Full result for the three-loop static quark potential

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- Introduction. Our calculational scheme
- Our techniques:
  reduction to master integrals by `FIRE`;
  evaluating master integrals by Mellin–Barnes representation;
  numerical checks by `FIESTA`
- Results

The QCD potential between a heavy static quark and its antiquark in an expansion in α_s and heavy-quark velocity $v \sim \alpha_s$.

Two alternative definitions of the potential:
in terms of a Wilson loop,
in the framework of an effective field theory.

The dynamics of a nonrelativistic quark-antiquark pair involves three well separated scales:
hard $\sim m_q$; soft $\sim m_q v$; ultrasoft $\sim m_q v^2$

Expansion by regions [M. Beneke & V.A. Smirnov '98]

Four regions are relevant:

hard (energy and momentum $\sim m_q$);
soft (energy and momentum $\sim m_q v$);
potential (energy $\sim m_q v^2$, momentum $\sim m_q v$);
ultrasoft (energy and momentum $\sim m_q v^2$)

'Integrating out' the hard modes: QCD $\rightarrow$ NRQCD
'Integrating out' the soft modes and potential gluons: NRQCD $\rightarrow$ pNRQCD.

The static potential as an operator in pNRQCD Lagrangian.

The static colour-singlet potential

$$
\begin{aligned}
V(|\boldsymbol{q}|) &= -\frac{4\pi C_F \alpha_s(|\boldsymbol{q}|)}{\boldsymbol{q}^2}\left[1+\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}a_1+\left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2\right. \\
&\quad \left. +\left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3\left(a_3+8\pi^2 C_A^3 \ln\frac{\mu^2}{\boldsymbol{q}^2}\right)+\cdots\right]
\end{aligned}
$$

with $C_A = N$ and $C_F = (N^2-1)/(2N)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N)$ colour gauge group, respectively, $T_F = 1/2$ is the index of the fundamental representation, and n_l is the number of light-quark flavours.

One loop

[W. Fischler '77; A. Billoire '80]

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_l$$

Two loops

[M. Peter'97; Y. Schröder'99]

$$a_2 = \left[\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right] C_A^2 - \left[\frac{1798}{81} + \frac{56}{3}\zeta(3)\right] C_A T_F$$

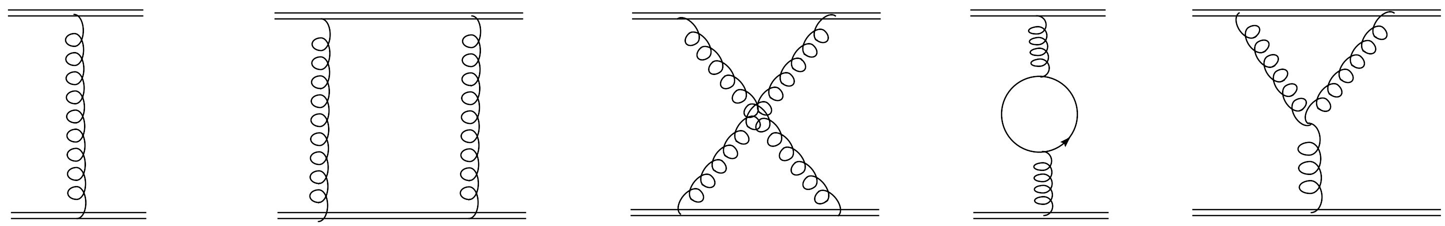
$$- \left[\frac{55}{3} - 16\zeta(3)\right] C_F T_F n_l + \left(\frac{20}{9} T_F n_l\right)^2$$

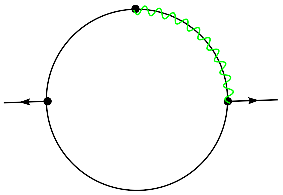
$1/m_q$-correction

[B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser'02]

Three loops: $a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} =?$
a_3 is one of a few missing NNNLO ingredients of the non-relativistic dynamics near threshold.

