Master Integrals for Massless Three-Loop Form Factors

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T. Gehrmann, G. Heinrich, C. Studerus, TH, PLB **640** (2006) 252, [hep-ph/0607185]

G. Heinrich, D. Maître, TH, PLB **662** (2008) 344, [0711.3590]

G. Heinrich, D. A. Kosower, V. A. Smirnov, TH, PLB **678** (2009) 359, [0902.3512]

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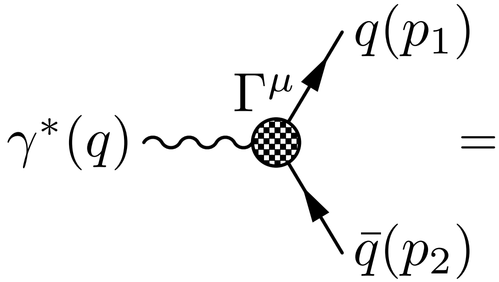
Outline

- Quark and gluon form factor in massless QCD
- Form factors: Status and applications
- Identification of three-loop master integrals
- Computation of masters: techniques and results
- Conclusion and outlook

Form factors

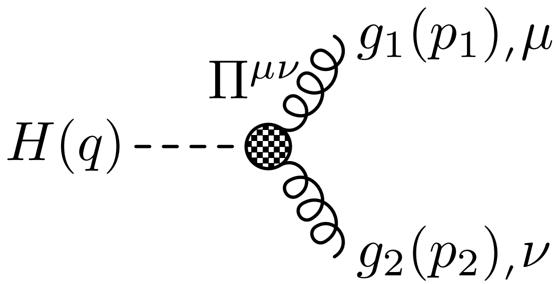
- Quark form factor F_q: $\gamma^* \to q\bar{q}\,,$ massless quarks



$$= -i\,e\,\bar{u}(p_1)\,\Gamma^\mu\,u(p_2)$$

$$\Gamma^\mu = F_q\,\gamma^\mu = \gamma^\mu\left[1 + \left(\frac{\alpha_s}{2\pi}\right)F_q^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 F_q^{(2)} + \left(\frac{\alpha_s}{2\pi}\right)^3 F_q^{(3)} + \ldots\right]$$

- Gluon form factor F_g: $H \to gg$, from effective vertex $\mathscr{L}_{eff} = -\frac{C_1(\alpha_s)}{4v}\,H\,G^{\mu\nu}\,G_{\mu\nu}$



$$= i\,\frac{C_1}{v}\,\Pi^{\mu\nu} = i\,\frac{C_1}{v}\,F_g\,\left(g^{\mu\nu}\,p_1\cdot p_2 - p_1^\nu\,p_2^\mu\right)$$

$$F_g = \left[1 + \left(\frac{\alpha_s}{2\pi}\right)F_g^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 F_g^{(2)} + \left(\frac{\alpha_s}{2\pi}\right)^3 F_g^{(3)} + \ldots\right]$$

History and status of the form factors

- General multi-loop strategies
  - Regulate UV and IR divergences of amplitude dimensionally, $D = 4 - 2\epsilon$
  - Apply algebraic reduction methods, reduction is exact in D dimensions
  - Obtain amplitude as a linear combination of a small set of master integrals
  - Poles up to 1/ϵ^{2L}

- Two-loop form factors through $\mathcal{O}(\epsilon^0)$ known since long
  - $F_q^{(2)}$ *[(Gonsalves'83); Kramer,Lampe'87; Matsuura,van Neerven'88; Matsuura,van der Maarck,van Neerven'89]*
  - $F_g^{(2)}$ *[Harlander'00; Ravindran,Smith,van Neerven'04]*
- Also extension of $F_q^{(2)}$ and $F_g^{(2)}$ to all orders in ϵ *[Gehrmann,Maitre,TH'05]*
  - $F_q^{(2)}$ and $F_g^{(2)}$ through order $\mathcal{O}(\epsilon^2)$: First step towards three-loop accuracy

- Three-loop form factors $F_q^{(3)}$ and $F_g^{(3)}$: Pole terms known through $\mathcal{O}(\epsilon^{-1})$, and also the finite pieces of the fermionic corrections to $F_q^{(3)}$ *[Moch,Vermaseren,Vogt'05]*
- Recently also the full $F_q^{(3)}$ and $F_g^{(3)}$ have become available
  - See talk by M. Steinhauser *[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]*

