Higher-order predictions from physical evolution kernels

Andreas Vogt (University of Liverpool)

Collaborations with S. Moch (DESY), G. Soar (UoL), J. Vermaseren (NIKHEF)

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- **Hard lepton-hadron processes in perturbative QCD**
  (status of) splitting functions P_{ij} and coefficient functions $C_{a,i\{j\}}$

- **Non-singlet physical evolution kernels, $\ln(1-x)$ behaviour**
  $\Rightarrow$ highest (two or three) logarithms in $C_{a,i}$ to all orders in α_s

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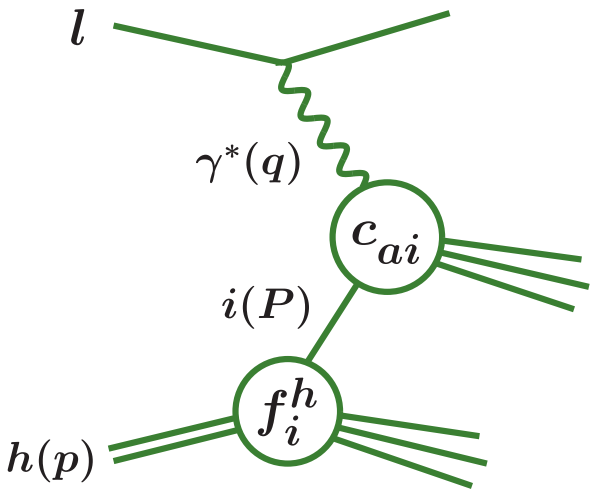
  $\Rightarrow$ highest (two or three) logarithms in $C_{a,i}$ to all orders in α_s
- **Scalar (ϕ) exchange DIS (Higgs for large m_{top}): $C_{\phi,i}$ to three loops**
- **Singlet physical kernels for the system (F_2, F_ϕ), large-x logs**
  $\Rightarrow$ leading three powers of $\ln(1-x)$ of P_{ij} at fourth order

MV, arXiv: 0902.2342 (JHEP), [0908.2746 (PLB)], 0909.2124; SMVV, to appear

Hard lepton-hadron processes in pQCD (I)

Inclusive deep-inelastic scattering (DIS), semi-incl. l^+l^- annihilation (SIA)



Left $\longrightarrow$ right: DIS, q spacelike, $Q^2 = -q^2$

$P = \xi p\,,\ f_i^h =$ **parton distributions**

Top $\longrightarrow$ bottom: l^+l^-, q timelike, $Q^2 = q^2$

$p = \xi P\,,$ **fragmentation distributions**

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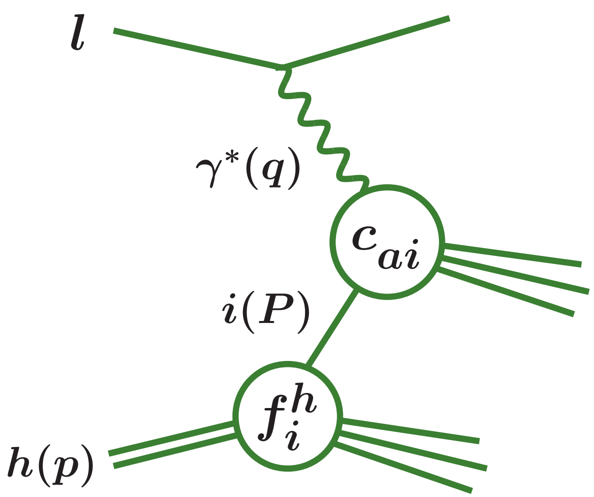

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$p = \xi P\,,$ fragmentation distributions

Drell-Yan (DY) l^+l^- production: bottom $\longrightarrow$ top, 2nd hadron from right ($\{\ldots\}$)

Structure functions/normalized cross sections F_a: coefficient functions

$$F_a(x,Q^2) \;=\; \Big[c_{a,i\{j\}}(\alpha_s(\mu^2),\mu^2/Q^2) \otimes f_i^h(\mu^2)\{\otimes f_j^{h'}(\mu^2)\}\Big](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables: $x = Q^2/(2p\cdot q)$ in DIS etc. μ: renorm./mass-fact. scale

Hard lepton-hadron processes in pQCD (II)

Parton / fragmentation distributions f_i : (renorm. group) evolution equations

$$\frac{d}{d\ln\mu^2} f_i(\xi,\mu^2) \;=\; \left[P_{ik}^{(S,T)}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

$\otimes$ **= Mellin convolution. Initial conditions incalculable in perturbative QCD**

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Expansion in α_s: splitting functions P, coefficient fct's c_a of observables

$$P \;=\; \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \ldots$$

$$c_a \;=\; \underbrace{\alpha_s^{n_a} \Big[c_a^{(0)} + \alpha_s c_a^{(1)}}\, + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \ldots \Big]$$

NLO: first real prediction of size of cross sections

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NLO: first real prediction of size of cross sections

NNLO, $P^{(2)}$, $c_a^{(2)}$: first serious error estimate of pQCD predictions

N^{3}LO: for high precision (α_s from DIS), slow convergence (Higgs in $pp/p\bar{p}$)

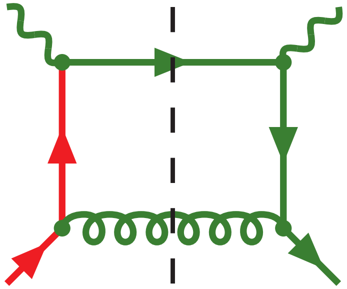
For our higher-order $\ln^k(1-x)$ predictions: at least two-loop results needed

Flavour singlet – non-singlet decomposition

Quark-quark splitting functions:

$$P_{q_i q_k} = P_{\bar{q}_i \bar{q}_k} = \delta_{ik} P_{qq}^{v} + P_{qq}^{s}$$

$$P_{q_i \bar{q}_k} = P_{\bar{q}_i q_k} = \delta_{ik} P_{q\bar{q}}^{v} + P_{q\bar{q}}^{s}$$



$$P_{qq}^{v} = \mathcal{O}(\alpha_s)$$

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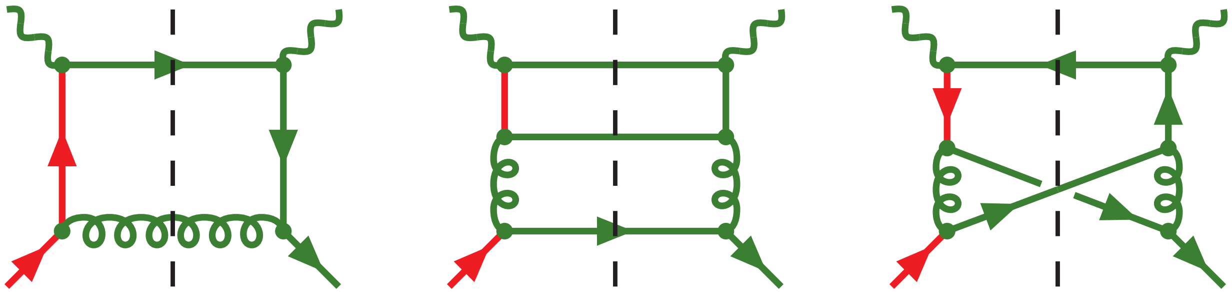

