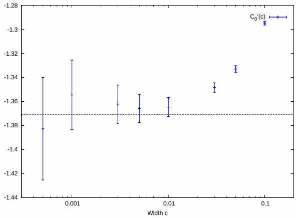

Outline

Bhabha Scattering

Application of FTT to QED Bhabha Scattering at NLO as Proof of Principle

- Includes 10 loop graphs, 2pt, 3pt and 4pt functions.
- Test subtration scheme for UV/IR divergences and internal singularities
- Two different scales: $m_e \approx 500\,$keV, $\sqrt{s} \approx 500\,$GeV

$\Rightarrow$ Compare with automated packages; FeynArts/FormCalc [Hahn ea, '98]

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In Mathematica

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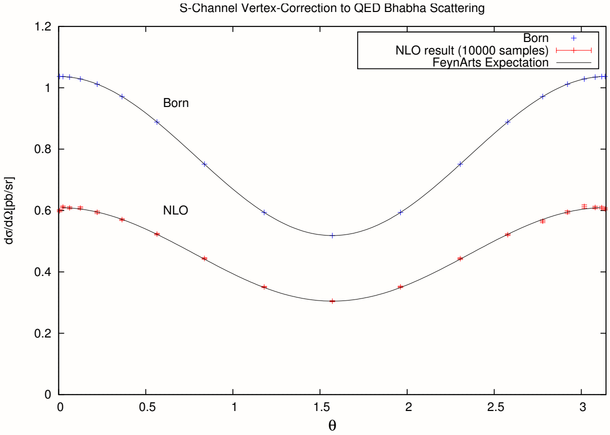

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- $\Delta E_s = 0.5$ GeV

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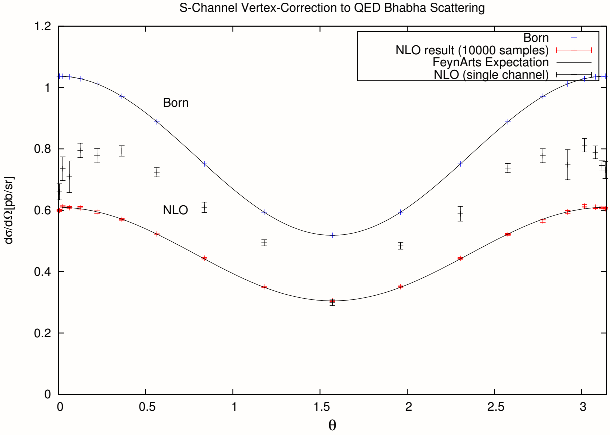

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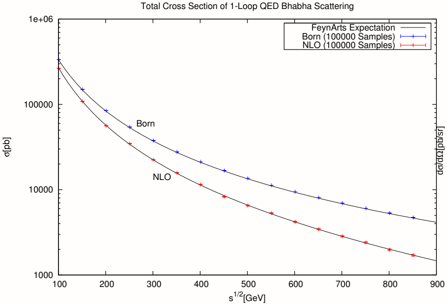
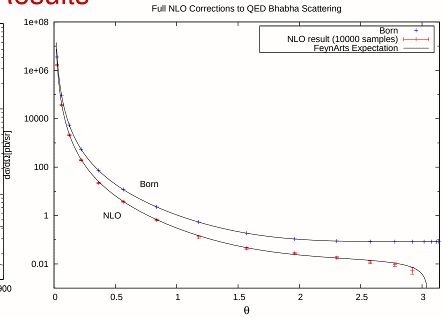
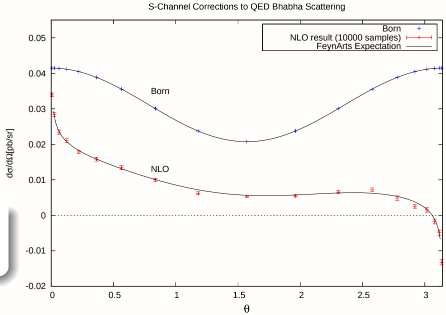




- $\Delta E_s = 5\,\text{GeV}$
- $\Delta\theta = 1\text{deg}$



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Monte Carlo Event Generation

Include NLO by using FTT

- Additional 3 inclusive variables k_i from phase space integral over additional particles in tree graphs
- Define event by x_i *and* k_i. For each set of external momenta an internal momentum is chosen simultaneously.
  $\Rightarrow$ Expect gain in computation speed compared to (semi-)analytical methods.

Negative Weights

- Integrand not positive definite
- Need to incorporate events with negative weights
  - Accept event if:

$$r \leq \frac{|w_i|}{w^{\pm}_{\text{max}}} \qquad w^{\pm}_{\text{max}} = \max(|w_{\text{max}}|, |w_{\text{min}}|)$$

  - Assign additional flag (± 1) to event, dependent on sign of w_i
- Expectation value, error:

$$\langle n_i \rangle = \langle n_i^+ \rangle - \langle n_i^- \rangle \qquad s_i = \sqrt{\langle n_i^+ \rangle + \langle n_i^- \rangle}$$

- Efficiency decreases.