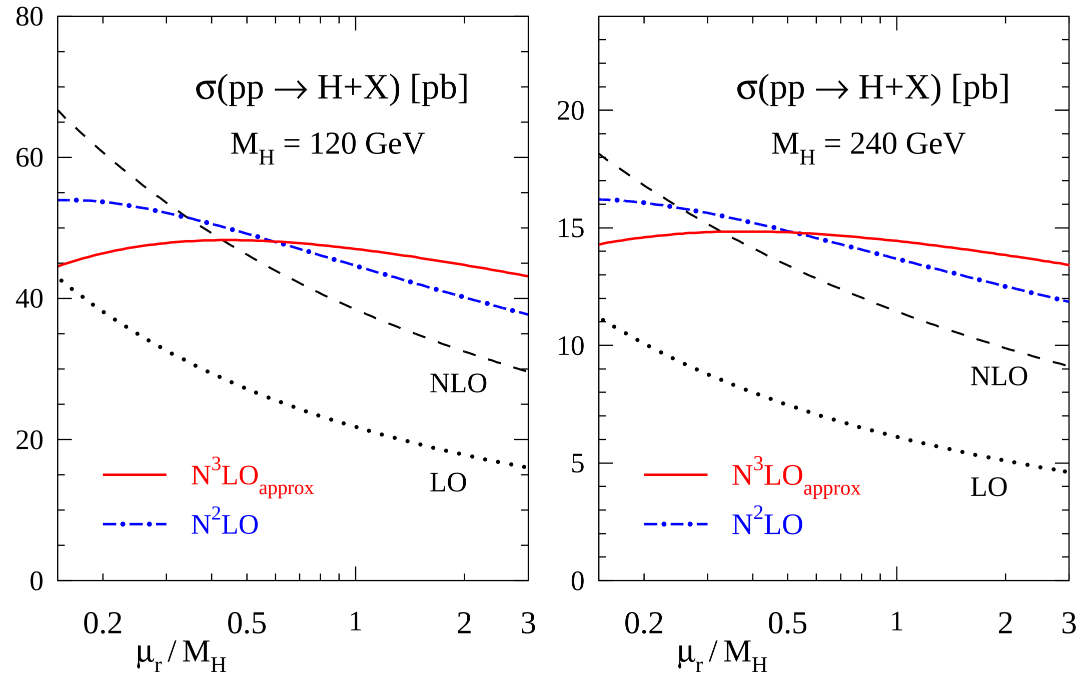
Applications of the form factors

- Both form factors have applications in many collider processes
- Quark form factor
  - Deep-inelastic scattering [Moch,Vermaseren,Vogt'04-'05]
  - Drell-Yan process $q\bar{q} \to W^{\pm}\,, Z^0\,, \gamma^*$ [Hamberg,Matsuura,van Neerven'91]
  - Two-parton contribution to $e^+e^- \to$ jets
- Gluon form factor
  - Higgs-production: $gg \to H$

[Dawson'91; Djouadi,Graudenz,Spira,Zerwas'91-'93]
[Harlander,Kilgore'01-'02; Catani,de Florian,Grazzini'01]
[Anastasiou,Melnikov'02; Ravindran,Smith,van Neerven'03]
[Anastasiou,Melnikov,Petriello'05; Moch,Vogt'05]

  - N^3LO without finite term
  - $\sigma_{\rm tot.}$ approximated to $\mathcal{O}(1\%)$ by $\sigma_{\rm tot.}^{m_t\to\infty}$ up to $M_H \approx 2m_t$

[Krämer,Laenen,Spira '96, see also e.g. Harlander,Ozeren'09]
[Pak,Rogal,Steinhauser'09; Anastasiou,Bucherer, Kunszt'09]



Applications of the form factors

- The quark and gluon form factor are the simplest quantities with IR divergences at higher orders in massless QFT $\Rightarrow$ Analytic result most desirable.
  - Prediction of the IR pole structure of QCD amplitudes

    [Magnea,Sterman'90; Catani'98; Sterman,Tejeda-Yeomans'02; Gehrmann,Gehrmann-de Ridder,Glover'04-'05]
    [Becher,Neubert'09; Gardi,Magnea'09; Dixon'09; Dixon,Gardi,Magnea'09]
  - Determination of resummation coefficients

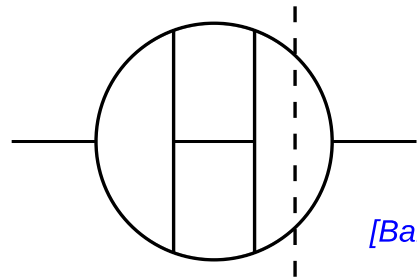
    [Collins,Soper,Sterman'84-'85; Magnea'00; Moch,Vermaseren,Vogt'05]
  - Check of exponential ansatz for planar n-point MHV amplitudes in $N = 4$ Super-Yang-Mills *[Anastastiou,Bern,Dixon,Kosower'03; Bern,Dixon,Smirnov'05]*

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

- $M_n^{(1)}(\epsilon)$: one-loop amplitude, exact in ϵ. $\quad a = \dfrac{N_c\, \alpha_s}{2\pi}(4\pi e^{-\gamma_E})^{\epsilon}$
- $f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\, \epsilon + f_2^{(l)}\, \epsilon^2$
- $C^{(l)}$ independent of n, and $E_n^{(l)}(\epsilon = 0) = 0\,.$

Identification of masters

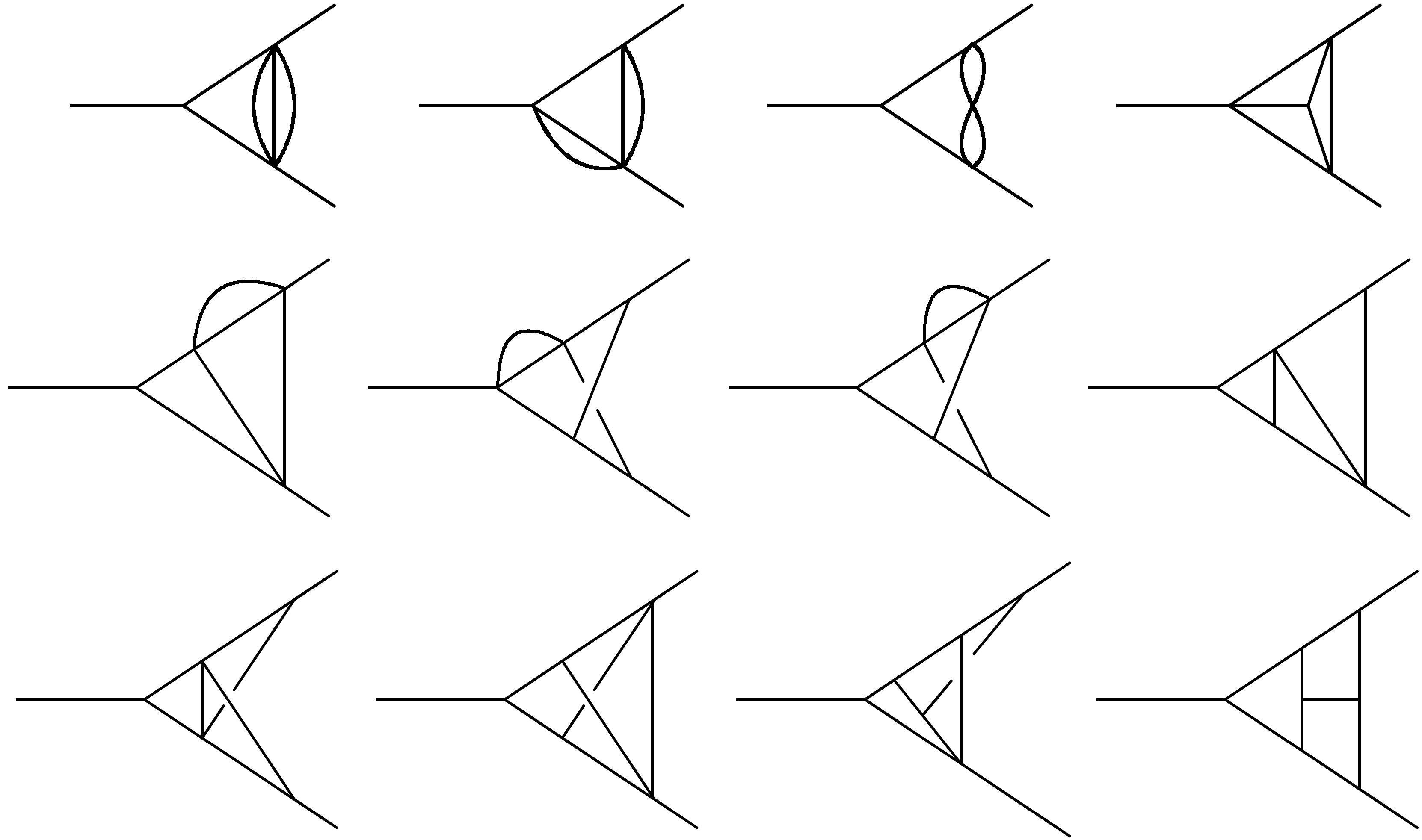
- Can identify the three-loop masters without the full form factor calculation
- Two-particle cuts of massless four-loop propagator-type master integrals *[Gehrmann,Heinrich,Studerus,TH'06]*