$P^{\mathrm{v}}_{\mathrm{qq}} = \mathcal{O}(\alpha_{\mathrm{s}})$ $\quad$ $P^{\mathrm{s}}_{\mathrm{qq}},\ P^{\mathrm{s}}_{\mathrm{q}\bar{\mathrm{q}}} : \alpha_{\mathrm{s}}^2$ $\quad$ $P^{\mathrm{v}}_{\mathrm{q}\bar{\mathrm{q}}} : \alpha_{\mathrm{s}}^2$

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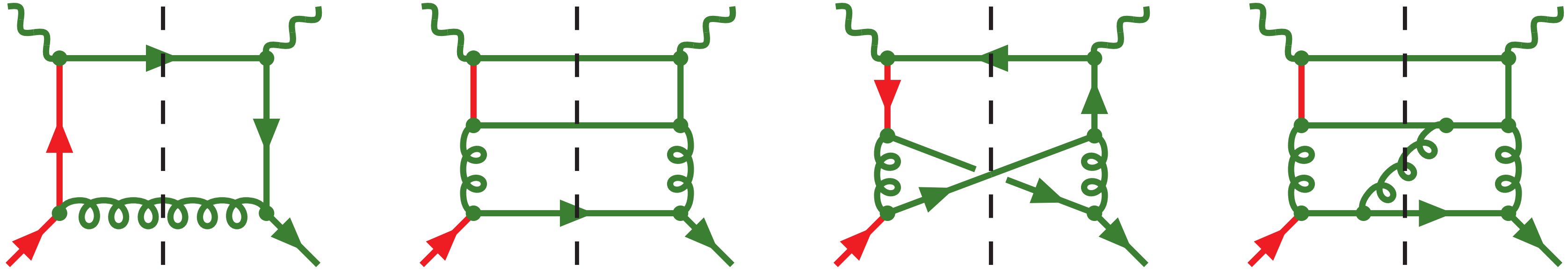

$P^{v}_{qq} = \mathcal{O}(\alpha_s)$ $\qquad$ $P^{s}_{qq},\, P^{s}_{q\bar{q}} \,:\, \alpha_s^2$ $\qquad$ $P^{v}_{q\bar{q}} \,:\, \alpha_s^2$ $\qquad$ $P^{s}_{q\bar{q}} \neq P^{s}_{qq} \,:\, \alpha_s^3$

Three types of difference (non-singlet) combinations: $P^{\pm}_{ns} = P^{v}_{qq} \pm P^{v}_{q\bar{q}}\,,\, P^{v}_{ns}$

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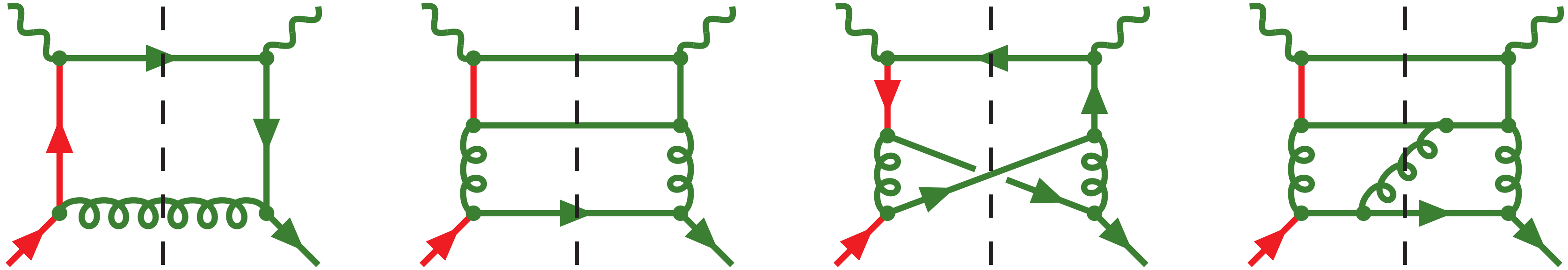

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Evolution of gluon and flavour-singlet quark distributions g and q_s

$$q_s = \sum_{r=1}^{n_f} (q_r + \bar q_r)\;, \quad \frac{d}{d\ln\mu^2}\begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix}$$

with (ps = ‘pure singlet’) $\quad P_{qq} = P^{+}_{\rm ns} + n_f\,(P^{\rm s}_{qq} + P^{\rm s}_{\bar q q}) \equiv P^{+}_{\rm ns} + P_{\rm ps}$

Quark coefficient fct’s: analogous decomposition $C_{a,q\{\bar q\}} = C_{a,\rm ns} + C_{a,\rm ps}$

Status of splitting and coefficient functions

NNLO splitting functions – no change since RADCOR 07

Spacelike $P_{ij}^{(2)}$ $\surd$, timelike $P_{\rm qq,gg}^{(2)}$ $\surd$, $P_{\rm qg,gq}^{(2)}$: only $N = 2$ (ζ_2 problem . . .)

MVV (04), MV (05,07)

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MVV (04), MV (05,07)

All-x higher-order coefficient functions – miminal progress

- DIS: 2-loop and 3-loop complete

  Zijlstra, van Neerven (91/92); Moch, Vermaseren (99); MVV (05,08)

- SIA: 2-loop complete and highest $(1-x)$ logs at 3-loop (new)

  Rijken, van Neerven (96); Mitov, Moch, A.V. (06); MV (09)

- DY: 2-loop complete

  Hamberg, van Neerven, Matsuura (91); Harlander, Kilgore (02)

- DIS, leading large-n_f to all orders

  Gracey (95); Mankiewicz, Maul, Stein (97)

Large-x logs in splitting and coefficient fct's

NS splitting functions (and $P_{\rm gg}$ for $C_F = 0$): $(1-x)$-suppressed single logs

$$P_{\rm ns}^{(l)}(x) \;=\; \frac{A_{l+1}}{(1-x)_+} + \widetilde{B}_{l+1}\,\delta(1-x) + \widetilde{C}_{l+1}\ln(1-x) + \mathcal{O}\Big((1-x)^{k\,\geq 1}\ln^{l}(1-x)\Big)$$

Korchemsky (89); MVV (04); Dokshitzer, Marchesini, Salam (05), . . .

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Singlet splitting functions: double logs at all powers of $(1-x)$

$P^{(l)}_{\rm ps,gg}/P^{(l)}_{\rm qg,gq}$: terms up to $(1-x)\ln^{2l-1}(1-x)/\ln^{2l}(1-x)$

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NS coefficient functions for $F_{1,2,3}$ in DIS, $F_{T,I,A}$ in SIA, $F_{\rm DY} = \frac{1}{\sigma_0}\,d\sigma_{\rm ns}/dQ^2$

$C_{a,\rm ns}^{(l)}$: terms up to $(1-x)^{-1}\ln^{2l-1}(1-x)$

$(1-x)^{-1}$ parts: threshold exponentiation   Sterman (87); Catani, Trentadue (89); . . .

