Three-loop vertex corrections

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Karlsruhe Institute of Technology (KIT)

Radcor 2009, October 26-30, Ascona

Outline

I. Massless quark and gluon form factor

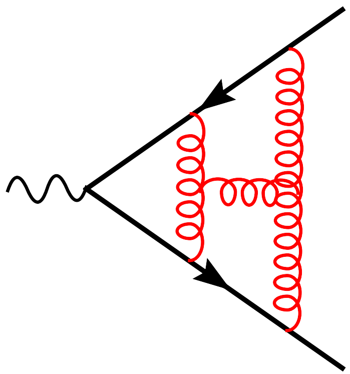
with P.A. Baikov, K.G. Chetyrkin, A.V Smirnov, V.A. Smirnov

II. Matching coefficient to the vector current

with P. Marquard, J.H. Piclum, D. Seidel

Outline

I. Massless quark and gluon form factor

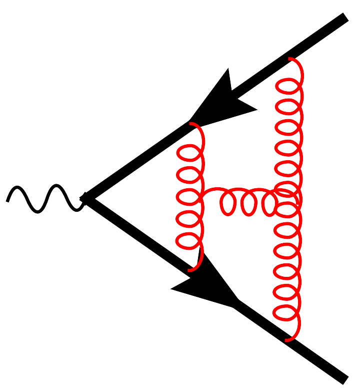


massless QCD

$q_1^2 = q_2^2 = 0$

$q^2 = (q_1 + q_2)^2$

II. Matching coefficient to the vector current



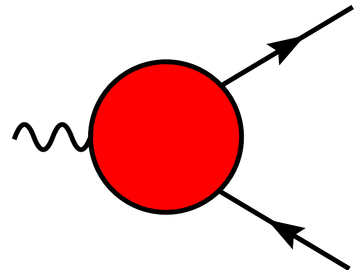
QCD, 1 heavy fermion, mass M_Q

$q_1^2 = q_2^2 = M_Q^2$

$q^2 = (q_1 + q_2)^2 = 4M_Q^2$

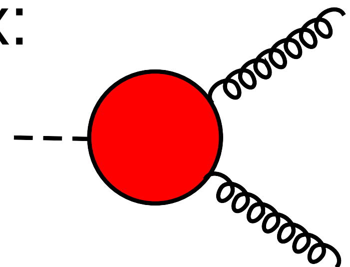
I. Form factors

Photon-quark vertex:



⇨ $\gamma^\mu F_q(q^2)$

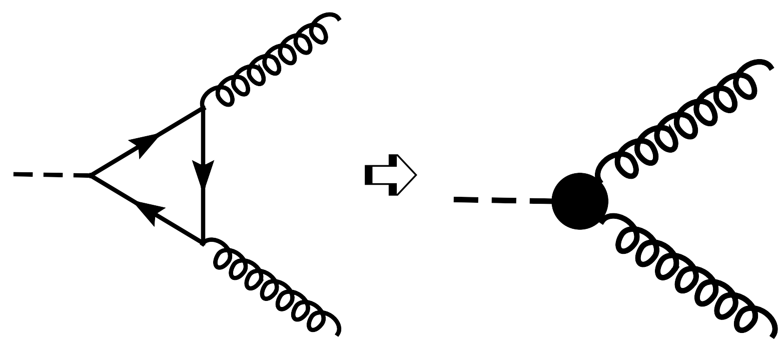
Higgs-gluon vertex:



⇨ $\left(g^{\mu\nu} q_1 q_2 - q_1^\nu q_2^\mu\right) F_g(q^2)$

F_g: construct effective theory for $m_t \to \infty$:

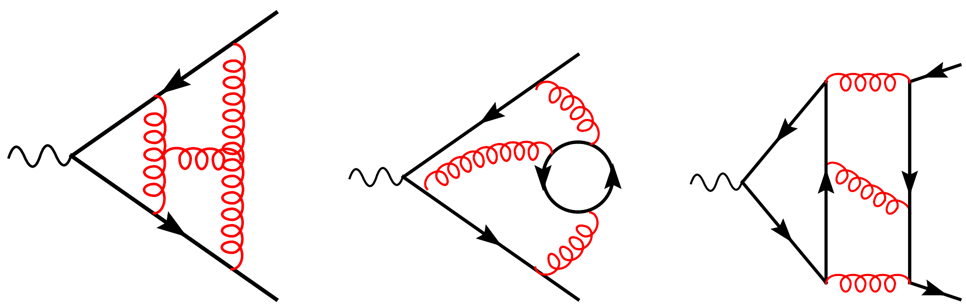
$\mathcal{L}^{\rm SM}$ ⇨ $\mathcal{L}_{\rm eff} = C_1 \frac{H}{v} G_{\mu\nu} G^{\mu\nu} + \ldots$



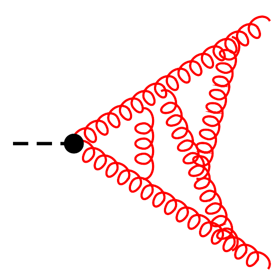
Form factor (2)

$$F = 1 + \frac{\alpha_s}{\pi} F^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 F^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 F^{(3)}$$

$$F_q^{(3)} = F_q^{(3),g} + F_q^{(3),n_f} + \sum_{q'} Q_{q'} F_q^{(3),sing}$$



$F_g^{(3)}$



Known results

- $F_q^{(2)}$ [(Gonsalves'83); Kramer,Lampe'86; Matsuura, van Neerven'87; Matsuura, van der Marck, van Neerven'88]
- $F_g^{(2)}$ [Harlander'00; Ravindran,Smith,van Neerven'04]
- $F_q^{(2)}$ and $F_g^{(2)}$ to $\mathcal{O}(\epsilon^2)$ [Gehrmann,Huber,Maitre'05]

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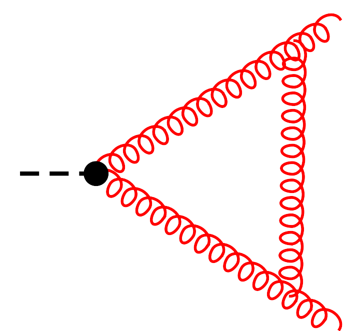
$$
\begin{aligned}
F_g^{(1)} \quad = \quad & C_A \Bigg\{ -\frac{2}{\epsilon^2} + \zeta_2 + \epsilon\left(-2 + \frac{14}{3}\zeta_3\right) + \epsilon^2\left(-6 + \frac{47}{20}\zeta_2^2\right) + \epsilon^3\Bigg(-14 + \zeta_2 \\
& -\frac{7}{3}\zeta_2\zeta_3 + \frac{62}{5}\zeta_5\Bigg) + \epsilon^4\left(-30 + 3\zeta_2 + \frac{14}{3}\zeta_3 + \frac{949}{280}\zeta_2^3 - \frac{49}{9}\zeta_3^2\right)\Bigg\}
\end{aligned}
$$



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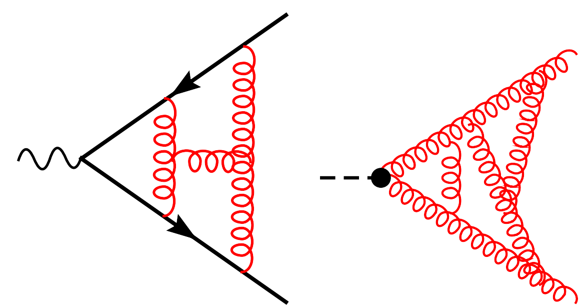
- $F_q^{(2)}$ and $F_g^{(2)}$ to $\mathcal{O}(\epsilon^2)$ [Gehrmann,Huber,Maitre'05]
- $1/\epsilon$ poles for $F_q^{(3)}$ and $F_g^{(3)}$ [Moch,Vermaseren,Vogt'05]

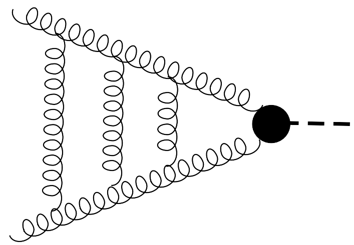
$$F^{(3)} = \frac{\#}{\epsilon^6} + \frac{\#}{\epsilon^5} + \frac{\#}{\epsilon^4} + \frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon^1} + \boxed{??}$$