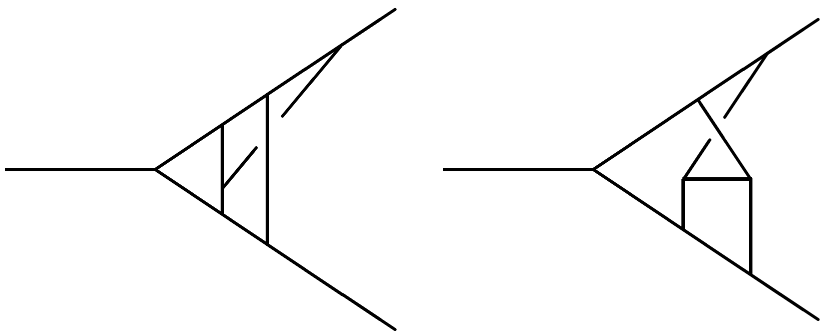


[Baikov'05]

- Each topology contains only one master integral
  - Choose unity for propagator powers
  - Numerator: Unity, or suitably chosen scalar product
- Obtain 22 masters in total
  - 8 of them are two-point functions or factorizable vertex diagrams (all known)
    [Tkachov'81; Chetyrkin,Tkachov'81; Gorishnii,Larin,Surguladze,Tkachov'89; Larin,Tkachov,Vermaseren'91; Bekavac'05]
  - 14 genuine three-loop vertex integrals

Three-loop vertex-type master integrals





Criteria: Number of propagators,
bubble insertions,
planar vs. crossed topologies,
Number of lines at outermost vertices.

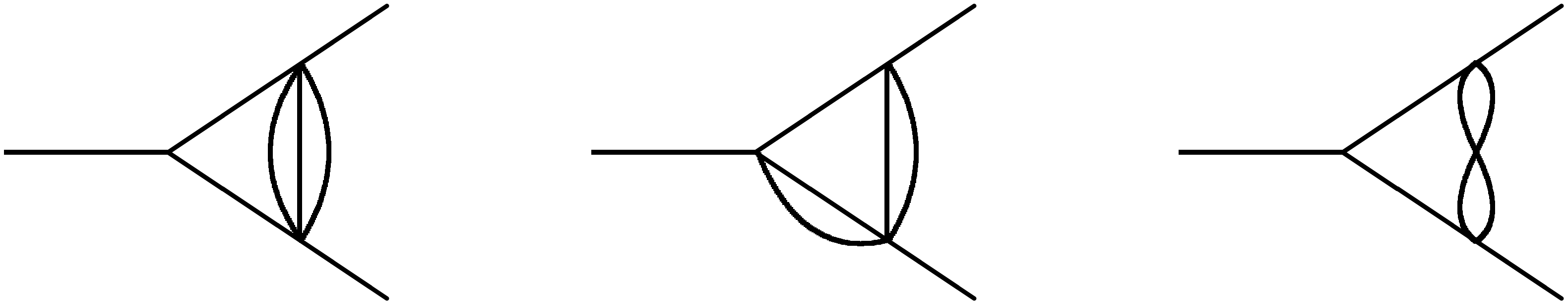
Properties of the masters

- Kinematics: $q = p_1 + p_2, \quad p_1^2 = p_2^2 = 0,$ all propagators massless
  - Hence: Only one scale: q^2, which has to factor out
  - General form of the result:

$$A = i^3 S_\Gamma^3 \left[- q^2 - i\,\eta \right]^{L \cdot D/2 - n_p} \cdot f(\epsilon)$$

- Required: Expansion of $f(\epsilon)$ about $\epsilon = 0$
  - Coefficients have increasing transcendentality T of Riemann ζ-function
  - Need all coefficients with $T \leq 6$, i.e. π^6 and ζ_3^2

Two-loop bubble insertions



$A_{5,1}$ $A_{5,2}$ $A_{6,1}$

- Only Γ-functions in the result
- Results for propagators raised to arbitrary powers ν_i. With $N = \sum_i \nu_i$,

