OPE and the NLL-resummed evolution for B-meson distribution amplitude

radiative and nonperturbative corrections to inverse moments for exclusive B decays

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with H. Kawamura (Univ. of Liverpool)

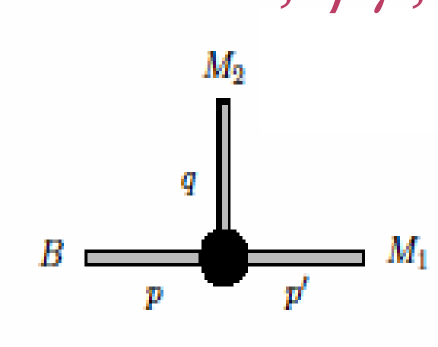
PLB673 ('09) 201 and more

QCD factorization for Exclusive B decays

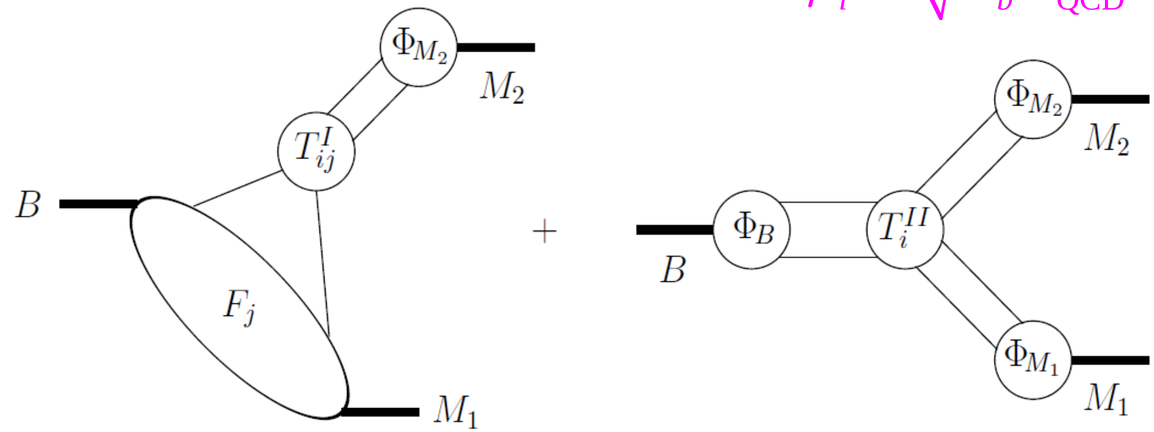
Beneke, Buchalla, Neubert, Sachrajda ('99)
Bauer, Pirjol, Stewart ('01)

$B \to \pi\pi,\ \rho\gamma,\ \pi l \nu, \cdots$

$\mu_i = \sqrt{m_b \Lambda_{\rm QCD}}$



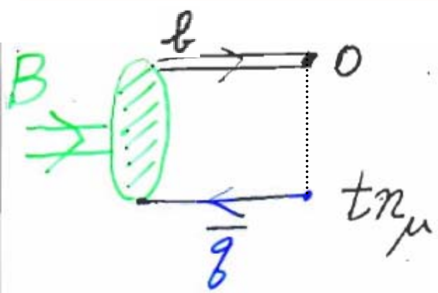
$m_b \to \infty$



B meson's LCDA in HQET

$b(x) = e^{-im_b v\cdot x} h_v(x) + O(1/m_b),\quad \not{v} h_v(x) = h_v(x)$

$$\tilde{\phi}_+(t,\mu) = \frac{1}{iF(\mu)} \langle 0| \left[\bar{q}(tn)\mathcal{P}\exp\left(ig\int_0^t d\lambda n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)\right]_\mu |\bar{B}(v)\rangle = \int d\omega e^{-i\omega t}\phi_+(\omega,\mu)$$



$n^\mu = (1,0,0,-1) \quad (n^2 = 0)$

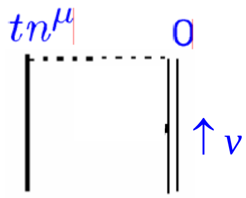
$p^\mu = m_B v^\mu \quad (v^2 = 1)$

$k^+ = \omega v^+$

$F(\mu)$: decay constant

$\langle 0|\bar{q}\not{n}\gamma_5 h_v|\bar{B}(v)\rangle = iF(\mu)$

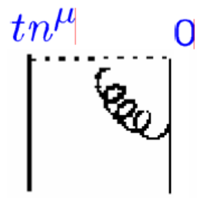
$$\bar{q}(tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0) \;=\; \sum_{k=0}^{\infty}\frac{t^k}{k!}\bar{q}(0)(\overleftarrow{D}\cdot n)^k\,\not{n}\gamma_5 h_v(0)$$



$$\left[\bar{q}(tn)\,P\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 q(0)\right]_\mu = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\bar{q}(0)(\overleftarrow{D}\cdot n)^k\,\not{n}\gamma_5 q(0)\right]_\mu$$