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Now known to next-to-next-to-next-to-leading log (N^3LL) accuracy (mod. A_4)

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05)
(+ more papers, esp. using SCET, from 2006), SIA: Blümlein, Ravindran (06); MV (09)

Non-singlet physical evolution kernels

Switch to Mellin moments wherever useful (convolutions $\otimes \longrightarrow$ products)

Manipulations of harmonic sums / polylogarithms, (inverse) Mellin transform

FORM 3 + packages: Vermaseren (00); TFORM: Tentyukov, Vermaseren (07)

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$$\frac{dF_a}{d\ln Q^2} = \frac{d\,C_a}{d\ln Q^2}\,q + C_a P\,q = \Big(\beta(a_s)\,\frac{d\,C_a}{da_s} + C_a P\Big) C_a^{-1} F_a$$
$$= \Big(P_a + \beta(a_s)\,\frac{d\ln C_a}{da_s}\Big) F_a = K_a F_a \equiv \sum_{l=0} a_s^{l+1} K_{a,l}\,F_a$$

K_a: physical kernel for the non-singlet observable F_a at $\mu = Q$.

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K_a: physical kernel for the non-singlet observable F_a at $\mu = Q$. Expansion

$$K_a = a_s P_{a,0} + \sum_{l=1} a_s^{\,l+1}\Big(P_{a,l} - \sum_{k=0}^{l-1} \beta_k\,\tilde{c}_{a,l-k}\Big)$$

with

$$\begin{aligned}
\tilde{c}_{a,1} &= c_{a,1} \\
\tilde{c}_{a,2} &= 2\,c_{a,2} - c_{a,1}^2 \\
\tilde{c}_{a,3} &= 3\,c_{a,3} - 3\,c_{a,2}\,c_{a,1} + c_{a,1}^3 \\
\tilde{c}_{a,4} &= 4\,c_{a,4} - 4\,c_{a,3}\,c_{a,1} - 2\,c_{a,2}^2 + 4\,c_{a,2}\,c_{a,1}^2 - c_{a,1}^4 \;\;, \;\; \ldots
\end{aligned}$$

Large-x logarithms in the physical kernels

Soft limit $1-x \ll 1$ $\Leftrightarrow$ large $L \equiv \ln N$: threshold exponentiation

$$C_a(N) \; = \; g_0 \exp\{Lg_1(a_s L) + g_2(a_s L) + \ldots\} \; + \; \mathcal{O}(1/N)$$

$\Rightarrow$ **single-logarithmic (SL) enhancement of physical evolution kernels K_a**

$$K_a(N) \; = \; -\sum_{l=1} A_l \, a_s^l \, L \; + \; \beta(a_s) \, \frac{d}{da_s} \{Lg_1(a_s L) + g_2(a_s L) + \ldots\} \; + \; \ldots$$

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Crucial observation: all K_a singly enhanced to all orders in N^{-1} or $(1-x)$

DIS/SIA leading terms, with $p_{qq}(x) = (1-x)_+^{-1} - 1 - x$ and HPLs H_0 etc

$$\begin{aligned}
K_{a,0}(x) &= 2\,C_F p_{qq}(x) + 3\,C_F \delta(1-x) \\
K_{a,1}(x) &= \ln(1-x)\, p_{qq}(x) \left[-2\,C_F \beta_0 \mp 8\,C_F^2\, H_0(x)\right] \\
K_{a,2}(x) &= \ln^2(1-x)\, p_{qq}(x) \left[\, 2\,C_F \beta_0^2 \pm 12\,C_F^2\, \beta_0\, H_0(x) + 32\,C_F^3\, H_{0,0}(x)\right] \\
K_{a,3}(x) &= \ln^3(1-x)\, p_{qq}(x) \left[-2\,C_F \beta_0^3 \mp 44/3\,C_F^2\, \beta_0^2\, H_0(x) \right. \\
&\qquad \left. -64\,C_F^3\, \beta_0\, H_{0,0}(x) + \xi_{P_3} C_F^4\, H_{0,0,0}(x)\right]
\end{aligned}$$

First term: leading large n_f, all orders via C_2 of Mankiewicz, Maul, Stein (97)

Higher-order non-singlet predictions

Conjecture: Single-log behaviour of K_a persists to (all) higher orders in α_s

$\Leftrightarrow$ **exponentiation of the coefficient functions beyond soft $(1-x)^{-1}$ terms**

Work towards a deductive proof: Laenen, Magnea, Stavenga, White (08/09)

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Recall
$$\underbrace{\tilde{c}_{a,4}}_{\text{SL}} = \underbrace{4\, c_{a,4}}_{\text{DL, new}} \underbrace{-\, 4\, c_{a,3}\, c_{a,1} - 2\, c_{a,2}^{\,2} + 4\, c_{a,2}\, c_{a,1}^{\,2} - c_{a,1}^{\,4}}_{\text{DL, known for DIS/SIA}}$$

$\Rightarrow$ coefficients of highest three powers of $\ln(1-x)$ from fourth order in α_s,

i.e., $\ln^{7,6,5}(1-x)$ at order α_s^4 for $F_{1,2,3}$ in DIS and $F_{T,I,A}$ in SIA

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Leading terms: $K_1 = K_2$, $K_{\mathrm{T}} = K_{\mathrm{I}}$ [total (‘integrated’) fragmentation fct.]

$\Rightarrow$ **also three logarithms for space- and timelike F_L: $\ln^{6,5,4}(1-x)$ at α_s^4 etc**

Alternative derivation: physical kernels for F_L, agreement non-trivial check

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Drell-Yan: only NNLO known $\Rightarrow$ only two logarithms fully predicted from α_s^3

Example: α_s^4 coefficient function for F_1 in DIS

$$
\begin{aligned}
c^{(4)}_{1,\mathrm{ns}}(x) \;=\; & \big(\ln^7(1-x)\; 8/3\, C_F^4 - \ln^6(1-x)\; 14/3\, C_F^3\, \beta_0 + \ln^5(1-x)\; 8/3\, C_F^2\, \beta_0^2 \big)\, p_{\mathrm{qq}}(x) \\
& + \ln^6(1-x)\; \big[C_F^4 \,\{ p_{\mathrm{qq}}(x)\,(-14 - 68/3\,\mathrm{H}_0) + 4 + 8\,\mathrm{H}_0 - (1-x)(6 + 4\,\mathrm{H}_0)\} \big] \\
& + \ln^5(1-x)\; \Big[C_F^4 \Big\{ p_{\mathrm{qq}}(x)\,(-9 - 8\,\widetilde{\mathrm{H}}_{1,0} + 448/3\,\mathrm{H}_{0,0} + 84\,\mathrm{H}_0 - 64\,\zeta_2) + 48\,\widetilde{\mathrm{H}}_{1,0} \\
& \qquad -22 - 96\,\mathrm{H}_{0,0} - 104\,\mathrm{H}_0 - (1-x)(13 + 24\,\widetilde{\mathrm{H}}_{1,0} - 48\,\mathrm{H}_{0,0} - 84\,\mathrm{H}_0 - 16\,\zeta_2) \Big\} \\
& \quad + C_F^3\,\beta_0\,\{ p_{\mathrm{qq}}(x)\,(41 + 316/9\,\mathrm{H}_0) - 10 - 32/3\,\mathrm{H}_0 + (1-x)(41/3 + 16/3\,\mathrm{H}_0)\} \\
& \quad + C_F^3\,C_A \Big\{ p_{\mathrm{qq}}(x)\,(16 + 8\,\widetilde{\mathrm{H}}_{1,0} + 8\,\mathrm{H}_{0,0} - 24\,\zeta_2) + 4 + (1-x)(28 - 8\,\zeta_2) \Big\} \\
& \quad + C_F^3\,(C_A - 2\,C_F)\, p_{\mathrm{qq}}(-x)\,(16\,\widetilde{\mathrm{H}}_{-1,0} - 8\,\mathrm{H}_{0,0}) \Big] \;+\; \mathcal{O}\big(\ln^4(1-x)\big)
\end{aligned}
$$