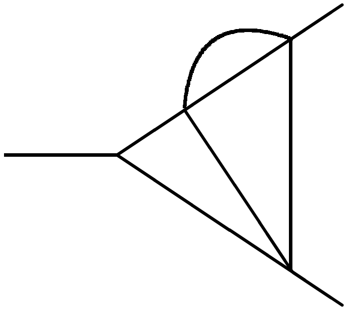
$$A_{5,1}[\nu_i] = \frac{i\,(-1)^{1-N}}{(4\pi)^{3D/2}}\left[-q^2 - i\,\eta\right]^{3D/2-N} \frac{\Gamma(\frac{D}{2}-\nu_3)\,\Gamma(\frac{D}{2}-\nu_4)\,\Gamma(\frac{D}{2}-\nu_5)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_4)\,\Gamma(\nu_5)}$$

$$\times \frac{\Gamma(N-\frac{3D}{2})\,\Gamma(\nu_{345}-D)\,\Gamma(\frac{3D}{2}-N+\nu_1)\,\Gamma(\frac{3D}{2}-N+\nu_2)}{\Gamma(\frac{3D}{2}-\nu_{345})\,\Gamma(2D-N)}$$

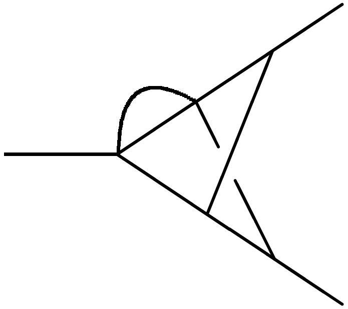
- ϵ-expansion simple

One-loop insertions and $A_{7,4}$

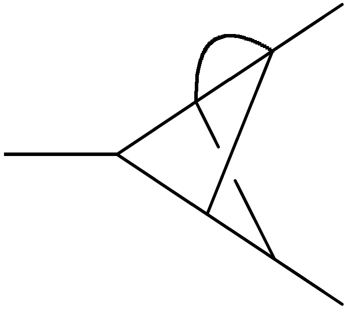
$A_{6,3}$



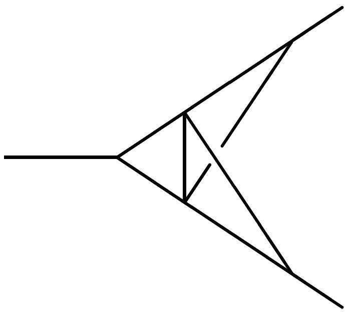
$A_{7,1}$



$A_{7,2}$



$A_{7,4}$



- Linear combination of Γ-functions and hypergeometric functions of unit argument, e. g.

$$
\begin{aligned}
A_{7,4} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D r}{(2\pi)^D} \; \frac{1}{k^2\;(k-q)^2\;(r+l-k)^2\;l^2\;(l-p_1)^2\;r^2\;(r-p_2)^2} \\
&= i\,S_\Gamma^3\,\left[-q^2 - i\eta\right]^{-1-3\,\epsilon} \cdot 2 \cdot \Gamma^4(1-\epsilon)\,\Gamma^2(-\epsilon) \\
&\quad \times \Bigg[\frac{\Gamma(1-\epsilon)\,\Gamma(3\,\epsilon)}{(1-3\,\epsilon)^2\;\Gamma(2-4\,\epsilon)}\;{}_4F_3(1,1-\epsilon,1-3\,\epsilon,2-6\,\epsilon;\,2-3\,\epsilon,2-3\,\epsilon,2-4\,\epsilon;\,1) \\
&\qquad - \frac{\Gamma(1-3\,\epsilon)\,\Gamma(2-3\,\epsilon)\,\Gamma(3\,\epsilon)\,\Gamma(1+2\,\epsilon)}{\Gamma(2-\epsilon)\,\Gamma(2-6\,\epsilon)} \\
&\qquad\quad \times {}_4F_3(1,1,1+2\,\epsilon,2-3\,\epsilon;\,2,2,2-\epsilon;\,1) \\
&\qquad + \frac{\Gamma^2(1-3\,\epsilon)\,\Gamma(1+2\,\epsilon)\,\Gamma(1+3\,\epsilon)}{\Gamma(2-\epsilon)\,\Gamma(2-6\,\epsilon)} \\
&\qquad\quad \times {}_4F_3(1,1,1+2\,\epsilon,1+3\,\epsilon;\,2,2,2-\epsilon;\,1) \Bigg] \\
&= i\,S_\Gamma^3\,\left[-q^2 - i\eta\right]^{-1-3\,\epsilon} \Bigg[\frac{6\,\zeta_3}{\epsilon^2} + \left(\frac{11\,\pi^4}{90} + 36\,\zeta_3\right)\frac{1}{\epsilon} + \left(\frac{11\,\pi^4}{15} + 216\,\zeta_3 - 2\,\pi^2\,\zeta_3\right. \\
&\qquad \left. + 46\,\zeta_5\right) + \left(\frac{22\,\pi^4}{5} - \frac{19\,\pi^6}{270} + 1296\,\zeta_3 - 12\,\pi^2\,\zeta_3 - 282\,\zeta_3^2 + 276\,\zeta_5\right)\epsilon + \mathcal{O}(\epsilon^2) \Bigg]
\end{aligned}
$$

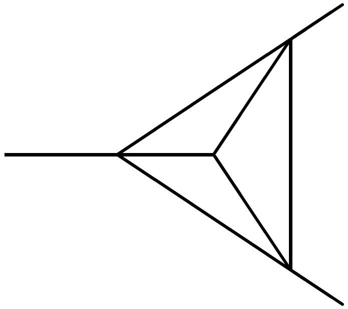
- ϵ-expansion with dedicated package

[Moch,Uwer (XSummer); Maître,TH (HypExp)]