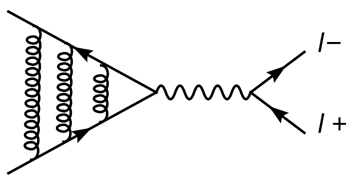
Applications

Virtual NNNLO corrections to

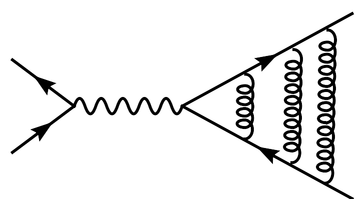
- $gg \longrightarrow H$



- DY



- $e^+e^- \longrightarrow 2$ Jets



- soft-gluon resummation
- cusp anomalous dimension

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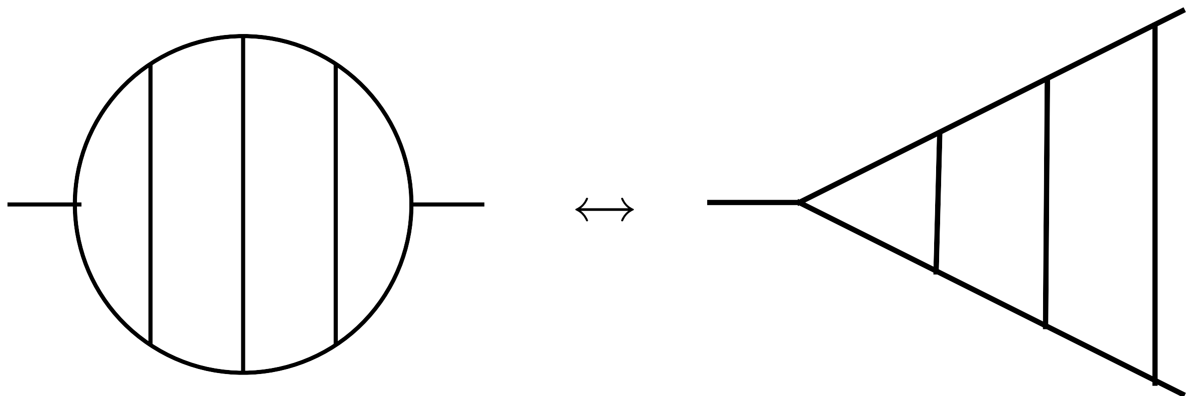
Some technical details

1. Reduction to MIs
2. Compute MIs

1. Reduction to MIs

Main approach:

- equivalence for recurrence relations between N-loop 2-point and $(N-1)$-loop 3-point functions [Baikov,Smirnov'96]



- similar to BAICER [Baikov,Chetyrkin,Kühn'02...'08]
- ParFORM, TFORM [Tentyukov,Vermaseren,...'04...'09]
- "Baikov-method"

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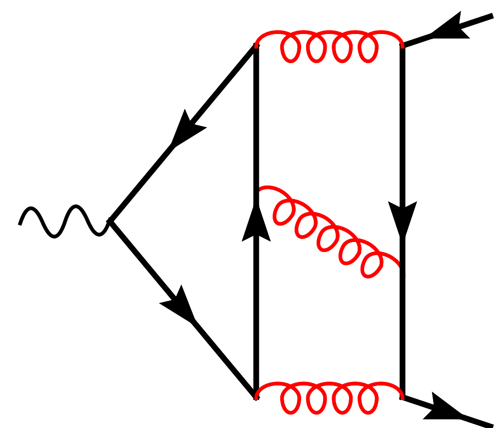
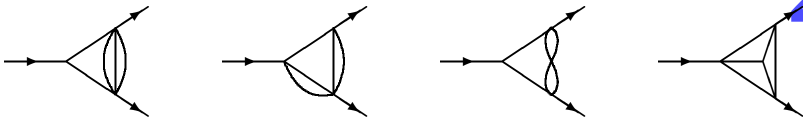
For singlet diagrams:

independent calculation with FIRE: Laporta $\oplus$ Gröbner [Smirnov'08]

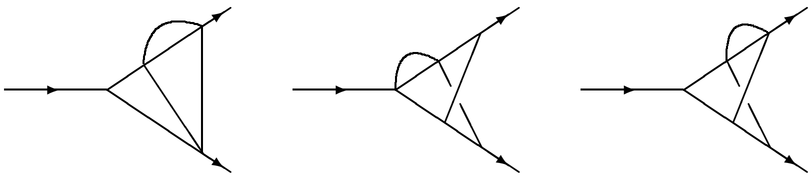


⇨ 22 MIs

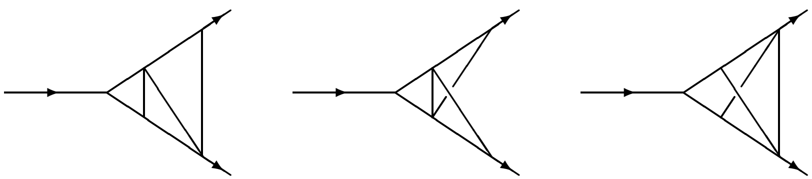
2. MIs



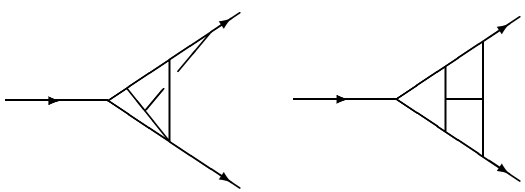
$A_{5,1}$ $A_{5,2}$ $A_{6,1}$ $A_{6,2}$



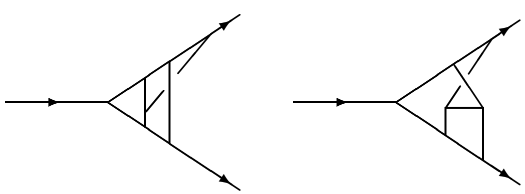
$A_{6,3}$ $A_{7,1}$ $A_{7,2}$



$A_{7,3}$ $A_{7,4}$ $A_{7,5}$



A_8 $A_{9,1}$

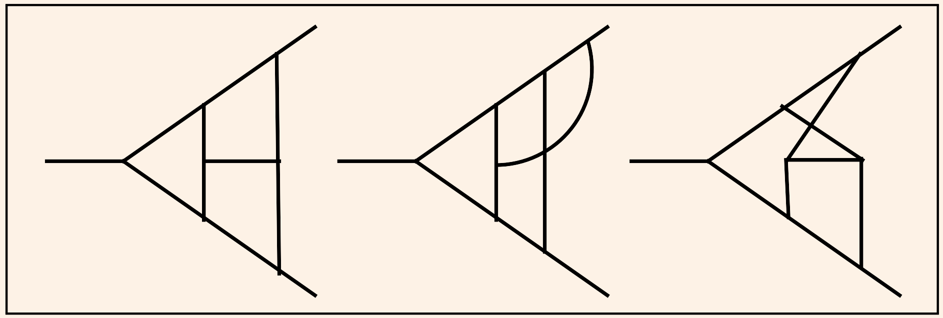


$A_{9,2}$ $A_{9,4}$

(+ 8 simple MIs)

22 master integrals

- 19 MIs: [Gehrmann,Heinrich,Huber,Studerus'06; Heinrich,Huber,Maitre'08]
- 3 most complicated MIs:

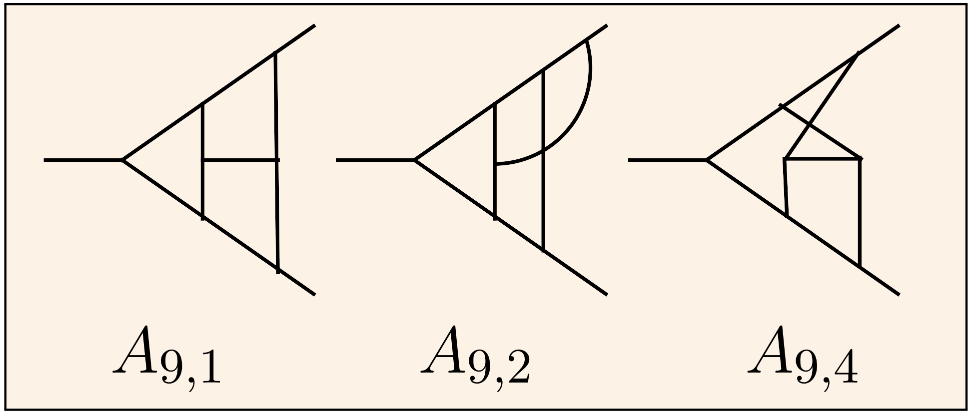


- extract by comparison with $F_q\Big|_{1/\epsilon-\text{poles}}$ [Moch,Vermaseren,Vogt'05]