Tree and one-loop approximations:

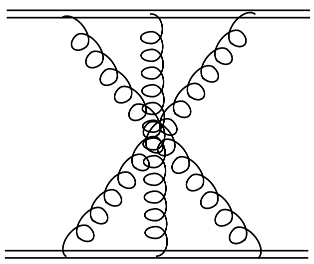


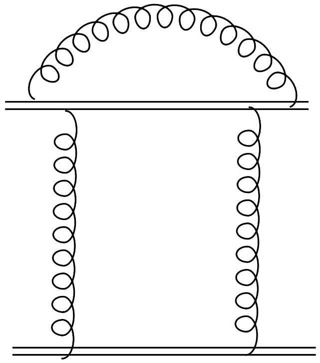


$$F(a_1, a_2, a_3) = \int \frac{\mathrm{d}^d k}{(-k^2 - i0)^{a_1}(-(q-k)^2 - i0)^{a_2}(-v \cdot k - i0)^{a_3}}$$

with $q \cdot v = 0, \quad v = (1, \vec{0})$

Two loops:





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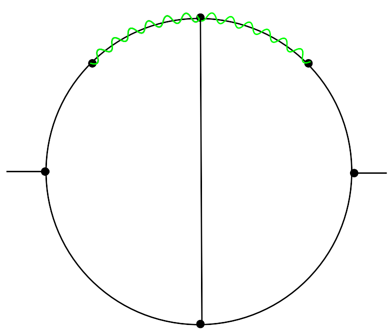
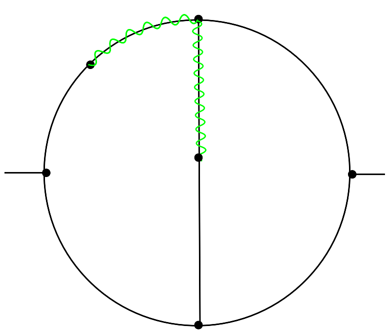
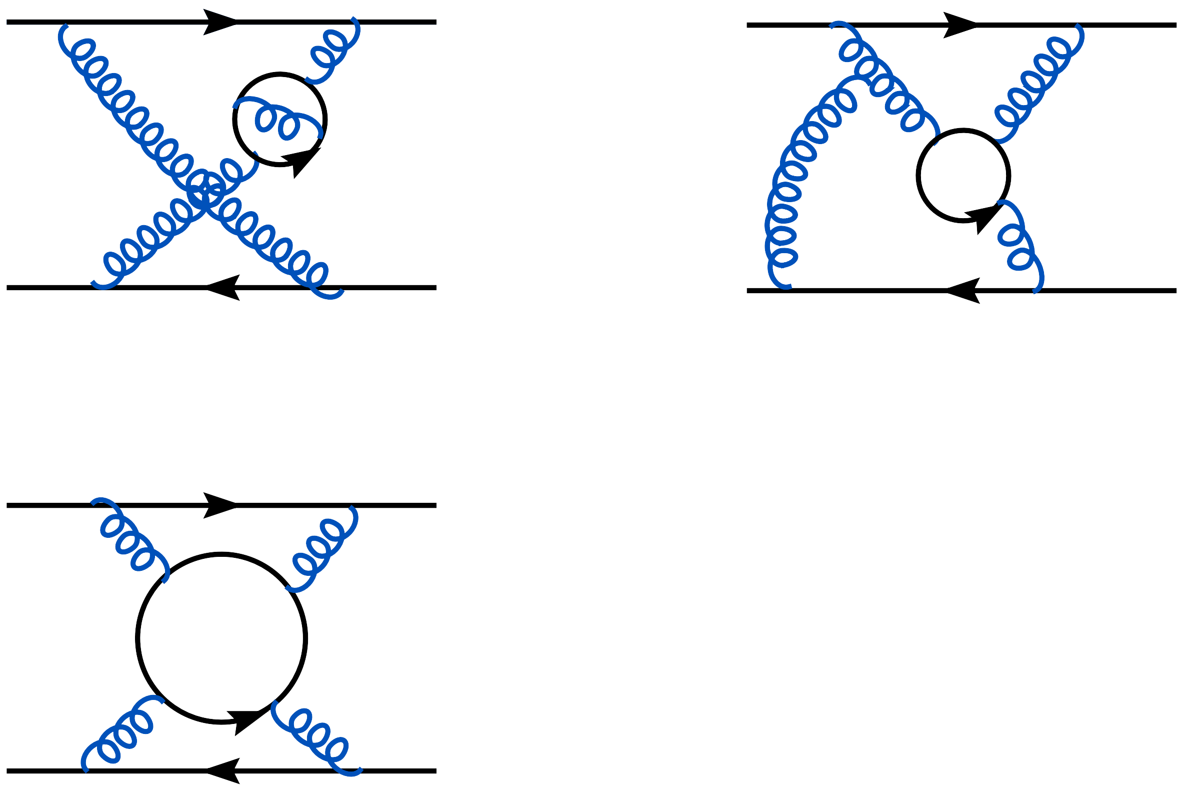
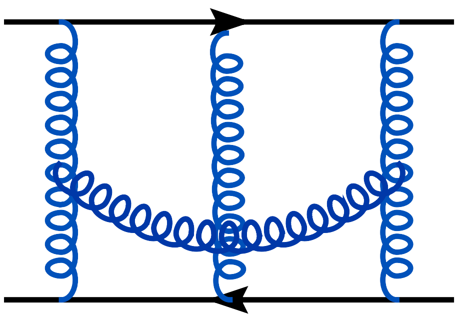
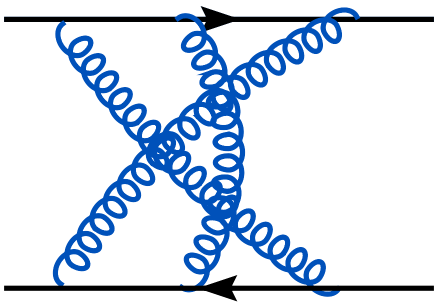
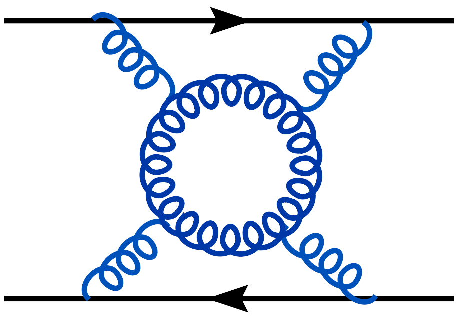
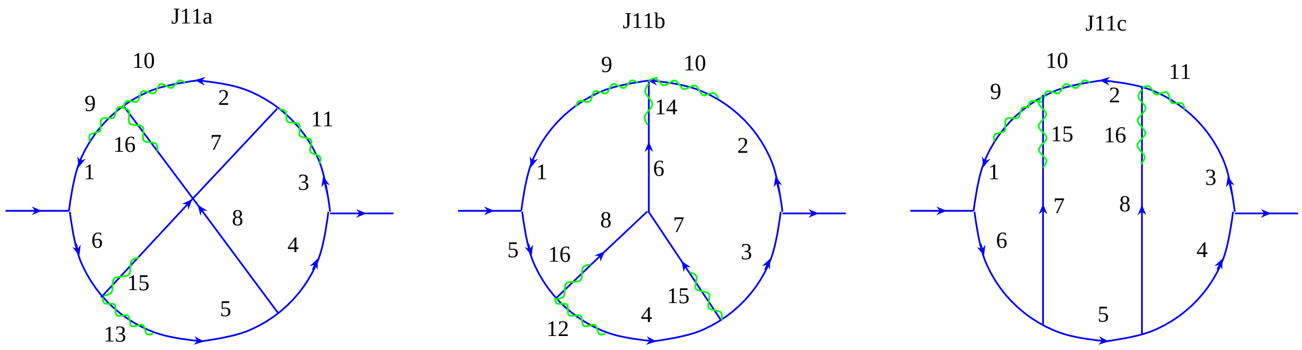