Example: α_s^4 coefficient function for F_1 in DIS

$$
\begin{aligned}
c^{(4)}_{1,\rm ns}(x) \;=\; & \Big(\ln^7(1-x)\, 8/3\, C_F^4 - \ln^6(1-x)\, 14/3\, C_F^3\, \beta_0 + \ln^5(1-x)\, 8/3\, C_F^2\, \beta_0^2\Big)\, p_{\rm qq}(x) \\
& + \ln^6(1-x)\;\Big[C_F^4\, \{\, p_{\rm qq}(x)\, (-14 - 68/3\, {\rm H}_0) + 4 + 8\, {\rm H}_0 - (1-x)(6 + 4\, {\rm H}_0)\}\Big] \\
& + \ln^5(1-x)\;\Big[C_F^4 \Big\{p_{\rm qq}(x)\, (-9 - 8\, \widetilde{\rm H}_{1,0} + 448/3\, {\rm H}_{0,0} + 84\, {\rm H}_0 - 64\, \zeta_2) + 48\, \widetilde{\rm H}_{1,0} \\
& \quad -22 - 96\, {\rm H}_{0,0} - 104\, {\rm H}_0 - (1-x)(13 + 24\, \widetilde{\rm H}_{1,0} - 48\, {\rm H}_{0,0} - 84\, {\rm H}_0 - 16\, \zeta_2)\Big\} \\
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\end{aligned}
$$

First line includes identity of coefficients of leading $\ln^k(1-x)$ and $\dfrac{\ln^k(1-x)}{x-1}$ terms

Conjectured by Krämer, Laenen, Spira (97)

Modified basis $\widetilde{\rm H}_{\rm m_1,m_2,\ldots} \equiv \widetilde{\rm H}_{\rm m_1,m_2,\ldots}(x)$ of harmonic polylogarithms, e.g.,

$$
\widetilde{\rm H}_{1,0} \;=\; {\rm H}_{1,0} + \zeta_2 \;, \quad \widetilde{\rm H}_{1,1,0} \;=\; {\rm H}_{1,1,0} - \zeta_2 \,\ln(1-x) - \zeta_3
$$

All $\ln(1-x)$ terms and ζ-functions taken out of expansions to all orders in $1-x$

All-order exponentiation of the $1/N$ terms (I)

For $F_{1,2,3}$, $F_{\mathrm{T,I,A}}$ and F_{DY}, up to terms of order $1/N^2$, with $L \equiv \ln N$

$$C_a(N) - C_a\Big|_{N^0 L^k} \;=\; \frac{1}{N}\left(\left[d^{(1)}_{a,1} L + d^{(1)}_{a,0}\right] a_{\mathrm{s}} + \left[\widetilde{d}^{(2)}_{a,1}\, L + d^{(2)}_{a,0}\right] a_{\mathrm{s}}^{2} + \ldots\right)$$

$$\exp\left\{L h_1(a_{\mathrm{s}} L) + h_2(a_{\mathrm{s}} L) + a_{\mathrm{s}} h_3(a_{\mathrm{s}} L) + \ldots\right\}$$

Exponentiation functions defined by expansions $h_k(a_{\mathrm{s}} L) \;\equiv\; \sum_{k=1} h_{kn}\,(a_{\mathrm{s}} L)^n$

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Coefficients for DIS/SIA (upper/lower sign) relative to $N^0 L^k$ exponentiation

$$\begin{aligned}
h_{1k} &= g_{1k} \\
h_{21} &= g_{21} + \frac{1}{2}\beta_0 \pm 6\, C_F \\
h_{22} &= g_{22} + \frac{5}{24}\beta_0^2 \pm \frac{17}{9}\beta_0\, C_F - 18\, C_F^2 \\
h_{23} &= g_{23} + \frac{1}{8}\beta_0^3 \pm \left(\frac{\xi_{\mathrm{K}_4}}{8} - \frac{53}{18}\right)\beta_0^2 C_F - \frac{34}{3}\beta_0\, C_F^2 \pm 72\, C_F^3
\end{aligned}$$

ξ_{K_4}: next-to-leading large-n_f coefficient at fourth order – should be feasible

First term of h_3 also known, but non-universal within DIS and SIA ($\Leftrightarrow F_L$)

All-order exponentiation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure/fragmentation functions F_L

$$C_L^{(\pm)}(N) \;=\; N^{-1}(d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \ldots)\,\exp\{L h_1(a_s L) + h_2(a_s L) + \ldots\}$$

with

$$h_{11} \;=\; 2\,C_F \;,\quad h_{12} \;=\; \frac{2}{3}\,\beta_0\,C_F \;,\quad h_{13} \;=\; \frac{1}{3}\,\beta_0^2\,C_F$$

$$h_{21} \;=\; \beta_0 \,+\, 4\,\gamma_e C_F \,-\, C_F \,+\, (4 - 4\,\zeta_2)(C_A - 2C_F)$$

$$h_{22} \;=\; \underbrace{\frac{1}{2}\,(\,\beta_0\,h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} \;-\; \underbrace{8\,(C_A - 2C_F)^2(1 - 3\,\zeta_2 + \zeta_3 + \zeta_2^2\,)}_{\text{Who ordered THIS?}}$$