Diagrams with Mellin-Barnes

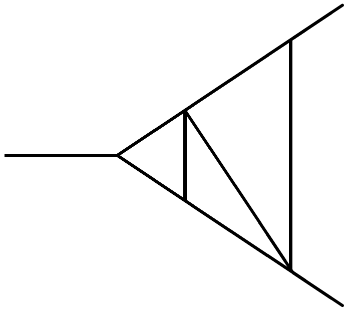
$A_{6,2}$,

$2\text{-}fold\,MB$



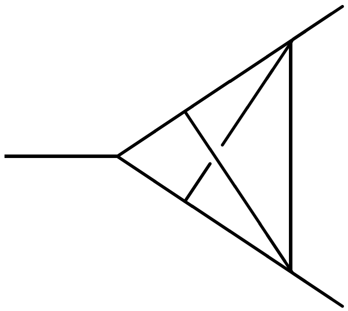
$A_{7,3}$,

$3\text{-}fold\,MB$



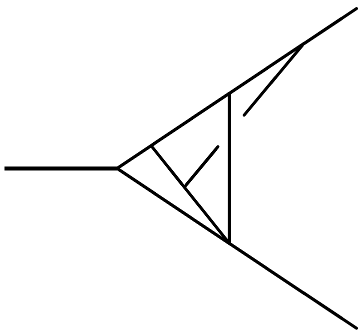
$A_{7,5}$,

$4\text{-}fold\,MB$



A_{8} ,

$4\text{-}fold\,MB$



- Derive multiple Mellin-Barnes representations, compute coefficients of ϵ-expansion order by order

$$A_{6,2} = -i\,S_\Gamma^3\,\left[-\,q^2 - i\,\eta\right]^{-3\,\epsilon}\frac{\Gamma^3(1-\epsilon)\,\Gamma(3\,\epsilon)\,\Gamma^2(1-3\,\epsilon)}{\Gamma(1-2\,\epsilon)\,\Gamma(2-4\,\epsilon)}\int_{c_1-i\,\infty}^{c_1+i\,\infty}\frac{dw_1}{2\pi i}\int_{c_2-i\,\infty}^{c_2+i\,\infty}\frac{dw_2}{2\pi i}$$
$$\times\frac{\Gamma(-1+3\,\epsilon-w_1)\,\Gamma(-1+2\,\epsilon-w_1)\,\Gamma(2-4\,\epsilon+w_1)\,\Gamma(-w_2)\,\Gamma(w_2-w_1)}{\Gamma(3\,\epsilon-w_1)\,\Gamma(2-4\,\epsilon+w_2)\,\Gamma(2-4\,\epsilon+w_1-w_2)}$$
$$\times\,\Gamma(1-\epsilon+w_2)\,\Gamma(1-\epsilon+w_1-w_2)\,\Gamma(1-2\,\epsilon+w_2)\,\Gamma(1-2\,\epsilon+w_1-w_2)$$

  - Analytic continuation to $\epsilon = 0$ *[Czakon'05; Smirnov,Smirnov'09]*
  - Barnes' Lemmas (e.g. Barnesroutines.m) *[Kosower]*

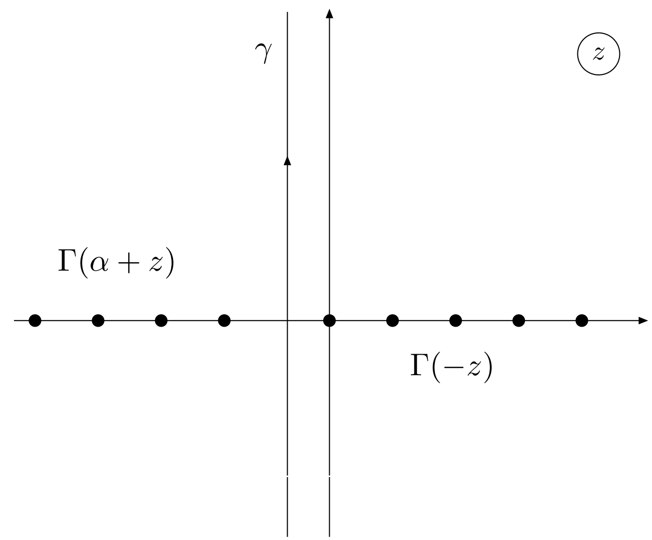
- Important: Numerical checks for all diagrams
  - Sector decomposition method, sub-percent precision (2-6 decimal digits) *[Hepp'66; Binoth,Heinrich'00-'04; Bogner,Weinzierl'07; Smirnov,Tentyukov'08]*
  - MB.m integration routines (4-10 decimal digits) *[Czakon'05; see also Anastasiou,Daleo'06]*

Computational techniques

- Multiple MB representations [Smirnov'99; Tausk'99]

$$\frac{1}{(A_1+A_2)^\alpha} = \int \frac{dz}{2\pi i}\, A_1^z\, A_2^{-\alpha-z}\, \frac{\Gamma(-z)\,\Gamma(\alpha+z)}{\Gamma(\alpha)}$$

- Analytic continuation to $\epsilon = 0$ [Czakon'05; Smirnov,Smirnov'09]
- Barnes' Lemmas (e.g. Barnesroutines.m) [Kosower]

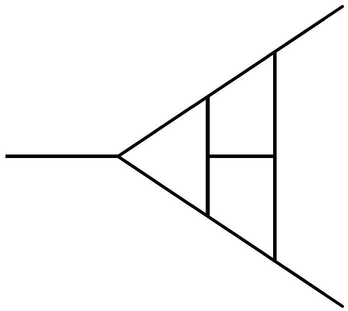


$$\int \frac{dz}{2\pi i}\, \Gamma(\lambda_1+z)\,\Gamma(\lambda_2+z)\,\Gamma(\lambda_3-z)\,\Gamma(\lambda_4-z) = \frac{\Gamma(\lambda_1+\lambda_3)\,\Gamma(\lambda_1+\lambda_4)\,\Gamma(\lambda_2+\lambda_3)\,\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$$

- Automation of 1- and (partially) 2-dim. MB integrations
- Use algebra of harmonic polylogarithms (HPL's) and multiple zeta values (MZV's)
  [Remiddi,Vermaseren'99; Gehrmann,Remiddi'01; Moch,Uwer'05; Maître'05,'07; Blümlein,Broadhurst,Vermaseren'09]
- Nested sums algorithm [Vermaseren'98; Moch,Uwer,Weinzierl'01]
- Series and integral representations of Γ, B, $\psi^{(i)}$, logs and polylogs, ${}_pF_{p-1}$, MeijerG.