  $F_g\Big|_{1/\epsilon-\text{poles}}$: consistency check
- 3 remaining coefficients in ϵ expansion: `FIESTA` [Smirnov,Tentukov'08], Mellin Barnes, `MB` [Czakon'06]
- independent (explicit) calculation: [Heinrich,Huber,Kosower,Smirnov'09]

[Talk by T. Huber]

2. MIs (2)



[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

[Heinrich,Huber,Kosower,Smirnov'09]

$$A_{9,4} = -\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \frac{1}{\epsilon^4}\left(1 + \frac{43\zeta(2)}{18}\right) + \frac{1}{\epsilon^3}\left(\frac{14}{9} + \frac{106\zeta(2)}{9} + \frac{109\zeta(3)}{9}\right)$$
$$+ \frac{1}{\epsilon^2}\left(-17 - \frac{311\zeta(2)}{18} + \frac{608\zeta(3)}{9} - \frac{481\zeta(4)}{144}\right)$$
$$+ \frac{1}{\epsilon}\left(84 + \frac{11\zeta(2)}{3} - \frac{949\zeta(3)}{9} + \frac{425\zeta(4)}{6} + \frac{3463\zeta(5)}{45} - \frac{2975\zeta(2)\zeta(3)}{18}\right)$$
$$+ X_{9,4} + \mathcal{O}(\epsilon)$$

Results

- $F_q^{(3)}$ and $F_g^{(3)}$: agreement with poles [Moch,Vermaseren,Vogt'05]
- $F_q^{(3)}\Big|_{n_f-}$part [Moch,Vermaseren,Vogt'05]
- $F_q^{(3),sing}$ can be extracted from [Moch,Vermaseren,Vogt'04]

Results

[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

$$
\boxed{F_q^{(3),g+n_f}\Big|_{\rm fin}} = C_F^3\left(\frac{26871}{8} - \frac{95137\zeta(2)}{60} + \frac{5569\zeta(3)}{5} + \frac{95375\zeta(4)}{48} + \frac{30883\zeta(2)\zeta(3)}{15} - \frac{16642\zeta(5)}{5} + \frac{2669(\zeta(3))^2}{3}\right.
$$
$$
\left. + \frac{1961387\zeta(6)}{2880} - \frac{24X_{9,1}}{5} + \frac{24X_{9,2}}{5} + \frac{6X_{9,4}}{5}\right) + C_A C_F^2\left(\frac{20003431}{29160} + \frac{4239679\zeta(2)}{1620} - \frac{121753\zeta(3)}{30}\right.
$$
$$
- \frac{11155817\zeta(4)}{4320} - \frac{92554\zeta(2)\zeta(3)}{45} + \frac{610462\zeta(5)}{225} - \frac{36743(\zeta(3))^2}{30} - \frac{1118529\zeta(6)}{640} + \frac{24X_{9,1}}{5}
$$
$$
\left. - \frac{16X_{9,2}}{5} - \frac{9X_{9,4}}{5}\right) + C_A^2 C_F\left(-\frac{88822328}{32805} - \frac{3486997\zeta(2)}{2916} + \frac{3062512\zeta(3)}{1215} + \frac{4042277\zeta(4)}{4320}\right.
$$
$$
\left. + \frac{5233\zeta(2)\zeta(3)}{12} - \frac{202279\zeta(5)}{450} + \frac{63043(\zeta(3))^2}{180} + \frac{4741699\zeta(6)}{11520} - X_{9,1} + \frac{2X_{9,2}}{5} + \frac{3X_{9,4}}{5}\right)
$$
$$
+ C_F^2 n_f T\left(-\frac{2732173}{1458} - \frac{45235\zeta(2)}{81} + \frac{102010\zeta(3)}{81} + \frac{40745\zeta(4)}{216} - \frac{686\zeta(3)\zeta(2)}{9} + \frac{556\zeta(5)}{45}\right)
$$
$$
+ C_A C_F n_f T\left(\frac{17120104}{6561} + \frac{442961\zeta(2)}{729} - \frac{90148\zeta(3)}{81} - \frac{5465\zeta(4)}{27} + \frac{736\zeta(3)\zeta(2)}{9} - \frac{416\zeta(5)}{3}\right)
$$
$$
+ C_F n_f^2 T^2\left(-\frac{2710864}{6561} - \frac{248\zeta(2)}{3} + \frac{12784\zeta(3)}{243} - \frac{166\zeta(4)}{27}\right)
$$

$$
X_{9,1} \approx 1428.99636786661835 91\,,
$$
$$
X_{9,2} \approx 528.0583 \pm 0.0326\,,\; X_{9,4} \approx -2085.380547 \pm 0.000025
$$

Results

[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

$$\left. F_q^{(3),sing}\right|_{\rm fin} = d^{abc}d^{abc}\left(\frac{2}{3} + \frac{5\zeta(2)}{3} + \frac{7\zeta(3)}{9} - \frac{\zeta(4)}{6} - \frac{40\zeta(5)}{9}\right)$$

$$\left. F_g^{(3)}\right|_{\rm fin} = C_A^3\left(\frac{14423912}{6561} + \frac{384479\zeta(2)}{2916} - \frac{370649\zeta(3)}{486} + \frac{280069\zeta(4)}{864} + \frac{1821\zeta(2)\zeta(3)}{4} - \frac{66421\zeta(5)}{90}\right.$$
$$\left. + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\right) + C_A^2 n_f T\left(-\frac{10021313}{6561} - \frac{75736\zeta(2)}{729} - \frac{1508\zeta(3)}{27}\right.$$
$$\left. + \frac{437\zeta(4)}{12} - \frac{878\zeta(3)\zeta(2)}{9} + \frac{6476\zeta(5)}{45}\right) + C_F C_A n_f T\left(-\frac{155629}{243} - \frac{82\zeta(2)}{3} + \frac{23584\zeta(3)}{81} - 16\zeta(4)\right.$$
$$\left. + 96\zeta(3)\zeta(2) + \frac{64\zeta(5)}{9}\right) + C_F^2 n_f T\left(\frac{608}{9} + \frac{592\zeta(3)}{3} - 320\zeta(5)\right) + C_F n_f^2 T^2\left(\frac{42248}{81} - \frac{64\zeta(2)}{3}\right.$$
$$\left. - \frac{2816\zeta(3)}{9} - \frac{224\zeta(4)}{3}\right) + C_A n_f^2 T^2\left(\frac{2958218}{6561} + \frac{304\zeta(2)}{27} + \frac{47296\zeta(3)}{243} + \frac{1594\zeta(4)}{27}\right)$$

$$X_{9,1} \approx 1428.9963678666183591\,,$$
$$X_{9,2} \approx 528.0583 \pm 0.0326\,,\; X_{9,4} \approx -2085.380547 \pm 0.000025$$

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Results

[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

SYM ($C_A = C_F = 2T$, $n_f = 1$):

$$F_q^{(3),g+n_f}\Big|_{\rm fin} = C_A^3\left(\frac{389216}{243} - \frac{155935\zeta(2)}{972} - \frac{54703\zeta(3)}{162} + \frac{23897\zeta(4)}{72} + \frac{15875\zeta(2)\zeta(3)}{36} - \frac{11279\zeta(5)}{10} + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\right)$$