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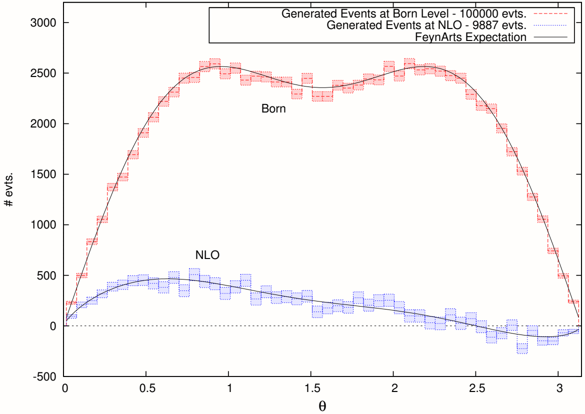
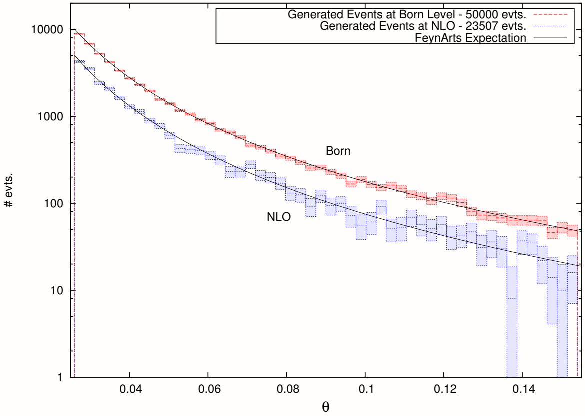


- $\sigma^{\text{tot}}_{\text{Born}} = 0.34744(29)$pb
- $\text{eff}_{\text{Born}} = 66\%$
- $\mathcal{L} = 290\text{fb}^{-1}$
- $\sigma^{\text{tot}}_{\text{NLO}} = 0.03434(91)$pb
- $\text{eff}^{\text{p+n}}_{\text{NLO}} = 1.8\%$
- $\text{eff}^{\text{hist}}_{\text{NLO}} = 0.14\%$

Results - Full Process in Forward Region



- $\sigma^{\text{tot}}_{\text{Born}} = 5981.3(3.3)\text{pb}$
- $\text{eff}_{\text{Born}} = 65\%$

- $\sigma^{\text{tot}}_{\text{NLO}} = 2812(24)\text{pb}$
- $\text{eff}^{\text{p+n}}_{\text{NLO}} = 3.0\%$
- $\text{eff}^{\text{hist}}_{\text{NLO}} = 0.8\%$

Conclusions

Summary

- Presented Method for computation of loop diagrams from tree graphs.
  $\Rightarrow$ allows fully numerical evaluation in matrix element/event generator framework
- Simple prescription for cancellation of UV-, IR-, internal singularities
- Proof of principle: Application to Bhabha scattering
- No further manipulations necessary
  $\Rightarrow$ Level of complexity rises solely due to increasing number of terms

Outlook

- Extension to full Standard Model
- Implementation in event generator package
- Far future: Extension to two loops

process	relevant for
$pp \to VV$ jet	$t\bar{t}H$, new physics
$pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$
$pp \to t\bar{t}+$ 2 jets	$t\bar{t}H$
$pp \to VVb\bar{b}$	VBF$\to H \to VV$, $t\bar{t}H$, new physics
$pp \to VV+$ 2 jets	VBF$\to H \to VV$
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The LHC *priority* wishlist, Les Houches '05. [hep-ph/0604120]

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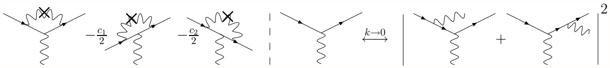

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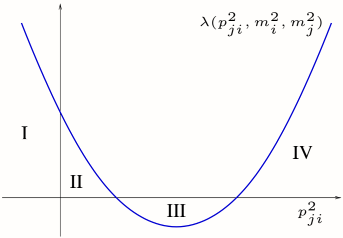

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II	dep. on loop momentum	✓
III	never	–
IV	always	!

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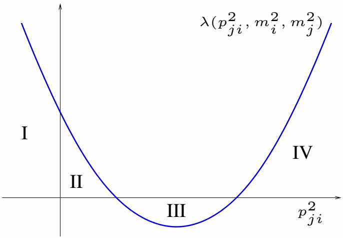

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Coordinate Mappings

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  - ⇒ No single parameterization allows for mapping the function into a constant
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  - WHIZARD has to find the *important* channels: The Feynman diagrams which have the strongest peaks ⇒ correspond to good parameterizations
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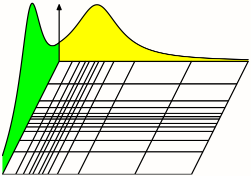

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