$$\psi^{(0)}(z) = -\gamma_E + (1-z)\int_0^1 dt\, t^{z-2}\, \ln(1-t)\,, \qquad \mathrm{Re}(z) > 0$$

The integral $A_{9,1}$



$$A_{9,1} = i\, S_\Gamma^3 \left[-\, q^2 - i\, \eta\right]^{-3-3\,\epsilon} \frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)} \int\frac{dw_1}{2\pi i}\int\frac{dw_2}{2\pi i}\int\frac{dw_3}{2\pi i}\int\frac{dw_4}{2\pi i}\int\frac{dw_5}{2\pi i}\int\frac{dw_6}{2\pi i}$$

$$\times \frac{\Gamma(-w_2)\,\Gamma(2+w_1+w_2)\,\Gamma(-w_3)\,\Gamma(w_3-w_2-w_4)\,\Gamma(-w_4)\,\Gamma(-w_5)\,\Gamma(1+w_3+w_5)}{\Gamma(-w_2-w_4)\,\Gamma(2+w_3+w_5)\,\Gamma(2+w_5+w_6)}$$

$$\times \frac{\Gamma(w_4+w_5-w_1)\,\Gamma(-w_6)\,\Gamma(1+w_5+w_6)\,\Gamma(-2-3\epsilon-w_3-w_5)}{\Gamma(-1-4\epsilon-w_5)\,\Gamma(-1-3\epsilon-w_2)\,\Gamma(3+\epsilon+w_1+w_2)\,\Gamma(3+3\epsilon+w_5)}$$

$$\times\,\Gamma(-1-2\epsilon-w_2-w_4)\,\Gamma(-2-3\epsilon-w_5-w_6)\,\Gamma(2+\epsilon+w_1+w_2)\,\Gamma(w_1-w_5-\epsilon)$$

$$\times\,\Gamma(3+2\epsilon+w_2+w_4+w_5+w_6)\,\Gamma(3+3\epsilon+w_3+w_5)\,\Gamma(3+3\epsilon+w_5+w_6)$$

$$\times\,\Gamma(-1-\epsilon-w_1)\,\Gamma(-1-\epsilon-w_2)$$

- Possesses a six-fold MB representation
- Analytic continuation to $\epsilon = 0$ requires $\mathcal{O}(200)$ iterations
- All calculational steps are carried out purely analytically

$$\begin{aligned} A_{9,1} \quad = \quad & i\, S_\Gamma^3 \left[-\, q^2 - i\, \eta\right]^{-3-3\,\epsilon} \times \Big[-\frac{1}{18\epsilon^5} + \frac{1}{2\epsilon^4} + \Big(-\frac{53}{18} - \frac{4\pi^2}{27}\Big)\frac{1}{\epsilon^3} + \Big(\frac{29}{2} + \frac{22\pi^2}{27} - 2\zeta_3\Big)\frac{1}{\epsilon^2} \\ & + \Big(-\frac{129}{2} - \frac{8\pi^2}{3} + \frac{158}{9}\zeta_3 - \frac{20\pi^4}{81}\Big)\frac{1}{\epsilon} + \Big(\frac{537}{2} + 6\pi^2 - \frac{578}{9}\zeta_3 + \frac{322\pi^4}{405} - \frac{14}{3}\pi^2\zeta_3 - \frac{238}{3}\zeta_5\Big) \\ & + \Big(-\frac{2133}{2} - 4\pi^2 + 158\zeta_3 - \frac{302\pi^4}{135} - \frac{26}{3}\pi^2\zeta_3 + \frac{826}{3}\zeta_5 - \frac{2398\pi^6}{5103} - \frac{466}{3}\zeta_3^2\Big)\epsilon + \mathcal{O}(\epsilon^2)\Big] \end{aligned}$$

- Derived also seven-folds with AMBRE and from BDS. Numerical checks with MB.m

[Gluza,Kajda,Riemann'07; Bern,Dixon,Smirnov'05]

$A_{9,1}^{(n)}$, homogeneous weight!

- Consider $A_{9,1}$ with an irreducible numerator, $A_{9,1}^{(n)}$

$$A_{9,1}^{(n)} = \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D r}{(2\pi)^D} \frac{r^2}{k^2\ (k+p_1)^2\ (k+l)^2\ (k-r)^2\ (l+r)^2\ (l+p_2)^2\ l^2\ (r+p_1)^2\ (r-p_2)^2}$$

- Derive a seven-fold MB representation from BDS [Bern,Dixon,Smirnov'05]
- Reduce everything to at most two-folds with Barnes' Lemmas
- Observe that the coeffcients in $A_{9,1}^{(n)}$ have homogeneous weight!
- Apply PSLQ algorithm, need $20 - 25$ digits [Ferguson,Bailey,Arno'99]

$$A_{9,1}^{(n)} = i\, S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-2-3\,\epsilon} \Big[-\frac{1}{36\epsilon^6} - \frac{\pi^2}{18\epsilon^4} - \frac{14\zeta_3}{9\epsilon^3} - \frac{47\pi^4}{405\epsilon^2} + \left(-\frac{85}{27}\pi^2\zeta_3 - 20\zeta_5\right)\frac{1}{\epsilon} + \left(-\frac{1160\pi^6}{5103} - \frac{137}{3}\zeta_3^2\right) + \mathcal{O}(\epsilon)\Big]$$

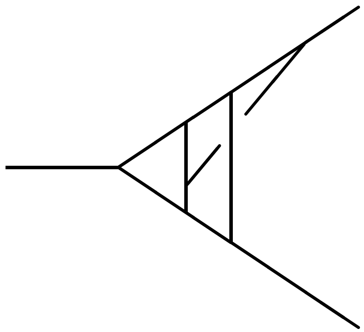
Relation between $A_{9,1}$ and $A_{9,1}^{(n)}$