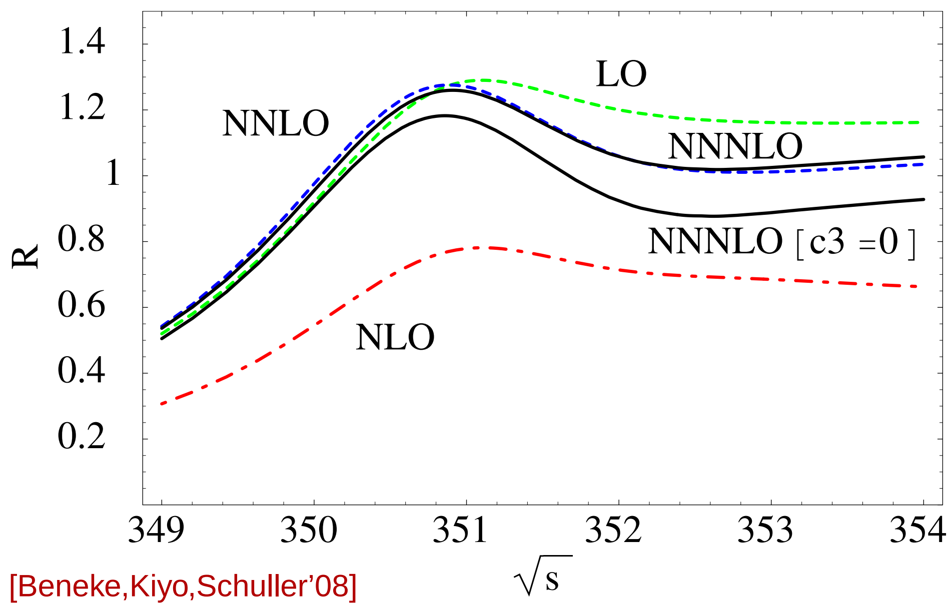
$$F_g^{(3)}\Big|_{\rm fin} = C_A^3\left(\frac{676219}{486} + \frac{61937\zeta(2)}{972} - \frac{93295\zeta(3)}{162} + \frac{95171\zeta(4)}{288} + \frac{16361\zeta(2)\zeta(3)}{36} - \frac{1645\zeta(5)}{2} + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\right)$$

II. c_v and NRQCD

QCD $\longrightarrow$ NRQCD ($\longrightarrow$ pNRQCD)

[Caswell,Lepage'86] [Bodwin,Braaten,Lepage'95] [Pineda,Soto'98;Brambilla,Pineda,Soto,Vairo'00]

- Quarkonia
- $t\bar{t}$ @ threshold



[Beneke,Kiyo,Schuller'08]

Missing ingredients:

- 3-loop matching of vector current (QCD $\rightarrow$ NRQCD) $c_v^{(3)}$
- 3-loop static potential: a_3

[V. Smirnov's talk]

- $\mathcal{O}(\epsilon)$ of 2-loop $1/m_q$ potential

- m_b from Υ sum rules (low-moment SRs: $m_b = 4.163 \pm 0.016$ GeV [Chetyrkin et al.'09])

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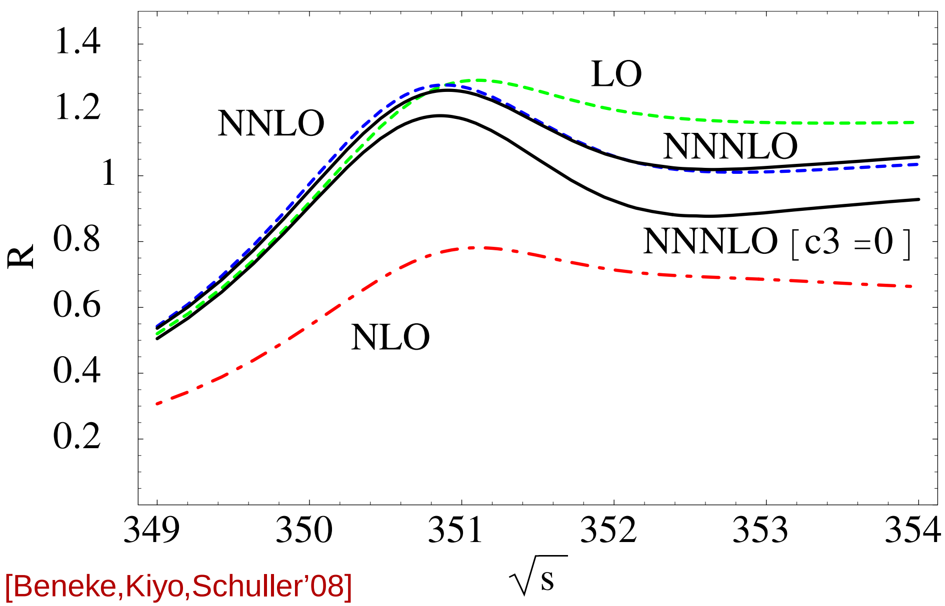

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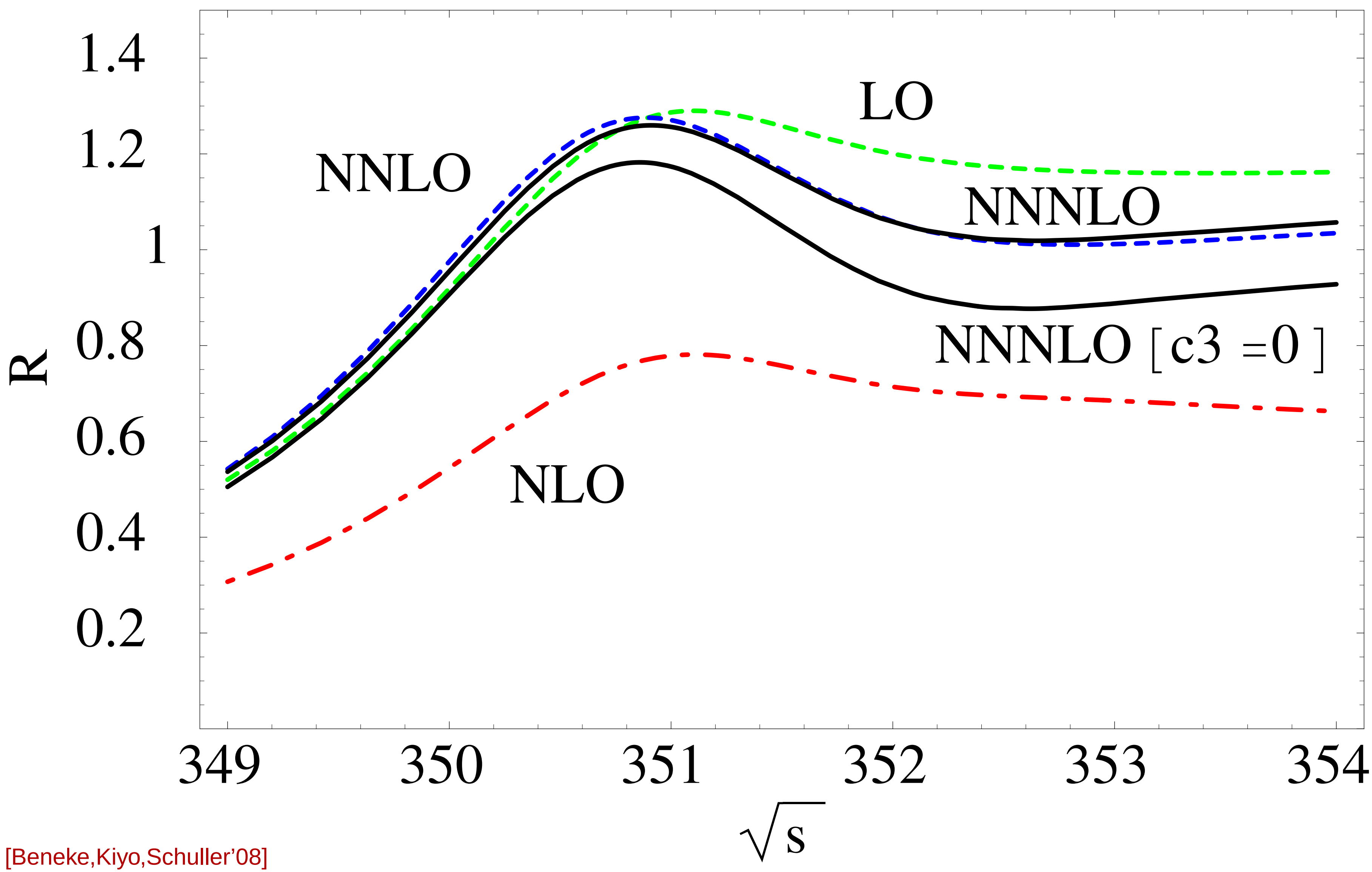
$e^+e^- \to t\bar{t}$ @ threshold