Generation of diagrams by QGRAPH



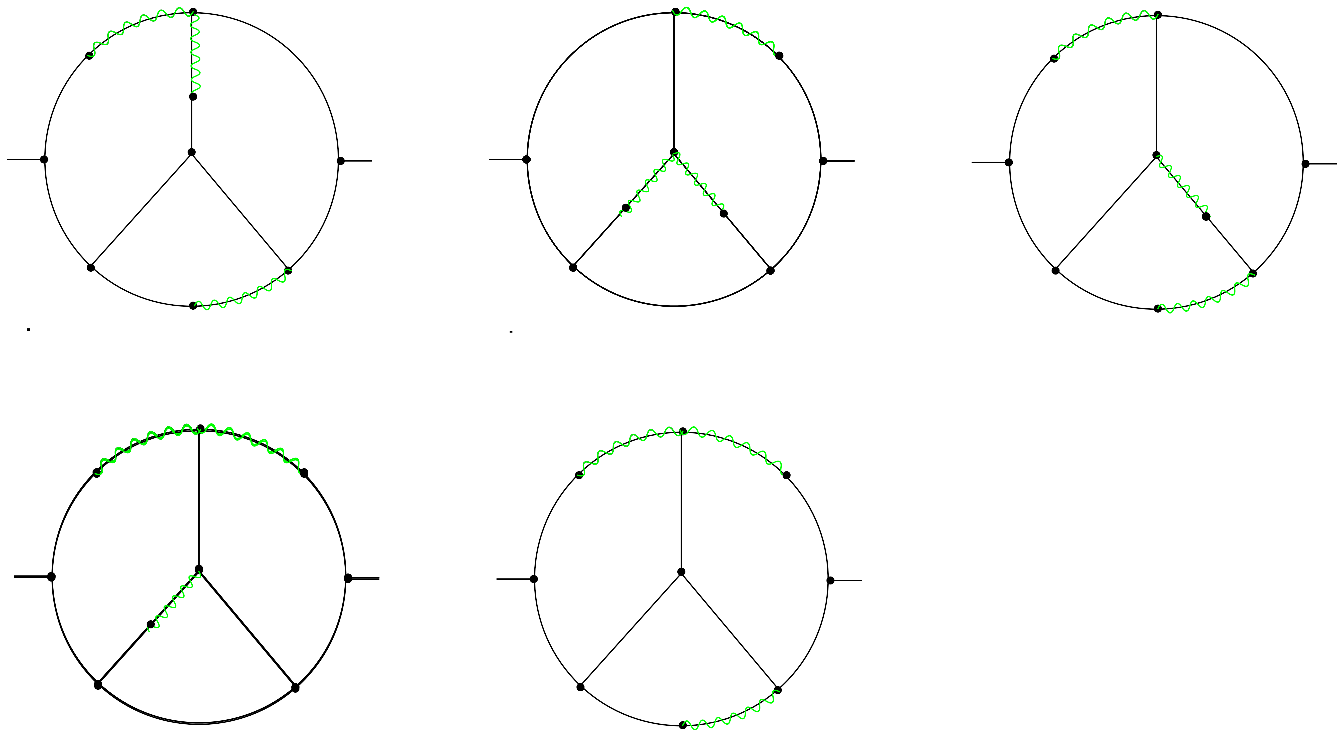


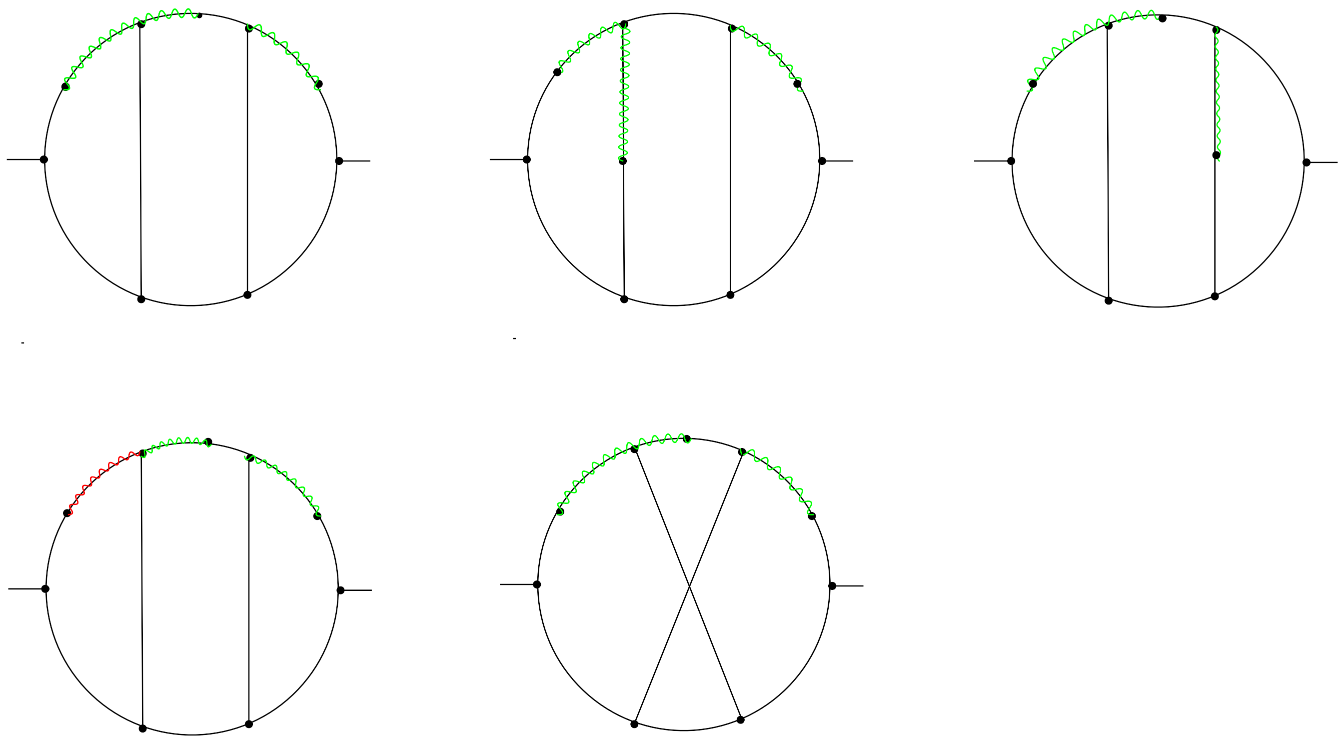
$$J_{11}^a(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0;n_9,n_{10},n_{11},n_{13},n_{15},n_{16};s_9,s_{10},s_{11},s_{13},s_{15},s_{16})$$
$$= \int\int\int \frac{(-p_0^2)^{-n_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{n_1}(-p_2^2)^{n_2}(-p_3^2)^{n_3}(-p_4^2)^{n_4}(-p_5^2)^{n_5}(-p_6^2)^{n_6}(-p_7^2)^{n_7}(-p_8^2)^{n_8}}$$
$$\times \frac{1}{(-v\cdot p_1 - s_9 i0)^{n_9}(-v\cdot p_2 - s_{10} i0)^{n_{10}}(-v\cdot p_3 - s_{11} i0)^{n_{11}}(-v\cdot p_5 - s_{13} i0)^{n_{13}}(-v\cdot p_7 - s_{15} i0)^{n_{15}}(-v\cdot p_8 - s_{16} i0)^{n_{16}}}$$
$$p_0 = k-r, p_1 = k, p_2 = l, p_3 = r, p_4 = q+r, p_5 = q+k-l+r, p_6 = q+k, p_7 = l-r, p_8 = k-l$$

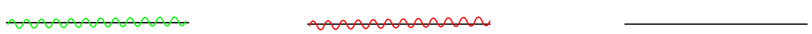
$$J_{11}^b(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0;n_9,n_{10},n_{12},n_{14},n_{15},n_{16};s_9,s_{10},s_{12},s_{14},s_{15},s_{16})$$
