From low-energy moments of $\Pi(q^2)$ to $R(s)$

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Introduction

vacuum polarization function

$$\Pi^{\mu\nu}(q^2) = (-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi(q^2) + q_\mu q_\nu \Pi_L(q^2)$$
$$= i\int dx\, e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle, \qquad j_\mu = \bar{\Psi}\gamma_\mu\Psi$$

related to $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ through

$$R(s) = 12\pi \mathrm{Im}\Pi(q^2 + \mathrm{i}\epsilon)$$

low-energy expansion

$$\Pi(q^2) = \frac{3}{16\pi^2}\sum_{n>0} C_n z^n, \qquad z = \frac{q^2}{4m^2}$$

main application: determination of heavy quark masses from low-energy sum-rules, cp. talk by J. Kühn

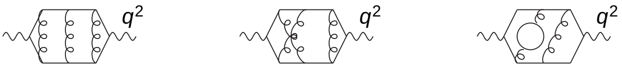
Outline

Calculation of low-energy expansion – Status

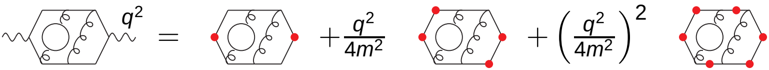
- calculate Taylor expansion of $\Pi(q^2)$ around $q^2 = 0$ in perturbative QCD up to NNNLO i.e. four loops
- at one and two loops $\Pi(q^2)$ is known analytically [Kallen et al '55]
- at three loops 30 terms are known in the low-energy expansion [Chetyrkin et al; Boughezal et al; Maier et al]
- at four loops first [Chetyrkin et al; Boughezal et al]
  second and third moment are known [Maier et al]
- terms $\propto n_f^2$ first 30 terms known at four loops [Czakon et al]
- terms $\propto \alpha_s^n n_l^{n-1}$ known to all orders in α_s [Grozin et al]

Calculation of low-energy expansion – Third moment

700 four-loop Feynman diagrams of the form



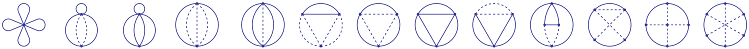
expansion around $q^2 = 0$



results in $4.5 \cdot 10^6$ four-loop vacuum integrals which have to be calculated

Calculation cont'd

- direct calculation of all these integrals not feasible
- but vacuum integrals are not independent, related by IBP-identities [Chetyrkin et al '81]
- perform reduction to master integrals using Laporta's algorithm
  needed: $4.5 \cdot 10^6$, calculated: $2 \cdot 10^8$, naive: 10^{13}
- in this case there are only 13 four-loop master integrals, all are known analytically



[Chetyrkin et al; Laporta; Kniehl et al; Schröder et al]

- Tools: `qgraf`[Nogueira], `(T)FORM`[Vermaseren], `Crusher`[PM, Seidel]

Third moment – Result

$$C_n^{(3)} = C_F T_F^2 n_l^2 C_{ll,n}^{(3)} + C_F T_F^2 n_h^2 C_{hh,n}^{(3)} + C_F T_F^2 n_l n_h C_{lh,n}^{(3)} + C_{n_f^0,n}^{(3)} + C_F T_F n_l \left(C_A C_{lNA,n}^{(3)} + C_F C_{lA,n}^{(3)} \right)$$

$$+ C_F T_F n_h \left(C_A C_{hNA,n}^{(3)} + C_F C_{hA,n}^{(3)} \right), \qquad \mu = m_q, \qquad c_4 = 24 a_4 + \log^4(2) - 6\zeta_2 \log^2(2), \qquad a_k = \mathrm{Li}_k(1/2)$$

$$C_{ll,3}^{(3),v} = \frac{31556642272}{49228003125} - \frac{256}{405}\zeta_3, \qquad C_{hh,3}^{(3),v} = \frac{56877138427}{12609717120} - \frac{6184964549}{1556755200}\zeta_3,$$