[Beneke,Kiyo,Schuller'08]

c_v

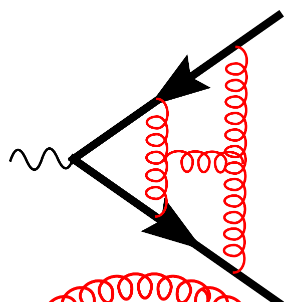
$$\text{QCD} \quad \longrightarrow \quad \text{NRQCD}$$

$$j_v^\mu = \bar{Q}\gamma^\mu Q \quad \longrightarrow \quad \tilde{j}^i = \phi^\dagger \sigma^i \chi$$

$$j_v^i \quad = \quad c_v(\mu)\tilde{j}^i + \frac{d_v(\mu)}{6m_Q^2}\phi^\dagger\sigma^i\vec{D}^2\chi + \ldots$$

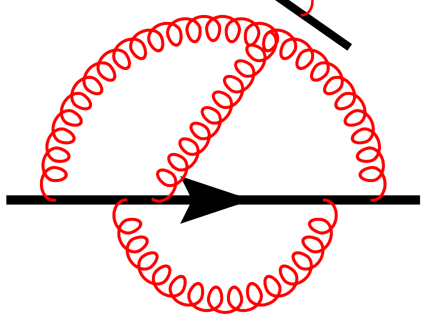
$$Z_2\Gamma_v \quad = \quad c_v\tilde{Z}_2\tilde{Z}_v^{-1}\tilde{\Gamma}_v + \ldots$$

Γ_v:



$\tilde{\Gamma}_v \equiv 1\,, \quad \tilde{Z}_2 \equiv 1$

Z_2:



$\tilde{Z}_v = 1 + \mathcal{O}(\alpha_s^2)$

[Melnikov,v.Ritbergen'00]

[Marquard,Mihaila,Piclum,Steinhauser'07]

c_v (2)

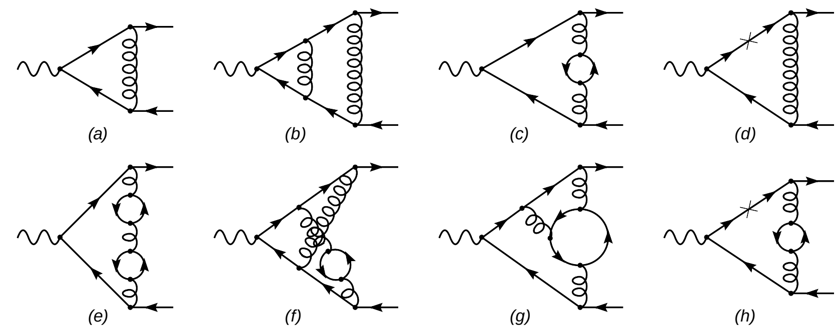
$$c_v = 1 + \frac{\alpha_s}{\pi} c_v^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_v^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_v^{(3)} + \mathcal{O}(\alpha_s^4)$$

$c_v^{(1)}$ [Källen, Sarby'55]

$c_v^{(2)}$ [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]

$c_v^{(3),n_l}$ [Marquard,Piclum,Seidel,Steinhauser'06]

$c_v^{(3),n_h}$ [Marquard,Piclum,Seidel,Steinhauser'08]

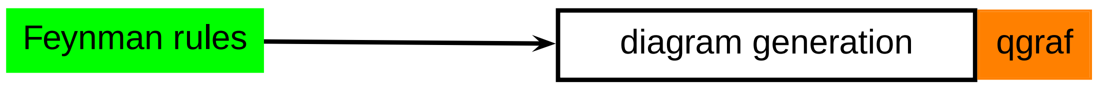


massive vertices

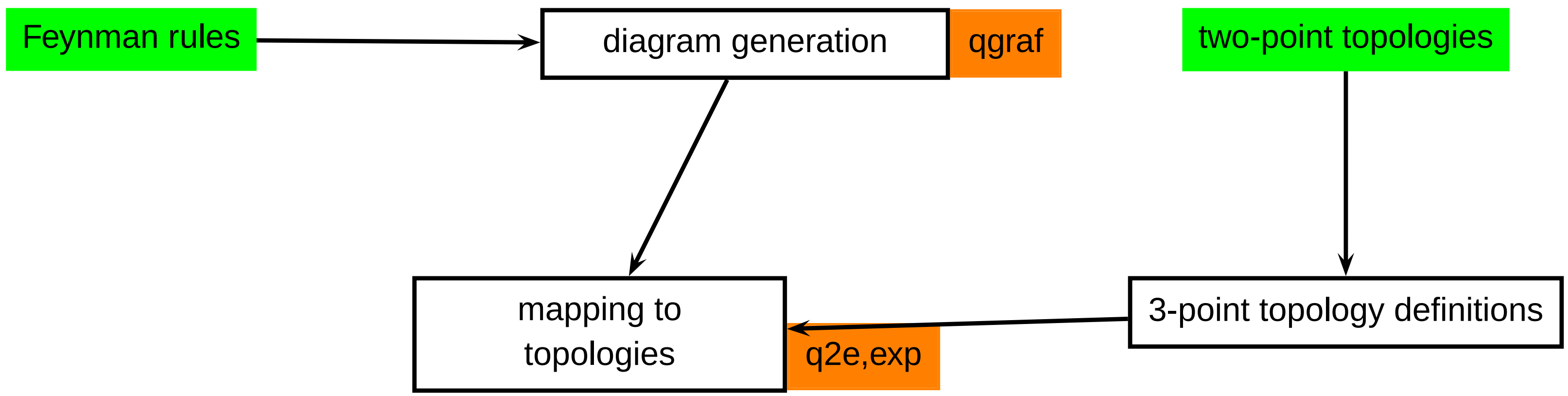
on-shell quarks: $q_1^2 = q_2^2 = M_Q^2$

$$(q_1 + q_2)^2 = 4M_Q^2$$

Automated calculation



Automated calculation



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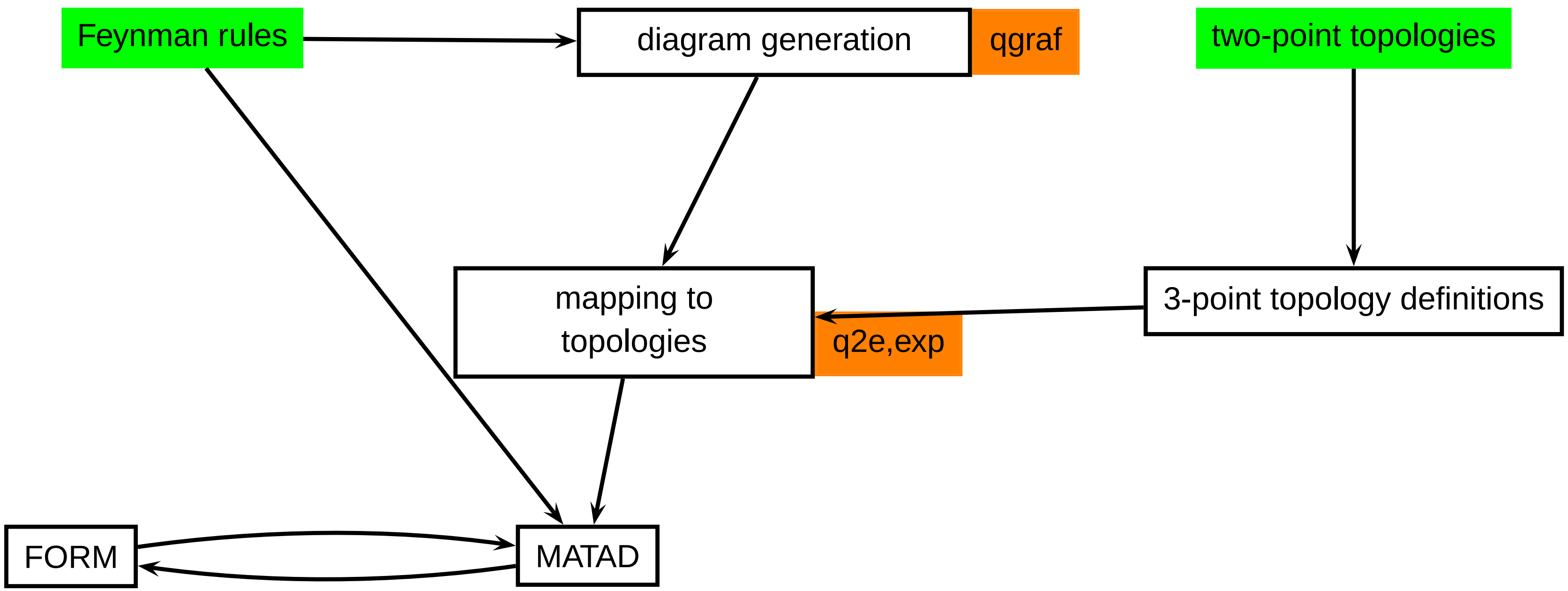