$$= \int\int\int \frac{(-p_0^2)^{-n_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{n_1}(-p_2^2)^{n_2}(-p_3^2)^{n_3}(-p_4^2)^{n_4}(-p_5^2)^{n_5}(-p_6^2)^{n_6}(-p_7^2)^{n_7}(-p_8^2)^{n_8}}$$
$$\times \frac{1}{(-v\cdot p_1 - s_9 i0)^{n_9}(-v\cdot p_2 - s_{10} i0)^{n_{10}}(-v\cdot p_4 - s_{12} i0)^{n_{12}}(-v\cdot p_6 - s_{14} i0)^{n_{14}}(-v\cdot p_7 - s_{15} i0)^{n_{15}}(-v\cdot p_8 - s_{16} i0)^{n_{16}}}$$
$$p_0 = l-q, p_1 = k-q, p_2 = r-q, p_3 = r, p_4 = l, p_5 = k, p_6 = k-r, p_7 = l-r, p_8 = k-l$$

$$J_{11}^c(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8;n_0;n_9,n_{10},n_{11},n_{15},n_{16};s_9,s_{10},s_{11},s_{15},s_{16})$$
$$= \int\int\int \frac{(-p_0^2)^{-n_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{n_1}(-p_2^2)^{n_2}(-p_3^2)^{n_3}(-p_4^2)^{n_4}(-p_5^2)^{n_5}(-p_6^2)^{n_6}(-p_7^2)^{n_7}(-p_8^2)^{n_8}}$$
$$\times \frac{1}{(-v\cdot p_1 - s_9 i0)^{n_9}(-v\cdot p_2 - s_{10} i0)^{n_{10}}(-v\cdot p_3 - s_{11} i0)^{n_{11}}(-v\cdot p_7 - s_{15} i0)^{n_{15}}(-v\cdot p_8 - s_{16} i0)^{n_{16}}}$$
$$p_0 = k-r, p_1 = k, p_2 = l, p_3 = r, p_4 = q+r, p_5 = q+l, p_6 = q+k, p_7 = k-l, p_8 = l-r$$