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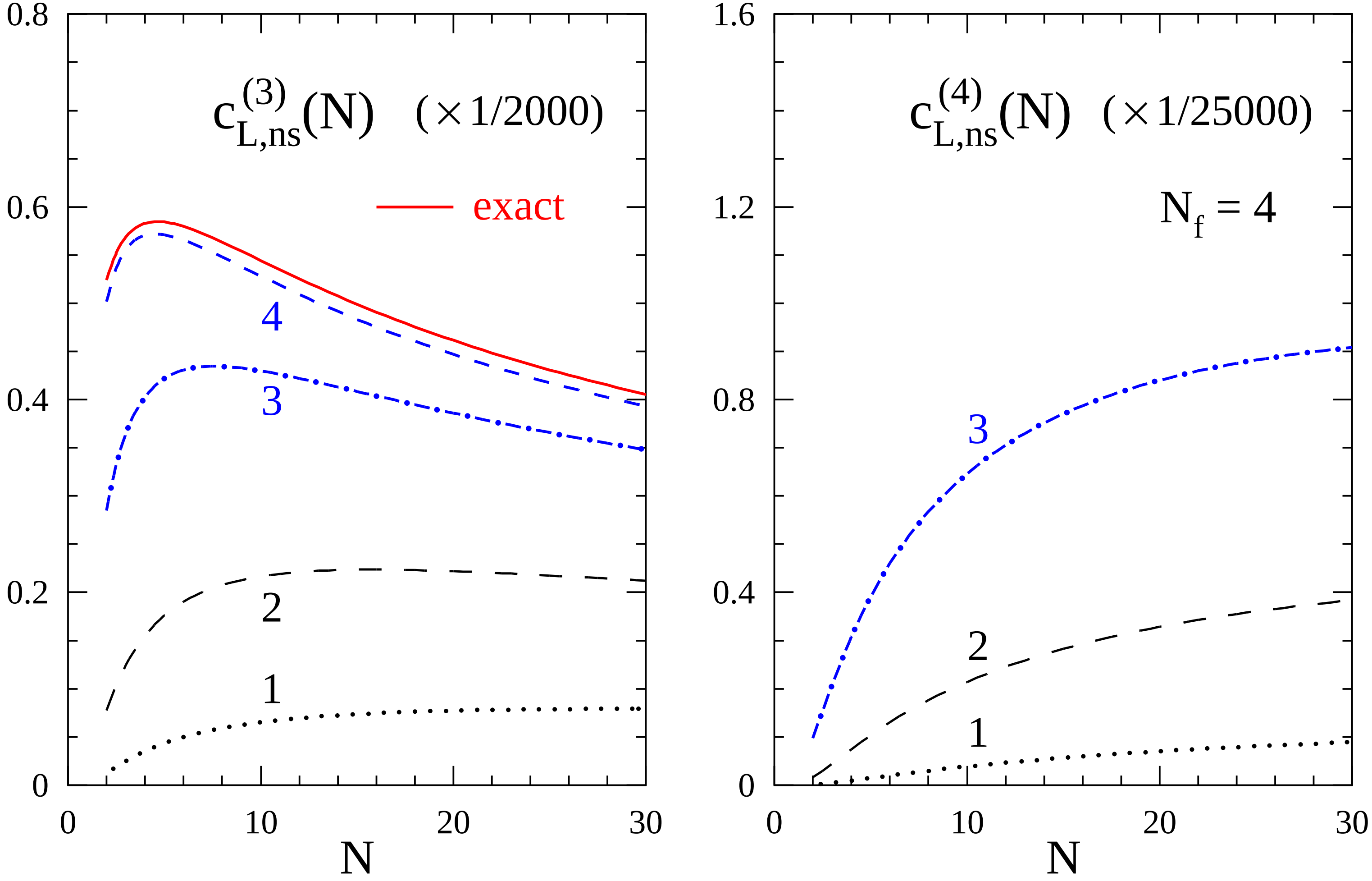
Remarks/questions

- **Less predictive than resum. of $N^0 L^k$ terms: nothing new, but A_2, in g_{22}**
- **Full NLL accuracy – complete $g_2(a_s L)$ – should be feasible for $F_{1,2,3}$ etc**
- **Full NNLL for $F_{1,2,3}$ etc, NLL for F_L: a log too far? h_{23} for F_L, anyone?**

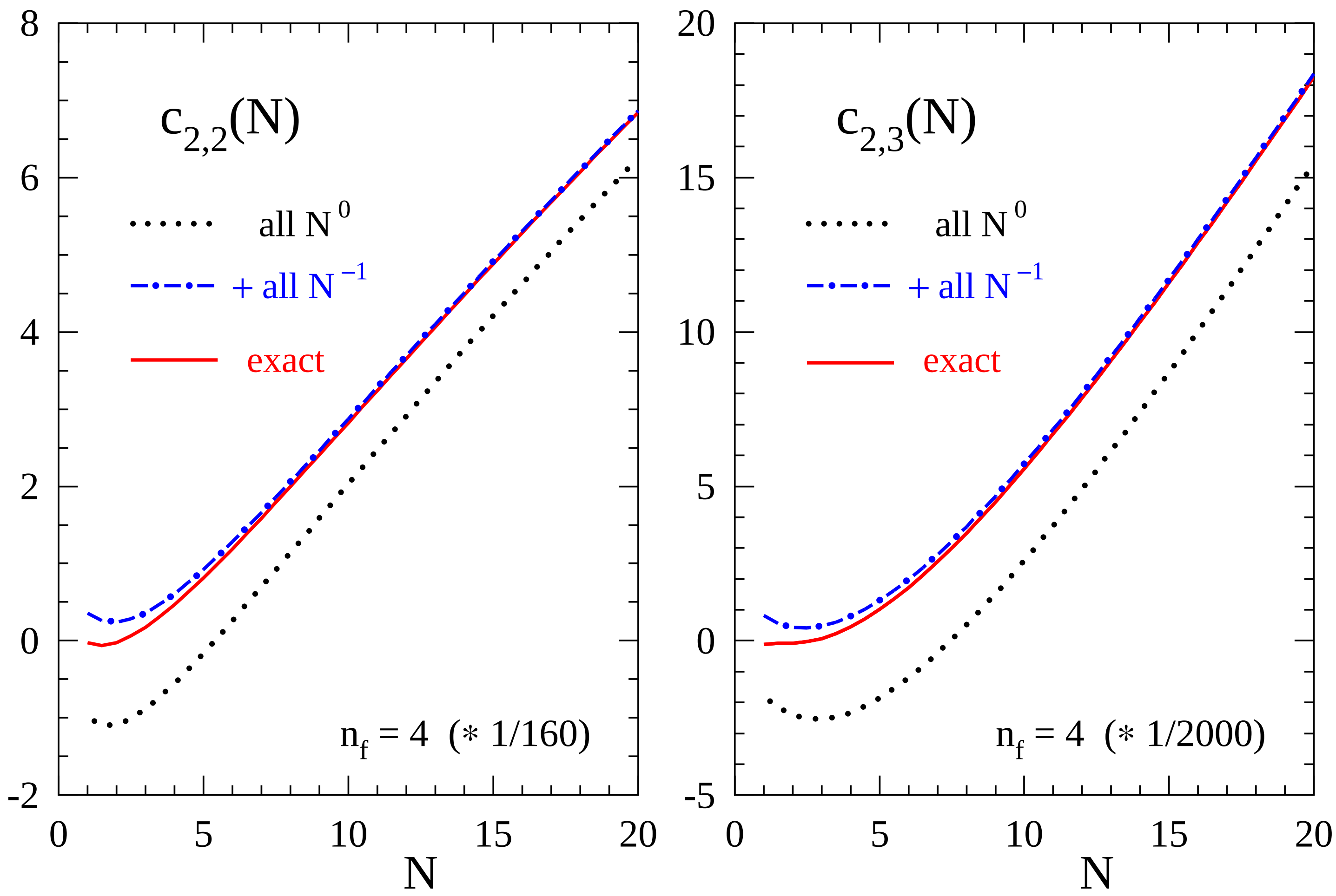
Third- and fourth-order C_L in DIS in N-space