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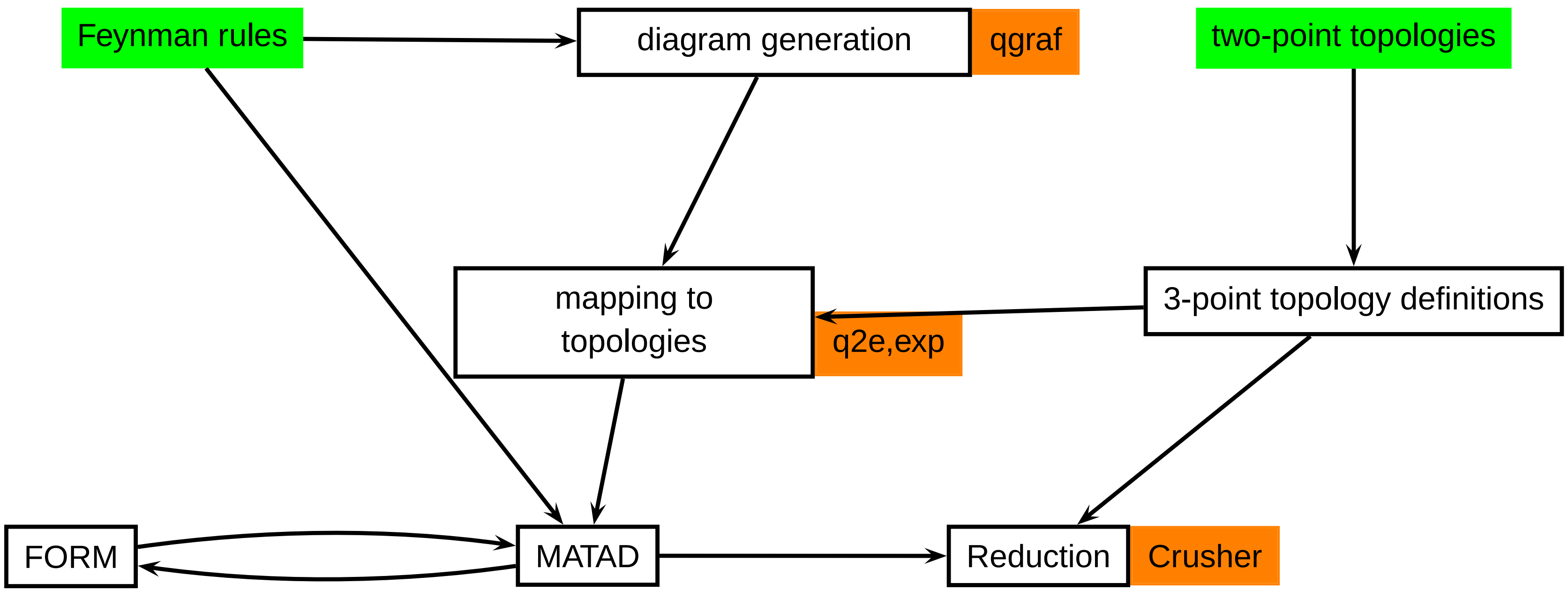

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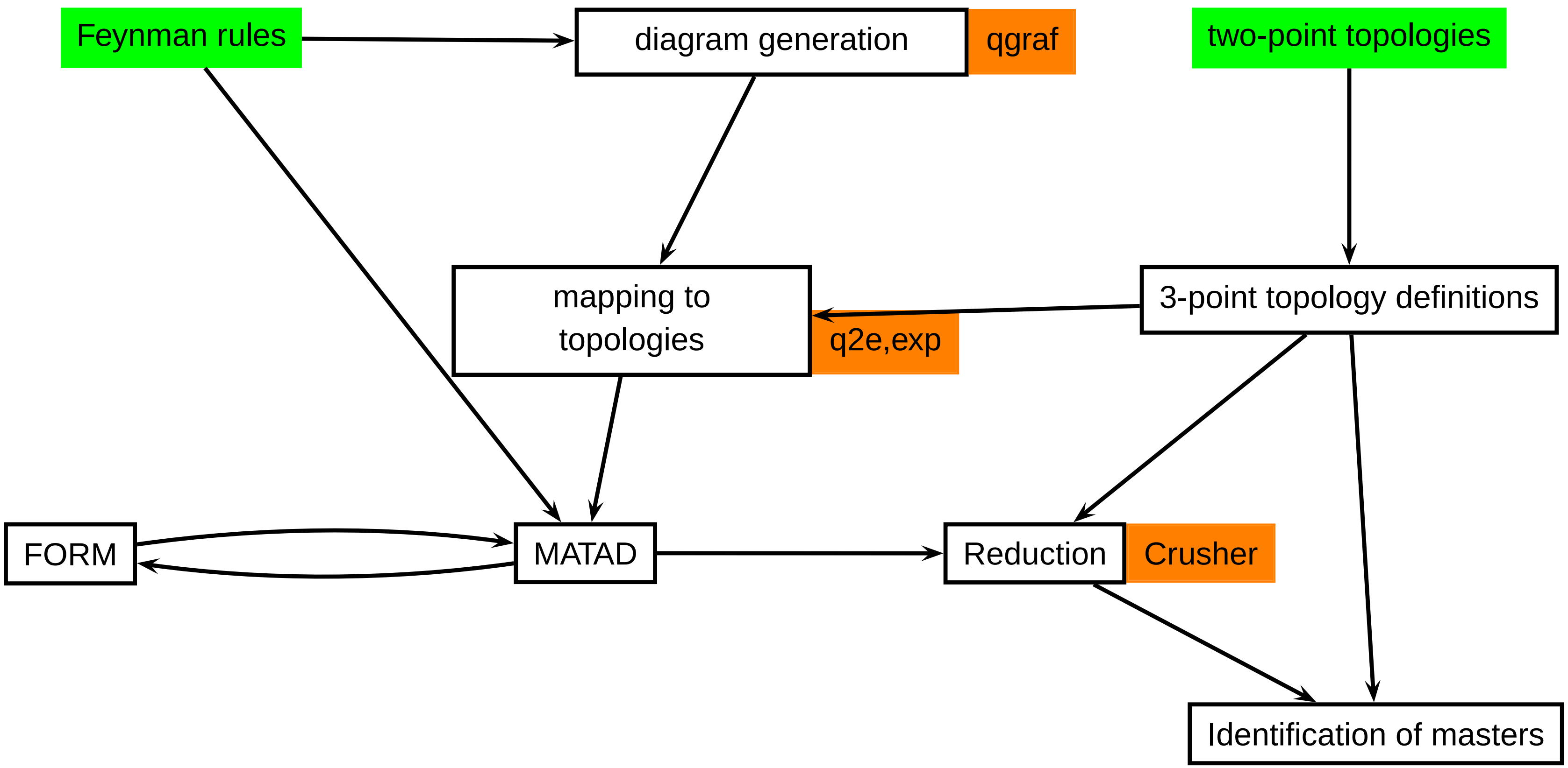
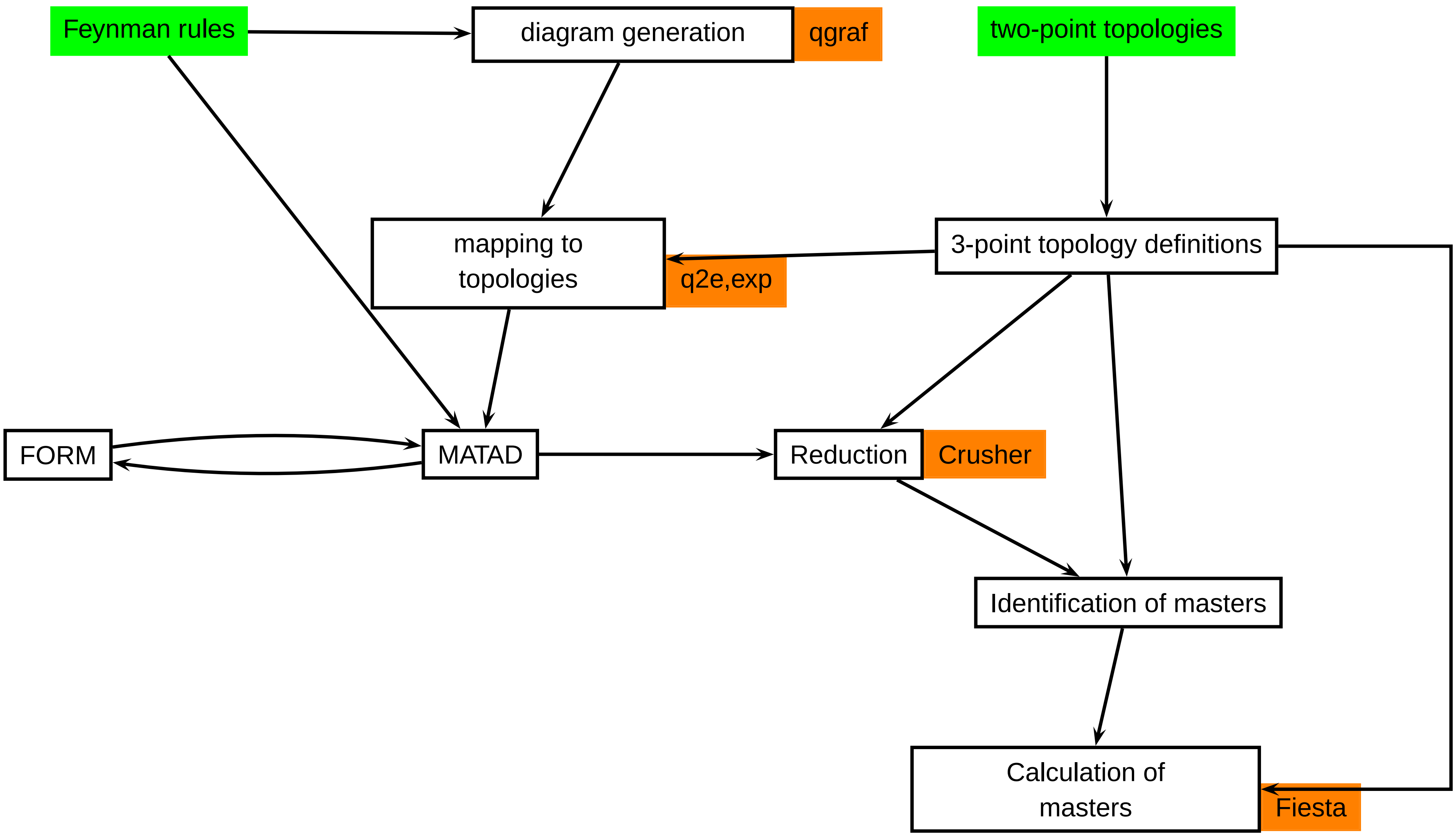