Various classes of Feynman integrals with 12 indices:

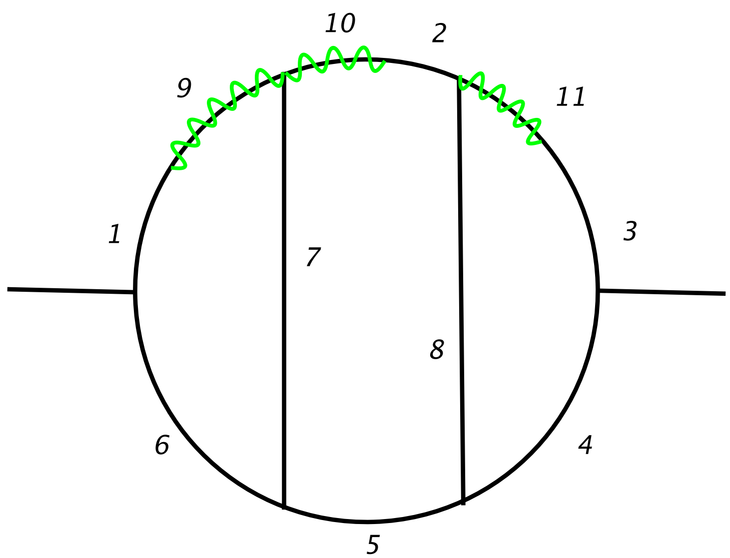






$$\frac{1}{(-v\cdot k-i0)^a} \qquad \frac{1}{(-v\cdot k+i0)^a} \qquad \frac{1}{(-k^2-i0)^a}$$

For example,



with the numerator chosen as $(-(k-r)^2)^{-a_{12}}$

Our techniques

IBP [K.G. Chetyrkin & F.V. Tkachov'81]

The whole problem of the evaluation of a given family of Feynman integrals $\rightarrow$

- constructing a reduction procedure using IBP
- evaluating master integrals

Solving reduction problems algorithmically:

- Laporta's algorithm

  [S. Laporta & E. Remiddi'96; S. Laporta'00; T. Gehrmann & E. Remiddi'01]

  A public version:

  AIR [C. Anastasiou & A. Lazopoulos'04]

  Private versions

  [T. Gehrmann & E. Remiddi, M. Czakon, Y. Schröder, A. Pak, C. Sturm, P. Marquard & D. Seidel, V. Velizhanin, . . .]

- Baikov's method [Baikov'96-09]
- Gröbner bases [O.V. Tarasov'98, A.V. Smirnov & V.A. Smirnov'05]
- Lee's approach [R.N. Lee'08]

A combination of Laporta's algorithm and Gröbner bases:
FIRE = Feynman Integrals REduction [A.V. Smirnov'08]
(implemented in `Mathematica`)

http://science.sander.su

or

http://www-ttp.particle.uni-karlsruhe.de/$\sim$asmirnov

Evaluating master integrals

~ 100 master integrals in the whole problem

Mellin-Barnes representation [V.A. Smirnov'99, J.B Tausk'99]

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z) \; .$$

- Derive an MB representation
- Resolve the singularity structure in ϵ
- Expand in a Laurent series in ϵ and evaluate expanded MB integrals

Evaluating Feynman Integrals (STMP 211);

Feynman Integrals Calculus (Springer 2006)

For planar diagrams, AMBRE can be used

[J. Gluza, K. Kajda & T. Riemann'07]

MB tools at http://projects.hepforge.org/mbtools/:

MB.m [M. Czakon] [M. Czakon'05]

MBresolve.m [A. Smirnov] [A.& V. Smirnovs'09]

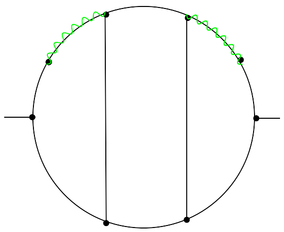
MBasymptotics.m [M. Czakon]

barnesroutines.m [D. Kosower]
(applying Barnes lemmas automatically)