1 = leading log etc. Good α_s^3 approximation by all four N^{-1} logarithms

As usual, cf. small-x: leading logs do not lead. Padé: ≈ 2.0 at $N = 20$

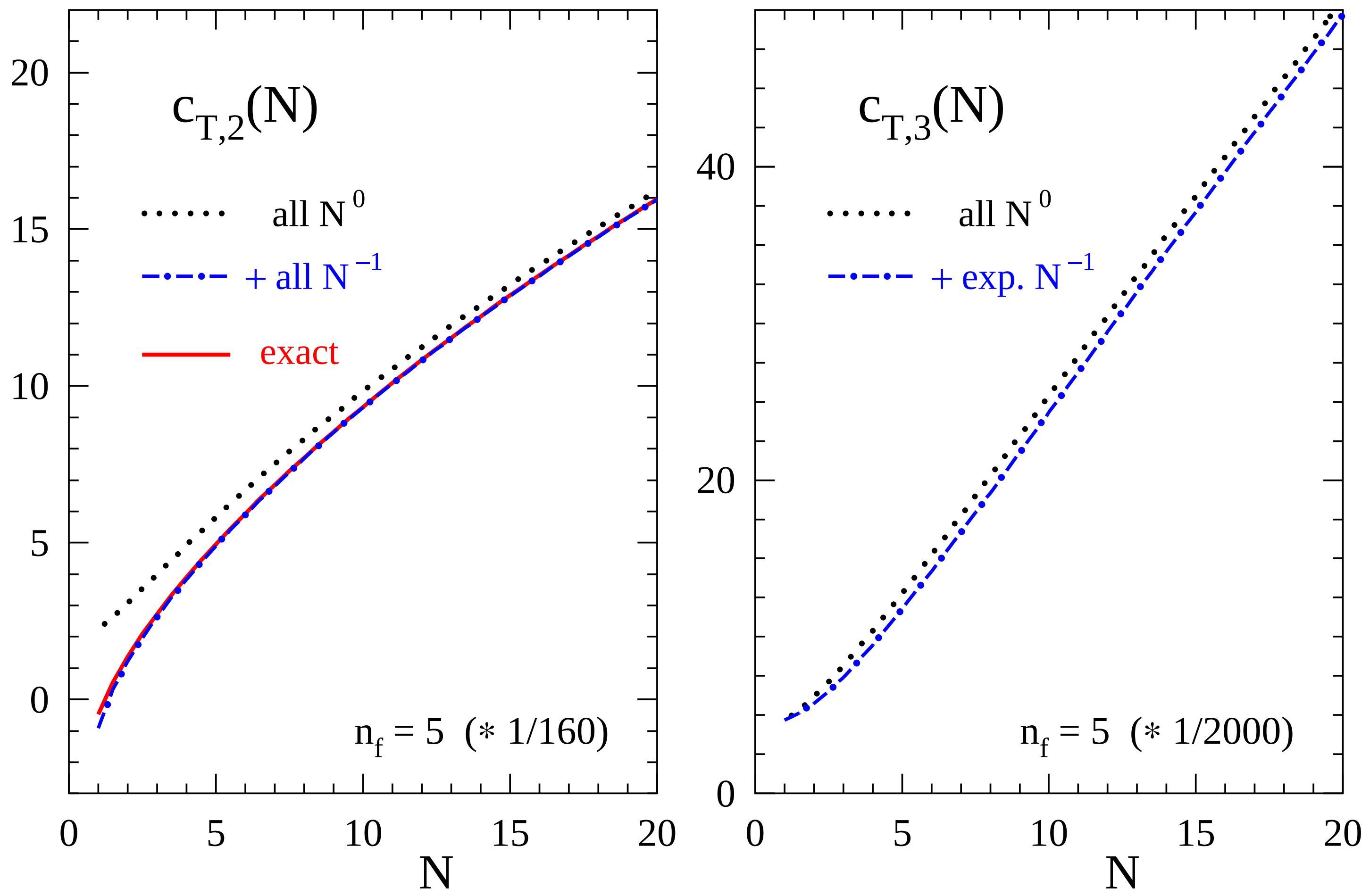
Second- and third-order C_2 in DIS in N-space



N^{-1} terms relevant over full range shown, $\mathcal{O}(N^{-2})$ sizeable only at $N < 5$

Sum of $N^{-1}\ln^k N$ looks almost constant: half of maximum only at $N \simeq 150$

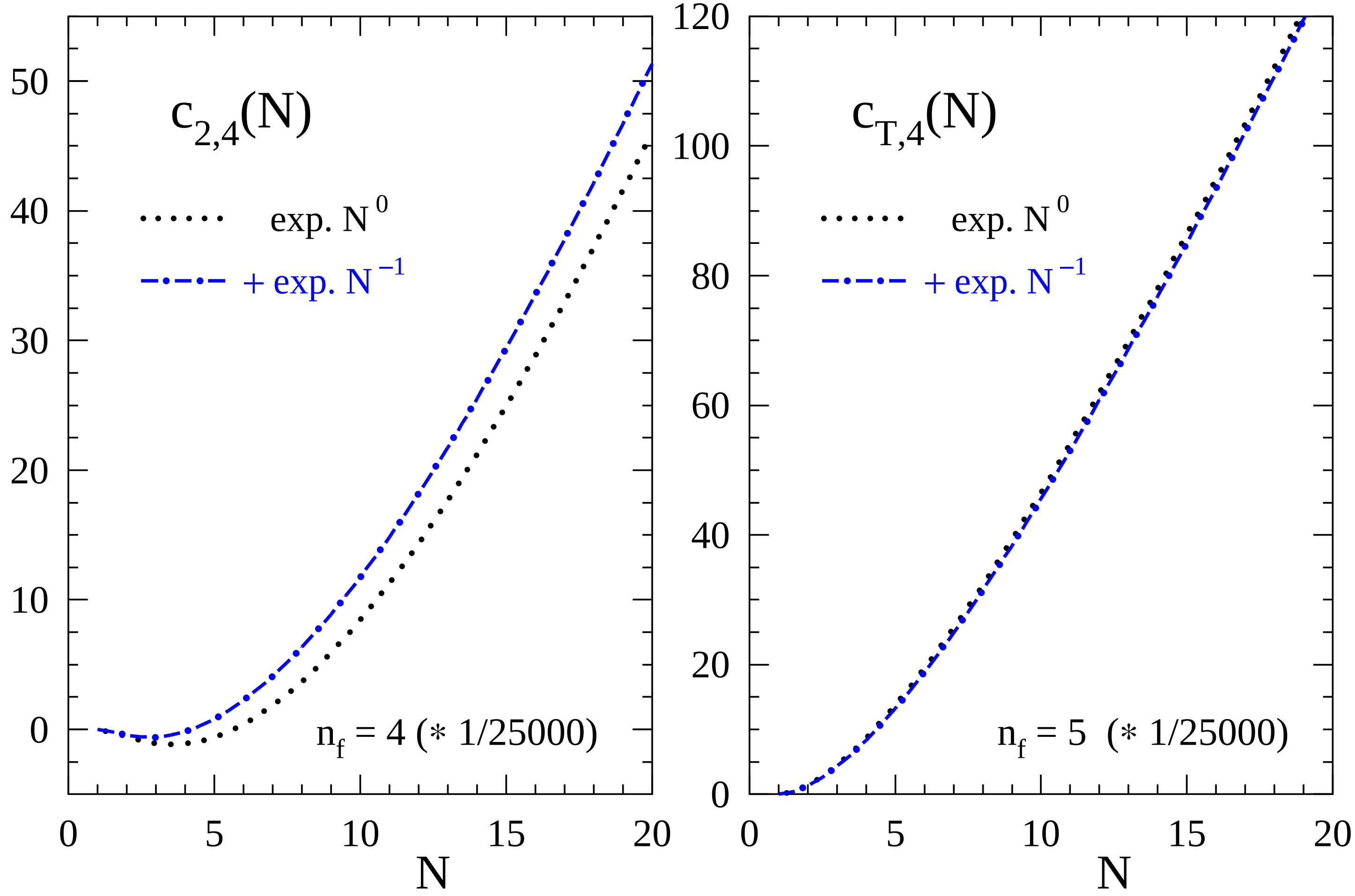
Second- and third-order C_T in SIA in N-space