Automated calculation



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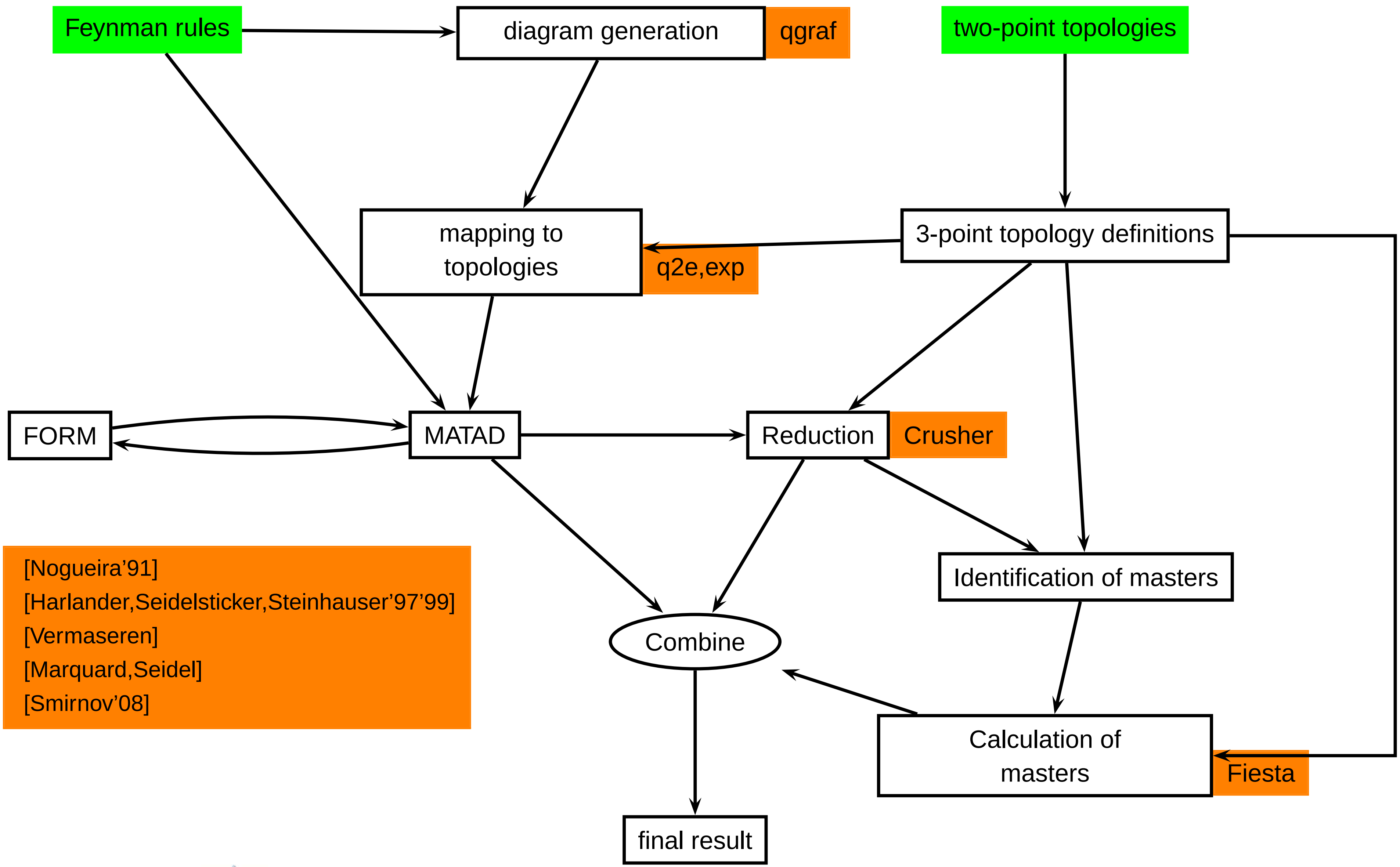

[Nogueira'91]
[Harlander,Seidelsticker,Steinhauser'97'99]
[Vermaseren]
[Marquard,Seidel]
[Smirnov'08]

Checks

- finiteness
- gauge parameter independece (ξ^1)
- `IfCut` dependence (n_h contribution)
- change basis of MIs (to be done for non-n_f term)
- compare with “exact” result