- $A_{9,1}$ and $A_{9,1}^{(n)}$ are related since only one of them is a master
- Perform Laporta reduction with `AIR` and `FIRE` [Laporta'01; Anastasiou,Lazopoulos'04; Smirnov'08]

$$
\begin{aligned}
A_{9,1}^{(n)} \quad = \quad & \frac{8(2D-7)(2D-5)(3D-10)(3D-8)(D-3)(1311D^2-11764D+26396)}{9(D-4)^5(3D-14)(5D-22)\left(q^2\right)^4} A_4 \\
& + \frac{80(2D-7)(3D-14)(3D-8)(D-3)^2}{3(D-4)^4(5D-22)\left(q^2\right)^3} A_{5,1} \\
& - \frac{64(2D-7)(D-3)^3(69D^2-580D+1220)}{9(D-4)^4(3D-14)(5D-22)\left(q^2\right)^3} A_{5,2} \\
& + \frac{8(3D-14)(3D-10)(3D-8)(D-3)^2}{(D-4)^4(5D-22)\left(q^2\right)^3} A_{5,1}^{(M)} - \frac{32(2D-7)(D-3)^3(45D-202)}{3(D-4)^4(5D-22)\left(q^2\right)^3} A_{5,2}^{(M)} \\
& + \frac{64(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)\left(q^2\right)^2} A_{6,1} - \frac{20(2D-7)(5D-18)}{9(D-4)^2\left(q^2\right)^2} A_{6,2} \\
& + \frac{8(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)\left(q^2\right)^2} A_{6,3} - \frac{2(3D-14)}{(5D-22)q^2} A_{7,3} - \frac{(3D-14)^2 q^2}{2(D-4)(5D-22)} A_{9,1}
\end{aligned}
$$

- Relation fulfilled ✓

$A_{9,2}$



- Possesses a six-fold MB representation
- Also here: Find $A_{9,2}^{(n)}$ of homogeneous weight, compute $A_{9,2}$ and $A_{9,2}^{(n)}$ independently, use PSLQ only for $A_{9,2}^{(n)}$
- No independent MB representation of $A_{9,2}^{(n)}$, use representation of $A_{9,2}$ and Laporta reduction

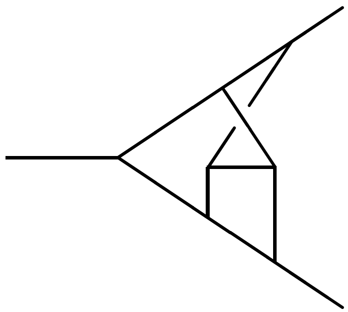
- So far, finite part only numerically

$$
\begin{aligned}
A_{9,2} \quad = \quad & i\, S_\Gamma^3 \left[- q^2 - i\,\eta\right]^{-3-3\,\epsilon} \\
& \times \Big[\frac{2}{9\epsilon^6} + \frac{5}{6\epsilon^5} + \Big(- \frac{20}{9} - \frac{7\pi^2}{27} \Big) \frac{1}{\epsilon^4} + \Big(\frac{50}{9} - \frac{17\pi^2}{27} - \frac{91}{9}\zeta_3 \Big) \frac{1}{\epsilon^3} \\
& \quad + \Big(- \frac{110}{9} + \frac{4\pi^2}{3} - \frac{166}{9}\zeta_3 - \frac{373\pi^4}{1080} \Big) \frac{1}{\epsilon^2} \\
& \quad + \Big(\frac{170}{9} - \frac{16\pi^2}{9} + \frac{494}{9}\zeta_3 - \frac{187\pi^4}{540} + \frac{179}{27}\pi^2\zeta_3 - 167\zeta_5 \Big) \frac{1}{\epsilon} \\
& \quad + (-670.0785 \pm 0.0326) + \mathcal{O}(\epsilon) \Big]
\end{aligned}
$$

$A_{9,4}$

$A_{9,4}$,

6-$fold\,MB$

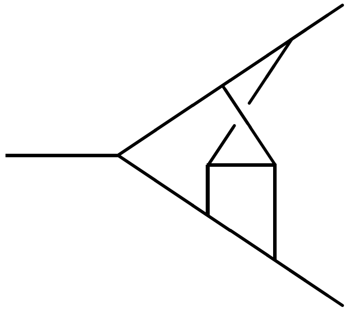


- Also possesses a six-fold MB representation
- Like before: Find $A_{9,4}^{(n)}$ of homogeneous weight, compute $A_{9,4}$ and $A_{9,4}^{(n)}$ independently, use PSLQ only for $A_{9,4}^{(n)}$
- No independent MB representation of $A_{9,4}^{(n)}$, use representation of $A_{9,4}$ and Laporta reduction

- So far, finite part and simple pole only numerically

$$
\begin{aligned}
A_{9,4} \quad = \quad & i\, S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-3-3\,\epsilon} \\
& \times \Big[\frac{1}{9\epsilon^6} + \frac{8}{9\epsilon^5} + \Big(-1 - \frac{10\pi^2}{27} \Big) \frac{1}{\epsilon^4} + \Big(-\frac{14}{9} - \frac{47\pi^2}{27} - 12\zeta_3 \Big) \frac{1}{\epsilon^3} \\
& \quad + \Big(17 + \frac{71\pi^2}{27} - \frac{200}{3}\zeta_3 - \frac{47\pi^4}{810} \Big) \frac{1}{\epsilon^2} \\
& \quad + \Big(117.3999538 \pm 0.0000032 \Big) \frac{1}{\epsilon} \\
& \quad + (1948.167043 \pm 0.000025) + \mathcal{O}(\epsilon) \Big]
\end{aligned}
$$