$$C_{lh,3}^{(3),v} = \frac{60361465477}{29393280000} - \frac{1765}{31104}c_4 + \frac{86485}{41472}\zeta_4 - \frac{57669161}{17418240}\zeta_3,$$

$$C_{lNA,3}^{(3),v} = -\frac{1475149211788337}{6452412825600000} - \frac{8529817}{77414400}c_4 + \frac{1510937903}{14745600}\zeta_4 - \frac{561258009401}{6193152000}\zeta_3,$$

$$C_{lA,3}^{(3),v} = \frac{983812946922223}{4389396480000} + \frac{8529817}{38707200}c_4 + \frac{21972351293}{17203200}\zeta_4 - \frac{28995540810097}{21676032000}\zeta_3,$$

$$C_{hNA,3}^{(3),v} = -\frac{454880458419083629}{585417045717504000} - \frac{7110196837}{1117670400}c_4 + \frac{1068488091383}{7451136000}\zeta_4 + \frac{4448}{315}\zeta_5 - \frac{438757401754772226 11}{433642256087040000}\zeta_3,$$

$$C_{hA,3}^{(3),v} = -\frac{2327115263308753}{2489610816000} - \frac{16870125343}{39916800}c_4 + \frac{286864384271}{26611200}\zeta_4 - \frac{377837317054807}{61471872000}\zeta_3$$

$$C_{n_f^0,3}^{(3),v} = \frac{8011001677156303009183663}{1270551061901721600000} - \frac{16091704629458603}{45731240755200}c_4 - \frac{1505000915688143609}{304874938368000}\zeta_4$$

$$- \frac{1781851011826}{310134825}\zeta_5 - \frac{859399602944}{310134825}a_5 - \frac{38830116184}{44304975}\log(2)\zeta_4 - \frac{214849900736}{930404475}\log^3(2)\zeta_2$$

$$+ \frac{107424950368}{4652022375}\log(2)^5 + \frac{1061162538194750079871}{128047474114560000}\zeta_3$$

What do we know about $\Pi(q^2)$

- at one and two loops known analytically [Kallen et al]

$$\Pi^{(0)}(z) = \frac{3}{16\pi^2}\left(\frac{20}{9} + \frac{4}{3z} - \frac{4(1-z)(1+2z)}{3z}G(z)\right),$$

$$\Pi^{(1)}(z) = \frac{3}{16\pi^2}\left[\frac{5}{6} + \frac{13}{6z} - (1-z)\frac{3+2z}{z}G(z) + (1-z)\frac{1-16z}{6z}G(z)^2 - \frac{1+2z}{6z}\left(1 + 2z(1-z)\frac{d}{dz}\right)\frac{I(z)}{z}\right]$$

$$G(z) = \frac{1}{2z}\frac{\log(u)}{\sqrt{1-\frac{1}{z}}}, \qquad u = \frac{\sqrt{1-\frac{1}{z}}-1}{\sqrt{1-\frac{1}{z}}+1}.$$

- at three loops full behavior reconstructed using Padé approximations [Chetyrkin et al]
- try Padé approximation at four loops [Hoang et al]

$\Pi(q^2)$ at four loops

Collect information of behavior in low-energy, threshold and high-energy region (example for vector current and $n_l = 3$)

- low-energy:

$$\Pi_3(z) = 6.95649z + 7.2478z^2 + 7.31855z^3 + \mathcal{O}(z^4)$$

$\Pi(q^2)$ at four loops

Collect information of behavior in low-energy, threshold and high-energy region (example for vector current and $n_l = 3$)

- low-energy: $\Pi_3(z) = \cdots + \mathcal{O}(z^4)$
- threshold:

$$\begin{aligned}\Pi_3(z) = {} & 2.63641/(1-z) \\ & + (-25.2331 - 7.75157\log(1-z))/\sqrt{1-z} \\ & - 11.0654\log(1-z) + 1.42833\log^2(1-z) \\ & - 0.421875\log^3(1-z) + K_0 + \mathcal{O}(\sqrt{1-z})\end{aligned}$$