Feynman gauge

$d = 4 - 2\varepsilon$



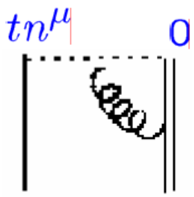
$$\sim \int d^{4-2\varepsilon}q\,\frac{1}{q^4} \sim \frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}$$

$$\left[\bar{q}(tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)\right]_\mu \neq \sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\bar{q}(0)(\overleftarrow{D}\cdot n)^k\,\not{n}\gamma_5 h_v(0)\right]_\mu$$

$$h_v(0) = \mathrm{P}\exp\left(ig\int_{-\infty}^{0} ds\, v_\mu A^\mu(sv)\right)h_v(-\infty v)$$

Feynman gauge

$d = 4 - 2\varepsilon$



$$= -\frac{\alpha_s C_F}{2\pi}\left(\frac{1}{2\varepsilon_{UV}^2} + \frac{\log(it\mu)}{\varepsilon_{UV}} + \log^2(it\mu) + \frac{5\pi^2}{24}\right)\bar{q}(tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)$$

$$= -\frac{\alpha_s C_F}{2\pi}\left(\frac{1}{2\varepsilon_{IR}} + \log(it\mu)\right)\int_0^1 d\xi\, \bar{q}(\xi tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0) + \cdots$$

$\bar{q} D\, \Gamma\, h_v$

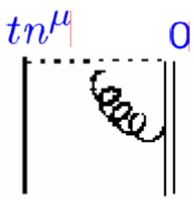
$\{\bar{q} D\, D\, \Gamma\, h_v,\ \bar{q} G\, \Gamma\, h_v\}$

$$\left[\bar{q}(tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right) \not{n}\gamma_5 h_v(0)\right]_\mu \neq \sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\bar{q}(0)(\overleftarrow{D}\cdot n)^k \not{n}\gamma_5 h_v(0)\right]_\mu$$

$$h_v(0) = \mathrm{P}\exp\left(ig\int_{-\infty}^{0} ds\, v_\mu A^\mu(sv)\right) h_v(-\infty v)$$

Feynman gauge

$d = 4 - 2\varepsilon$



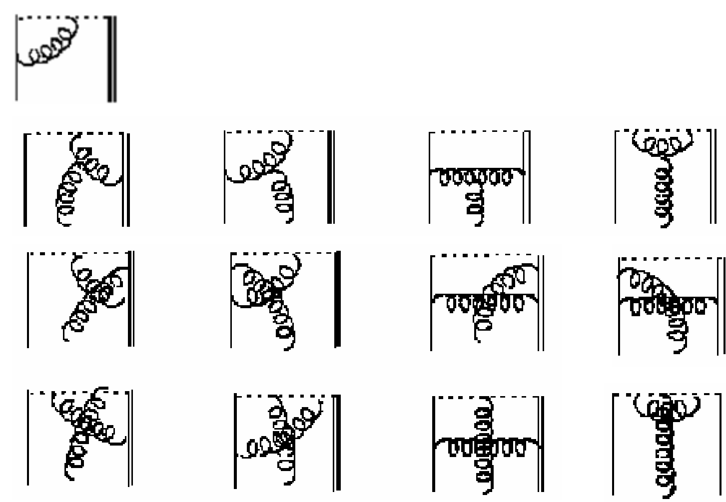
$$= -\frac{\alpha_s C_F}{2\pi}\left(\frac{1}{2\varepsilon_{UV}^2} + \frac{\log(it\mu)}{\varepsilon_{UV}} + \log^2(it\mu) + \frac{5\pi^2}{24}\right)\bar{q}(tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)$$

$$= -\frac{\alpha_s C_F}{2\pi}\left(\frac{1}{2\varepsilon_{IR}} + \log(it\mu)\right)\int_0^1 d\xi\, \bar{q}(\xi tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0) + \cdots$$

$\bar{q} D \Gamma h_v$

$\{\bar{q} D D \Gamma h_v,\ \bar{q} G \Gamma h_v\}$

OPE at $t \lesssim \dfrac{1}{\mu}$



$$\sum_i \tilde{C}_i(t,\mu) O_i(\mu)$$

$\tilde{C}_i$: up to $O(\alpha_s)$, O_i : up to dim.5

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$$\left[\bar{q}(tn)\,\mathrm{P}\exp\left(ig\int_0^t d\lambda\, n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)\right]_\mu$$

$$= \bar{q}\not{n}\gamma_5 h_v\left[1+\frac{\alpha_s}{4\pi}C_F\left(-2L^2-2L-\frac{5}{12}\pi^2\right)\right]$$

dim.3

$$+(-it)\left\{\bar{q}(in\cdot\overleftarrow{D})\not{n}\gamma_5 h_v\left[1+\frac{\alpha_s}{4\pi}C_F\left(-2L^2-L-\frac{5}{12}\pi^2\right)\right]\right.$$

$$\left.+\ \bar{q}(iv\cdot\overleftarrow{D})\not{n}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}C_F(-4L+3)\right]\right\}$$