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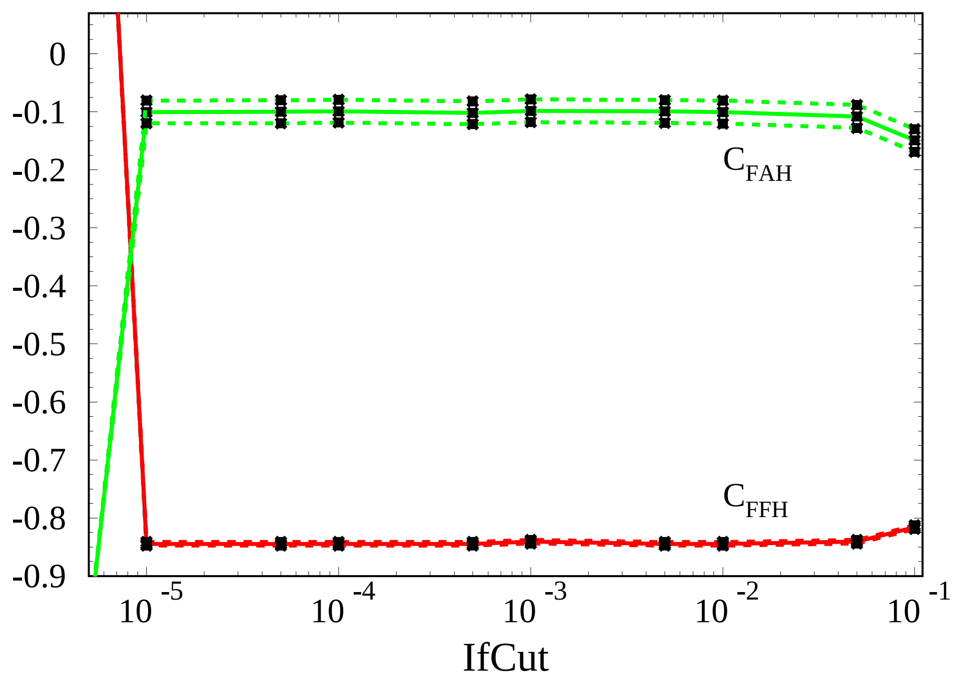

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Checks

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- change basis of MIs (to be done for non-n_f term)

	basis 1	basis 2	basis 3
$c_{\rm FFH}\vert_{\rm log}$	$-0.496(2)$	$-0.50(1)$	$-0.496(8)$
$\tilde{c}_{\rm FFH}$	$-0.841(3)$	$-0.85(6)$	$-0.86(2)$
$\tilde{c}_{\rm FAH}$	$-0.10(2)$	$-0.15(9)$	$-0.13(4)$
$\tilde{c}_{\rm FHH}$	$0.05126(1)$	$0.0513(1)$	$0.0513(1)$
$\tilde{c}_{\rm FHL}$	$-0.27029(4)$	$-0.27028(3)$	$-0.2703(2)$

- compare with “exact” result

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Checks

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- compare with "exact" result

	"numerical"	"exact"
$\tilde{c}_v^{(2)}$	$-42.5138(2)$	-42.5140
$\tilde{c}_{\mathrm{FFL}}$	$46.692(1)$	$46.7(1)$
$\tilde{c}_{\mathrm{FAL}}$	$39.623(1)$	$39.6(1)$
$\tilde{c}_{\mathrm{FHL}}$	$-0.27029(4)$	-0.27025
$\tilde{c}_{\mathrm{FLL}}$	$-2.46833(3)$	-2.46834
$\tilde{c}_v^{(3)}\|_{n_l}$	$n_l\,(120.660(3) - 0.8228 n_l)$	$n_l\,(121. - 0.8228 n_l)$

$\tilde{Z}_v$

[Marquard,Piclum,Seidel,Steinhauser]

$$\tilde{Z}_v = 1 + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^2 \left(\frac{1}{12}C_F^2 + \frac{1}{8}C_F C_A\right)\frac{\pi^2}{\epsilon} + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^3 C_F T \Bigg\{ n_l \left[\left(\frac{1}{54}C_F + \frac{1}{36}C_A\right)\frac{\pi^2}{\epsilon^2} - \left(\frac{25}{324}C_F + \frac{37}{432}C_A\right)\frac{\pi^2}{\epsilon}\right] + C_F n_h \frac{\pi^2}{60\epsilon}\Bigg\} + \ldots$$

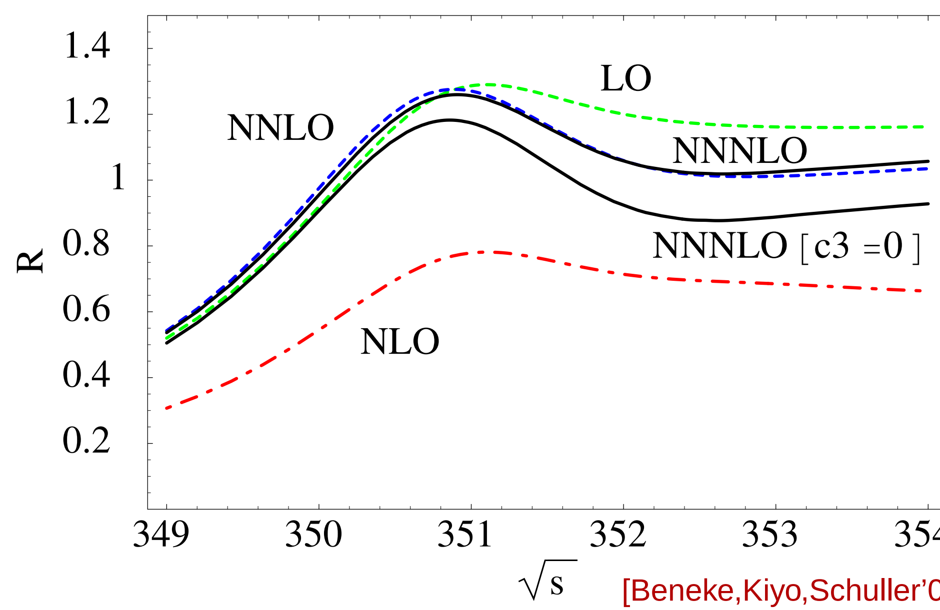
- analytic result: [Kniehl,Penin,Smirnov,Steinhauser'02, Beneke,Kiyo,Penin'07]
- (numerical) agreement better than 1%

Results

[Marquard,Piclum,Seidel,Steinhauser]

$$c_v(\mu = m_Q) = 1 - 2.67\frac{\alpha_s}{\pi} + (-44.55 + 0.41 n_l)\left(\frac{\alpha_s}{\pi}\right)^2 + (-2xxx.xx + 120.66 n_l - 0.82 n_l^2)\left(\frac{\alpha_s}{\pi}\right)^3$$

- $\alpha_s^3 n_l^0$ check missing (base change)
- large corrections
- $\overline{\rm MS}$ scheme
- combine with US contributions

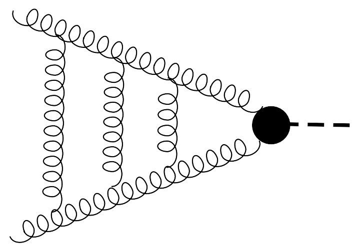


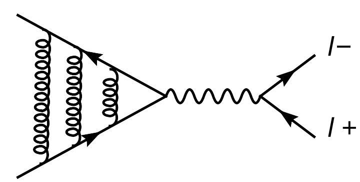
[Beneke,Kiyo,Schuller'08]

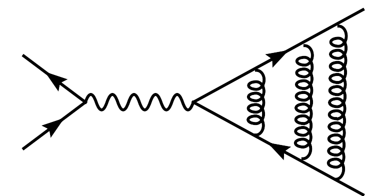
Summary

I. Massless quark and gluon form factor

- 3-loop corrections for F_q and F_g
- 3 most complicated MIs
- applications: NNNLO contribution to
  - $gg \to H$
  - DY
  - $e^+e^- \to 2$ Jets







- independent check of 3-loop cusp anomalous dimension

II. Matching QCD $\to$ NRQCD

- building block:
  $\sigma(e^+e^- \to t\bar{t})$, m_b from Υ sum rules
- large numercial corrections in $\overline{\text{MS}}$ scheme