$\Pi(q^2)$ at four loops

Collect information of behavior in low-energy, threshold and high-energy region (example for vector current and $n_l = 3$)

- low-energy: $\Pi_3(z) = \cdots + \mathcal{O}(z^4)$
- threshold: $\Pi_3(z) = \cdots + K_0 + \mathcal{O}(\sqrt{1-z})$
- high-energy:

$$\begin{aligned}\Pi_3(z) = {} & -6.172 - 0.070\log(-4z) \\ & + 0.121\log^2(-4z) - 0.037\log^3(-4z) \\ & + 1/z(-4.333 - 3.756\log(-4z) \\ & + 2.118\log^2(-4z) - 0.319\log^3(-4z)) \\ & + 1/z^2(D_2 - 5.130\log(-4z) + 0.318\log^2(-4z) \\ & + 0.401\log^3(-4z) - 0.079\log^4(-4z)) + \mathcal{O}(1/z^3)\end{aligned}$$

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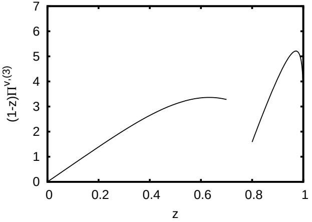

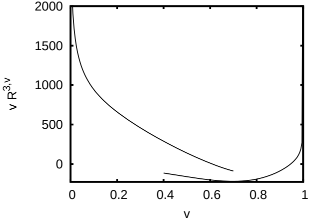

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Collect information of behavior in low-energy, threshold and high-energy region (example for vector current and $n_l = 3$)

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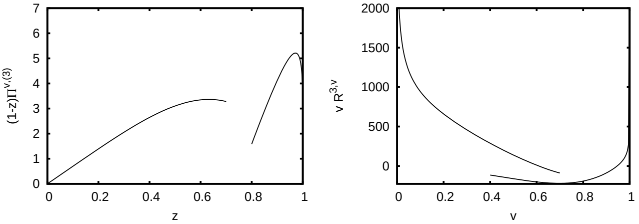


- Find approximating function and determine further terms in low-energy expansion and the missing constants K_0 and D_2.

Padé approximation I

- Next Step: Construct the Padé approximation
- But: $\Pi(z)$ logarithmically divergent for $z \to 1$ and $z \to \infty$, can not be approximated by rational function
- split $\Pi(z)$ in two parts

$$\Pi(z) = \Pi_{reg}(z) + \Pi_{log}(z),$$

where $\Pi_{log}(z)$ contains all logarithmic contributions

Padé approximation II – $\Pi_{log}(z)$

- How to construct $\Pi_{log}(z)$?
- Use one- and two-loop results as building blocks
  - threshold behavior

$$G(z) = \frac{\pi}{2}\frac{1}{\sqrt{1-z}} + \mathcal{O}(\sqrt{1-z})$$

$$\Pi^{(1)}(z) = -\frac{3}{16}\log(1-z) + const + \mathcal{O}(\sqrt{1-z})$$

  - high-energy behavior

$$G(z) = \frac{-\log(-4z)}{2z} + \mathcal{O}(z^{-2})$$

- Ansatz for $\Pi_{log}(z)$

$$\Pi_{log}(z) = \sum k_{ij}\Pi^{(1)}(z)^i G(z)^j + \sum d_{mn}(zG(z))^m(1-\frac{1}{z})^{\frac{m}{2}}\frac{1}{z^n}$$

threshold high-energy

Padé approximation III – $\Pi_{reg}(z)$

- $\Pi_{reg}(z) = \Pi(z) - \Pi_{log}(z)$ is free of logarithmic singularities
- perform a conformal mapping to the unit circle
  $z \to 4\omega/(1+\omega)^2$
- fit by Padé approximation of the form