$A_{9,4}$



- Also possesses a six-fold MB representation
- Like before: Find $A_{9,4}^{(n)}$ of homogeneous weight, compute $A_{9,4}$ and $A_{9,4}^{(n)}$ independently, use PSLQ only for $A_{9,4}^{(n)}$
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$$
\begin{aligned}
A_{9,4} \quad = \quad & i\, S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-3-3\,\epsilon} \\
& \times \Big[\frac{1}{9\epsilon^6} + \frac{8}{9\epsilon^5} + \Big(-1 - \frac{10\pi^2}{27}\Big)\frac{1}{\epsilon^4} + \Big(-\frac{14}{9} - \frac{47\pi^2}{27} - 12\zeta_3\Big)\frac{1}{\epsilon^3} \\
& \quad + \Big(17 + \frac{71\pi^2}{27} - \frac{200}{3}\zeta_3 - \frac{47\pi^4}{810}\Big)\frac{1}{\epsilon^2} \\
& \quad + \Big(-84 - \pi^2 - \frac{671\,\pi^4}{540} + \frac{940\,\zeta_3}{9} + \frac{652\,\pi^2\,\zeta_3}{27} - \frac{692}{9}\zeta_5\Big)\frac{1}{\epsilon} \\
& \quad + (1948.167043 \pm 0.000025) + \mathcal{O}(\epsilon) \Big]
\end{aligned}
$$

[from Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

Conclusion and Outlook

- We computed all vertex-type master integrals for 3-loop form factors in massless QCD
- Extraction of poles up to $1/\epsilon^6$
- Calculation requires several techniques: Γ, ${}_pF_q$, Mellin-Barnes $\ldots$
- So far, all results derived by purely analytic steps!
- All but two coefficients known analytically
- Independent numerical checks performed with sector decomposition

- $A_{9,2}$: Only $\mathcal{O}(20)$ terms left, but all four or five-fold
- $A_{9,4}$: $\mathcal{O}(10^4)$ terms left, at most three-folds
- here PSLQ will be unavoidable, but numerical precision not yet good enough

Backup slides

MB representation of $A_{9,2}$

$$
\begin{aligned}
A_{9,2} \quad = \quad & i\, S_\Gamma^3 \, \left[- \, q^{\,2} - i\,\eta\right]^{-3-3\,\epsilon} \frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)} \int\frac{dw_1}{2\pi i}\int\frac{dw_2}{2\pi i}\int\frac{dw_3}{2\pi i}\int\frac{dw_4}{2\pi i}\int\frac{dw_5}{2\pi i}\int\frac{dw_6}{2\pi i} \\
& \times \frac{\Gamma(-w_1)\,\Gamma(2+w_1+w_2)\,\Gamma(-w_5)\,\Gamma(-w_2+w_3+w_4+1)\,\Gamma(w_5-w_2)\,\Gamma(w_5-w_4)}{\Gamma(1-w_4+w_5)\,\Gamma(-w_1-w_3+w_6)\,\Gamma(1-w_2+w_5+w_6)\,\Gamma(2-\epsilon+w_1+w_3+w_4)} \\
& \times \frac{\Gamma(-w_6)\,\Gamma(w_6+1)\,\Gamma(w_6-w_3)\,\Gamma(1+w_5+w_6)\,\Gamma(-2-2\epsilon-w_1-w_3)}{\Gamma(2-2\epsilon+w_1+w_3+w_4)\,\Gamma(-1-w_1-3\epsilon)\,\Gamma(1-\epsilon+w_4-w_5)\,\Gamma(3+\epsilon+w_1+w_2)} \\
& \times \Gamma(-1-\epsilon-w_2)\,\Gamma(1-\epsilon+w_1+w_3)\,\Gamma(w_2-w_4-\epsilon)\,\Gamma(1+w_4-\epsilon)\,\Gamma(-1-\epsilon-w_1) \\
& \times \Gamma(1-\epsilon+w_1+w_3+w_4-w_5-w_6)\,\Gamma(\epsilon-w_1-w_3-w_4+w_5+w_6) \\
& \times \Gamma(w_4-w_5-\epsilon)\,\Gamma(2+\epsilon+w_1+w_2)\,\Gamma(3+2\epsilon+w_1+w_3+w_4)
\end{aligned}
$$

MB representation of $A_{9,4}$

$$
\begin{aligned}
A_{9,4} \quad = \quad & i\, S_\Gamma^3 \, \left[- q^{\,2} - i\,\eta\right]^{-3-3\,\epsilon} \frac{\Gamma^3(1-\epsilon)}{\Gamma(-2\,\epsilon)\Gamma(-1-4\,\epsilon)} \\
& \times \int \frac{dw_1}{2\pi i} \int \frac{dw_2}{2\pi i} \int \frac{dw_3}{2\pi i} \int \frac{dw_4}{2\pi i} \int \frac{dw_5}{2\pi i} \int \frac{dw_6}{2\pi i} \\
& \times \frac{\Gamma(-w_1)\,\Gamma(1+w_1+w_2)\,\Gamma(-w_3)\,\Gamma(1-w_1+w_3)\,\Gamma(w_3-w_2)\,\Gamma(1+w_4)\,\Gamma(1+w_5)}{\Gamma(1-w_1)\,\Gamma(w_1+w_2-w_3-w_4+w_5-2\epsilon)\,\Gamma(1-2\epsilon+w_1+w_2)} \\
& \times \frac{\Gamma(-w_5)\,\Gamma(w_4-w_5+1)\,\Gamma(w_5-w_4)\,\Gamma(-w_6)\,\Gamma(1+w_3+w_4+w_6-w_5)}{\Gamma(2-w_1+w_3+w_4)\,\Gamma(1-w_2+w_3+w_4-w_5)\,\Gamma(2+w_3+w_4+w_6)} \\
& \times \Gamma(-2-3\epsilon-w_4)\,\Gamma(w_1+w_2-w_3-2\epsilon)\,\Gamma(-w_1-\epsilon)\,\Gamma(w_1-\epsilon)\,\Gamma(-1-\epsilon-w_2) \\
& \times \Gamma(-2-3\epsilon-w_3-w_4+w_5-w_6)\,\Gamma(1+\epsilon-w_1-w_2+w_3+w_4) \\
& \times \Gamma(1+w_2-\epsilon)\,\Gamma(2+\epsilon+w_1+w_2+w_6)\,\Gamma(3+3\epsilon+w_3+w_4+w_6)
\end{aligned}
$$