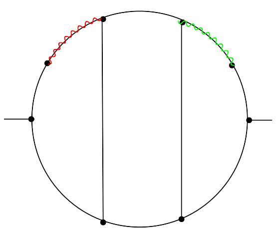
$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)$$

$$= \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}$$

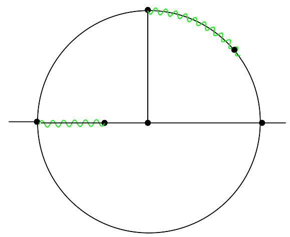
For example,



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}\left[\frac{56\pi^4}{135\epsilon}+\frac{112\pi^4}{135}+\frac{16\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}\left[-\frac{64\pi^4}{135\epsilon}-\frac{128\pi^4}{135}-\frac{32\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$



$$\frac{(i\pi^{d/2})^3}{q^2 v^2}\left[\frac{32\pi^4}{135\epsilon} - \frac{128\pi^4}{135} + \frac{88\pi^2\zeta(3)}{9} + \frac{188\zeta(5)}{3} + O(\epsilon)\right]$$

Transcendentality level 5 (and, sometimes, 6) is needed.
The constants of level 6 that we encounter:

$$\zeta(6),\ \zeta(4)\ln^2 2,\ \zeta(3)^2,\ \zeta(3)\zeta(2)\ln 2,\ \zeta(2)\ln^4 2,$$
$$\ln^6 2,\ \mathrm{Li}_6\left(\tfrac{1}{2}\right),\ \mathrm{Li}_4\left(\tfrac{1}{2}\right)\ln^2 2, \ldots\ s_6 = S(\{-5,-1\},\infty)$$

PSLQ [H.R.P. Ferguson & D.H. Bailey'91]

Only three constants are not known analytically.

Numerical checks by `FIESTA` [A.V. Smirnov & M.N. Tentyukov'08]

(Feynman Integral Evaluation by a Sector decomposiTion Approach)

A tool to evaluate Feynman integrals numerically by sector decompositions.

[T. Binoth & G. Heinrich'00]

Strategies that are guaranteed to terminate

[C. Bogner & S. Weinzierl'07]

A, B, C, X

Strategy S [A.V. Smirnov & M.N. Tentyukov'08]

http://science.sander.su
http://www-ttp.particle.uni-karlsruhe.de/∼asmirnov

`FIESTA 2` [A.& V. Smirnovs & M.N. Tentyukov'09]

- Expanding integrals automatically in limits of momenta and masses (with the use of Mellin–Barnes representation)

  [M.C. Bergère, C. de Calan and A.P.C. Malbouisson,'78; K. Pohlmeyer,'82; V. Pilipp'08]
- complete parallelization (even to multiple computers)
- high-precision arithmetics
- new integrators
- Speer sectors as a strategy [A.& V. Smirnovs'09]

Results

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} \,,$$

$a_3^{(3)}, a_3^{(2)}, a_3^{(1)}$ [A.V. Smirnov, V.A. Smirnov, and M. Steinhauser'08]

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3 \,,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 \,, + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2 \,,$$

$$a_3^{(1)} = (-709.717)\, C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5) \right) C_A C_F T_F$$

$$+ \left(\frac{286}{9} + \frac{296\zeta(3)}{3} - 160\zeta(5) \right) C_F^2 T_F + (-56.83(1))\, \frac{d_F^{abcd} d_F^{abcd}}{N_A}\,,$$

$$C_A = N_c\,, \quad C_F = \frac{N_c^2 - 1}{2N_c}\,, \quad T_F = \frac{1}{2}\,,$$

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{18 - 6N_c^2 + N_c^4}{96 N_c^2}\,.$$

UV poles in ϵ are removed by renormalization.

The remaining simple pole is canceled by the pole of the US contribution:

$$\frac{4\pi C_F \alpha_s}{|\vec{q}|^2} \left(\frac{\alpha_s}{4\pi}\right)^3 16\pi^2 C_A^3 \left(\frac{1}{6\epsilon} + \cdots\right) + \cdots$$

$$a_3^{(0)} \;=\; (\ldots)\, C_A^3 + (-136.39(6))\, \frac{d_F^{abcd} d_A^{abcd}}{N_A} \,,$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c^3 + 6 N_c}{48}$$

An essential cancelation of the n_l^1 and n_l^0 contributions is expected.

Some additional checks are in progress.

to be completed