Larger soft-gluon enhancement, mainly from low powers of $\ln N$ (neg. in DIS)

Compensation between $N^{-1}\ln^k N$ terms (unknown for $k = 0, 1$ at order α_s^3)

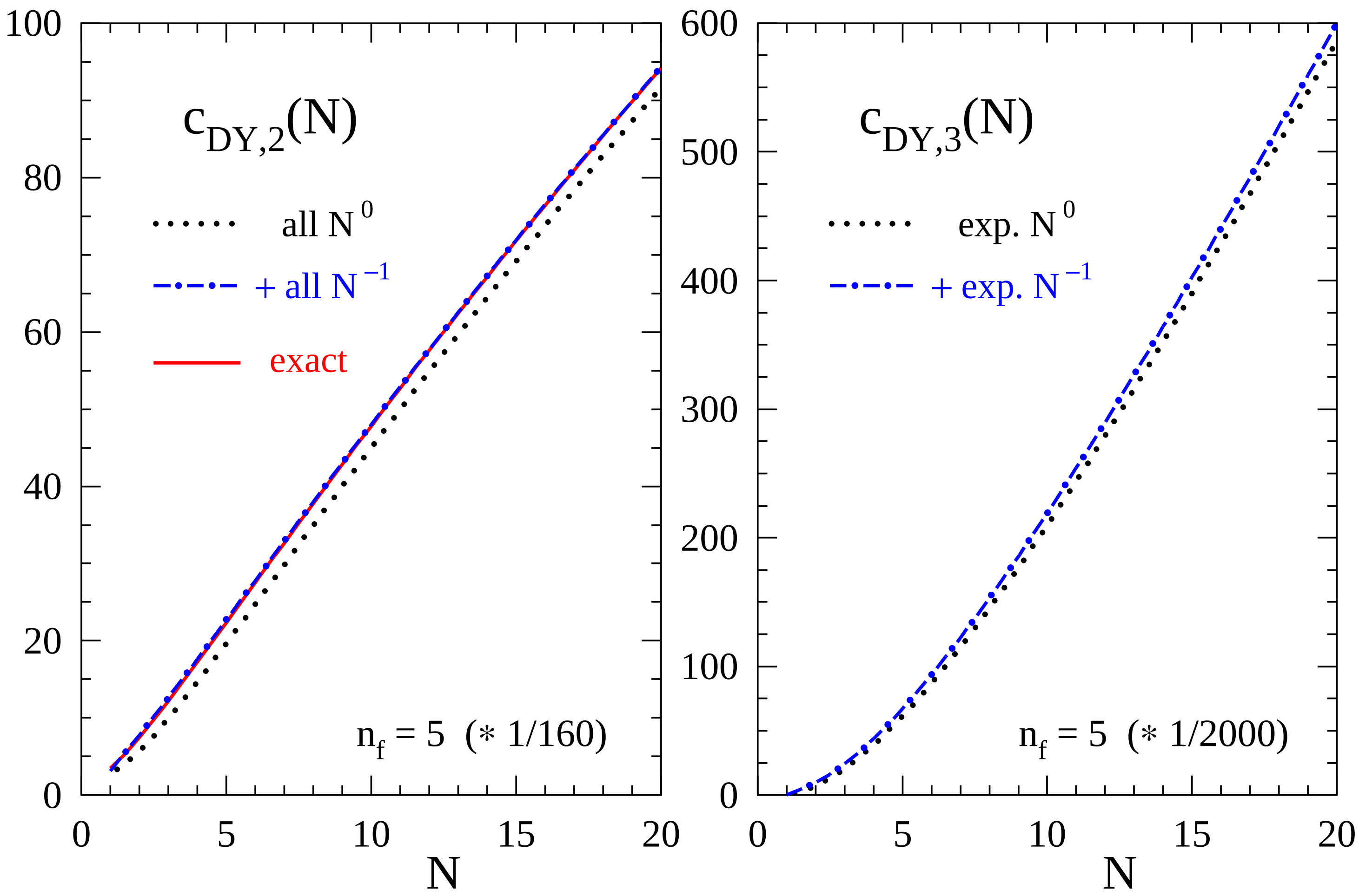
Fourth-order C_2 (DIS) and C_T (SIA) at large N



Exp. N^0: 7 of 8 logs, exp. N^{-1}: 4 of 7 logs – ξ_{K_4} numerically suppressed

N^{-1} contributions again relevant for F_2, but small for F_T at least at $N > 5$

Second- and third-order C_{DY} in N-space

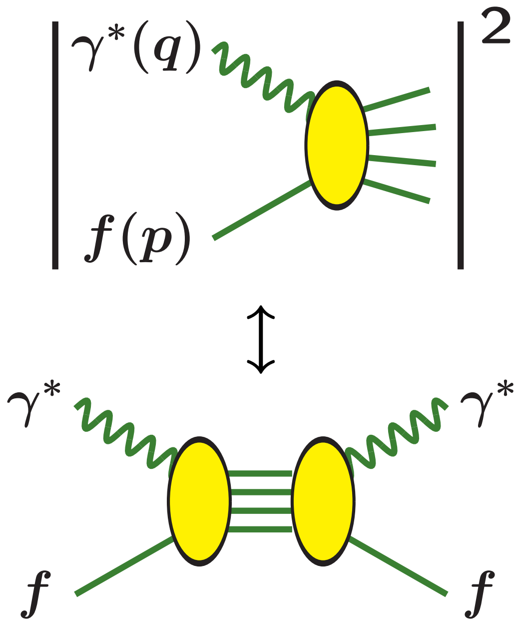


Exp. N^0: all logs, exp. N^{-1}: 3 of 5 logs – ξ_{DY_3} numerically insignificant

N^{-1} contributions small down to even lower moments than in the SIA case

Three-loop calculation of structure functions

Optical theorem: $(\gamma^*, W, H)\, f$ **total cross sections** $\longleftrightarrow$ **forward amplitudes**



	tree	1-loop	2-loop	3-loop
qγ	1	3	25	359
gγ		2	17	345
			2	56
qW	1	3	32	589
qH		1	23	696
gH	1	8	218	6378
		1	33	1184
sum	3	18	350	9607