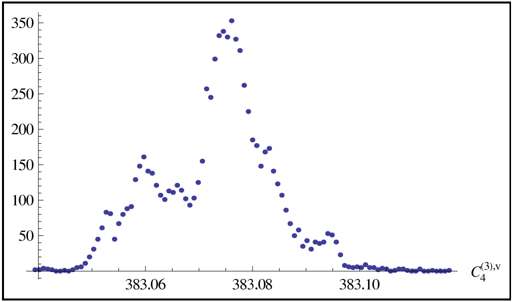
$$p_{nm}(\omega) = \frac{\sum_{i=0}^{n} a_i \omega^i}{1 + \sum_{i=1}^{m} b_i \omega^i}$$

  with $n + m + 1 = 9$ degrees of freedom
- error estimate: modify $\Pi_{log}(z)$, vary a_i and b_m

$$\Pi_{log}(z) = \sum k_{ij} \Pi^{(1)}(z)^i G(z)^j (a_i + \frac{1}{z}) + \sum d_{mn} (zG(z))^m (1 - \frac{1}{z})^{\frac{m}{2}} \frac{1}{z^n} (b_m + \frac{1}{z})$$

Padé approximation III – Quality of Fit

- Distribution of the $\mathcal{O}(8000)$ reconstructed values for the first missing low-energy constant C_4



- strongly peaked, very narrow distribution ($383.04 < C_4 < 383.12$)
- choose standard deviation as measure for error

Padé approximation IV – Results

	$n_l = 3$	$n_l = 4$	$n_l = 5$
$C_1^{(3),v}$	366.1748	308.0188	252.8399
$C_2^{(3),v}$	381.5091	330.5835	282.0129
$C_3^{(3),v}$	385.2331	338.7065	294.2224
$C_4^{(3),v}$	383.073(11)	339.913(10)	298.576(9)
$C_5^{(3),v}$	378.688(32)	338.233(32)	299.433(27)
$C_6^{(3),v}$	373.536(61)	335.320(63)	298.622(54)
$C_7^{(3),v}$	368.23(9)	331.90(10)	296.99(9)
$C_8^{(3),v}$	363.03(13)	328.33(14)	294.94(12)
$C_9^{(3),v}$	358.06(17)	324.78(18)	292.72(16)
$C_{10}^{(3),v}$	353.35(20)	321.31(22)	290.44(19)

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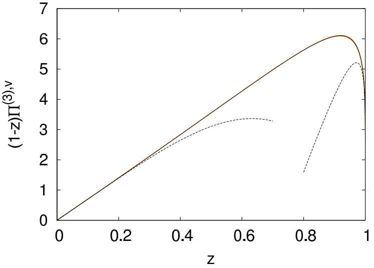
384.79 ± 0.57
[Hoang et al]

-10 ± 11
[Hoang et al]

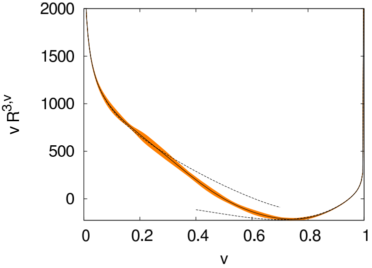
Padé approximation V – $R(s)$

- reconstruction of $\Pi(q^2)$ below and $R(s)$ above threshold
- red error band corresponds to three times the local standard deviation

$(1-z)\Pi(z)$
below threshold



$vR(v)$
above threshold



Conclusion

- calculated third low-energy moment at NNNLO
- reconstructed $\Pi(q^2)$ and therefore $R(s)$
- very precise prediction for the low-energy moments of $\Pi(q^2)$ up to $n = 10$
- no precise predictions for missing threshold and high-energy constants K_0 and D_2
- for the discussion of the impact on the determination of heavy quark masses see talk by J. Kühn