Coefficient of $(2p\cdot q)^N$ $\longleftrightarrow$ N**-th moment** $A^N = \int_0^1 dx\, x^{N-1} A(x)$

New: some $\alpha_s^3 n_f$ parts independently rederived **Bierenbaum, Blümlein, Klein (09)**

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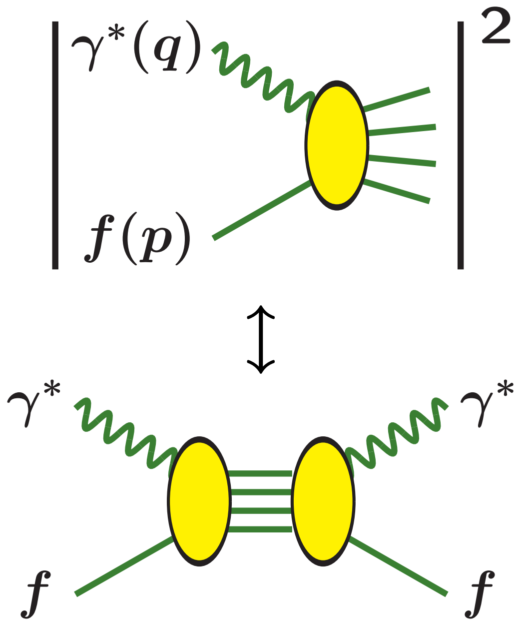

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P_{gg}, P_{gq}: **DIS by Higgs exchange in heavy-top limit** ($\phi G^a_{\mu\nu} G_a^{\mu\nu}$ **coupling)**

Calculation to $\alpha_s^3 \varepsilon^0$ **(dim. reg.,** $D = 4 - 2\,\varepsilon$**):** **coefficient functions** $c^{(1,2,3)}_{\phi,i}$

Properties of the coefficient functions $C_{\phi,i}$

To order α_s^2 also derived, independently with an entirely different method by

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09, to appear)

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- Second moments, for $n_f = 4$ light flavours

$$C_{\phi,g}(N=2) = 1 + 2.3387\,\alpha_s + 3.2956\,\alpha_s^2 + 5.3704\,\alpha_s^3$$
$$C_{\phi,q}(N=2) = -0.1415\,\alpha_s - 0.7378\,\alpha_s^2 - 2.6791\,\alpha_s^3$$

Electromagnetic DIS, for comparison

$$C_{2,q}(N=2) = 1 + 0.0354\,\alpha_s - 0.0785\,\alpha_s^2 - 0.1986\,\alpha_s^3$$
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- Large-x: usual double-logarithmic enhancements
- Small-x: double logarithmic up to $\alpha_s^n \frac{1}{x} \ln^{2n-1} x$ in both $C_{\phi,g}$ and $C_{\phi,q}$

  Catani, Ciafaloni, Hautman (91); Hautman (02)

  Artefact of using the heavy-top effective vertex in the high-energy limit

  Marzani, Ball, Del Duca, Forte, Vicini (08)

Singlet physical evolution kernel for (F_2, F_ϕ)

As in the non-singlet case above, but with 2-vectors$/2\times 2$ matrices P_{ij} and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,q} & C_{2,g} \\ C_{\phi,q} & C_{\phi,g} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}$$

Furmanski, Petronzio (81) – together with other options such as (F_2, F_L) studied by Blümlein et al. (00)

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$$\frac{dF}{d\ln Q^2} = \frac{dC}{d\ln Q^2}\, q + CP\, q = \left(\beta(a_s)\,\frac{dC}{da_s} + CP\right) C^{-1} F$$

$$= \Bigg(\underbrace{\beta(a_s)\,\frac{d\ln C}{da_s}}_{\text{DL (ns + ps)}} + \underbrace{[C, P]\, C^{-1} + P}_{\text{DL (singlet only)}}\Bigg) F = KF$$

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Observation at NLO, NNLO: single logarithmic enhancement

$$K_{ab}^{(n)} \sim \ln^n(1-x) + \ldots\,, \quad \text{leading } K_{22/\phi\phi}^{(n)} \text{ same as NS}/C_F = 0$$

Conjecture: this behaviour persists to N^{3}LO

$\Rightarrow$ **prediction of $\ln^{6,5,4}(1-x)$ of $P^{(3)}_{\rm qg,gq}$ and $\ln^{5,4,3}(1-x)$ of $P^{(3)}_{\rm ps,gg|C_F}$**

Example: α_s^4 splitting function $P_{qg}^{(3)}(x)$

For brevity: only $(1-x)^0$ part shown – known to all powers, $C_{AF} \equiv C_A - C_F$

$$
\begin{aligned}
P_{qg}^{(3)}(x) \quad = \quad & \ln^6(1-x) \cdot 0 \\
& + \ln^5(1-x)\left[\frac{22}{27} C_{AF}^3 n_f - \frac{14}{27} C_{AF}^2 C_F n_f + \frac{4}{27} C_{AF}^2 n_f^2\right] \\
& + \ln^4(1-x)\left[\left(\frac{293}{27} - \frac{80}{9}\zeta_2\right) C_{AF}^3 n_f + \left(\frac{4477}{16} - 8\zeta_2\right) C_{AF}^2 C_F n_f \right. \\
& \quad \left. - \frac{13}{81} C_{AF} C_F^2 n_f - \frac{116}{81} C_{AF}^2 n_f^2 + \frac{17}{81} C_{AF} C_F n_f^2 - \frac{4}{81} C_{AF} n_f^3\right] \\
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&+ \mathcal{O}\left(\ln^3(1-x)\right)
\end{aligned}
$$

- **Vanishing of the coefficient of the leading term (only) at order α_s^4:**
  accidental cancellation of contributions, for all four splitting functions
- **Remaining terms vanish in the supersymmetric case $C_A = C_F = n_f$**
  Nontrivial check: same as for $P_{qg}^{(2)}$, not obvious from above construction

Summary and outlook

- **Non-singlet physical kernels for nine observables in DIS, SIA and DY:**
  single-log large-x enhancement at NNLO/N^{3}LO to all orders in $(1-x)$
- **Conjecture: same for higher orders – work towards a proofs in progress**
  Predict highest three (DY: two) $\ln^k(1-x)$ terms of higher-order C_a
- **Exponentiation in N-space, less predictive than threshold resummation**
  Progress via subleading large-n_f, complementary top-down studies
- **No double logs either in singlet kernel for structure functions (F_2, F_ϕ)**
  Coeff. fct's $C_{\phi,i}$ by two groups, predict three logs in α_s^4 splitting fct's

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- **A small contribution towards uncovering iterative structures in pQCD**
  Limited phenomenol. relevance now: assess importance of N^{-1} terms
- **Near/mid future: combine with other results, e.g., fixed-N calculations**
  (close to) feasible now, cf. Baikov, Chetyrkin (06)
- **Far future: proven, use to check all-N/all-x fourth-order calculations**