dim.4

$$+\frac{(-it)^2}{2}\left\{\bar{q}(in\cdot\overleftarrow{D})^2\not{n}\gamma_5 h_v\left[1+\frac{\alpha_s}{4\pi}C_F\left(-2L^2-\frac{2}{3}L-\frac{5}{12}\pi^2\right)\right]\right.$$

dim.5

$$+\ \bar{q}(iv\cdot\overleftarrow{D})(in\cdot\overleftarrow{D})\not{n}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}C_F\left(-4L+\frac{10}{3}\right)\right]$$

$$+\ \bar{q}(iv\cdot\overleftarrow{D})^2\not{n}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}C_F\left(-4L+\frac{10}{3}\right)\right]$$

$$+\ \bar{q}igG_{\mu\nu}v^\mu n^\nu\not{n}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}\left\{C_F\left(-4L+\frac{10}{3}\right)+C_G\left(7L-\frac{13}{2}\right)\right\}\right]$$

$$+\ \bar{q}igG_{\mu\nu}\gamma^\mu n^\nu\not{\bar{n}}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}\left\{C_F\left(-\frac{4}{3}L+\frac{4}{3}\right)+C_G(L-1)\right\}\right]$$

$$+\ \bar{q}igG_{\mu\nu}\gamma^\mu v^\nu\not{\bar{n}}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}\left\{C_F\left(-\frac{2}{3}L+\frac{2}{3}\right)+C_G(L-1)\right\}\right]$$

$$\left.+\ \bar{q}gG_{\mu\nu}\sigma^{\mu\nu}\not{n}\gamma_5 h_v\left[\frac{\alpha_s}{4\pi}\left\{C_F\left(-\frac{L}{3}+\frac{1}{3}\right)+C_G\left(\frac{L}{4}-\frac{1}{4}\right)\right\}\right]\right\}$$

$$L\equiv\log(it\mu e^{\gamma_E})$$

$$C_G = N_c$$

$$v_\mu \equiv \frac{n_\mu+\bar{n}_\mu}{2}$$

$$n^2=\bar{n}^2=0$$

Complete OPE result ($\overline{\mathrm{MS}}$) with $\{C_i\}$ up to $O(\alpha_s)$ and $\{O_i\}$ up to dim.5

double logs due to cusp singularity

LCDA from OPE

$L \equiv \log(it\mu e^{\gamma_E})$

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$$\begin{aligned}\tilde{\phi}_+(t,\mu) &= 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-2L^2 - 2L - \frac{5}{12}\pi^2\right) \\ &+ (-it)\frac{4}{3}\bar{\Lambda}\left[1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-2L^2 - 4L + \frac{9}{4} - \frac{5}{12}\pi^2\right)\right] \\ &+ \frac{(-it)^2}{2}\left(2\bar{\Lambda}^2\left[1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-2L^2 - \frac{16}{3}L + \frac{35}{9} - \frac{5}{12}\pi^2\right)\right]\right. \\ &+ \frac{2}{3}\lambda_E^2(\mu)\left[1 + \frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2 - 2L + \frac{2}{3} - \frac{5}{12}\pi^2\right) + C_G\left(\frac{3}{4}L - \frac{1}{2}\right)\right\}\right] \\ &\left. + \frac{1}{3}\lambda_H^2(\mu)\left[1 + \frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2 - \frac{2}{3} - \frac{5}{12}\pi^2\right) + C_G\left(-\frac{1}{2}L + \frac{1}{2}\right)\right\}\right]\right)\end{aligned}$$

dim.3

dim.4

dim.5

$\overline{\text{MS}}$

HQET parameters

$\bar{\Lambda} = m_B - m_b$

$$\langle 0|\bar{q}(iv\cdot \overleftarrow{D})\not{n}\gamma_5 h_v|\bar{B}(v)\rangle = \frac{3}{4}\langle 0|\bar{q}(in\cdot \overleftarrow{D})\not{n}\gamma_5 h_v|\bar{B}(v)\rangle = iF(\mu)\bar{\Lambda}$$

$$\langle 0|\bar{q}\alpha\cdot \mathbf{gE}\gamma_5 h_v|\bar{B}(v)\rangle = F(\mu)\lambda_E^2(\mu)$$ "Chromo-electronic"

$$\langle 0|\bar{q}\sigma\cdot \mathbf{gH}\gamma_5 h_v|\bar{B}(v)\rangle = iF(\mu)\lambda_H^2(\mu)$$ "Chromo-magnetic"

completely represented by HQET parameters $\bar{\Lambda},\ \lambda_E^2,\ \lambda_H^2$

$\int_0^{\Lambda_{UV}} d\omega\, \omega^N \phi_+(\omega)$: dim.3&4 **terms reproduce the results by Lee & Neubert ('05)**

LCDA from OPE

$L \equiv \log(it\mu e^{\gamma_E})$

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$$
\begin{aligned}
\tilde{\phi}_+(t,\mu) = 1 &+ \frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2-2L-\frac{5}{12}\pi^2\right) && \text{dim.3}\\
&+(-it)\frac{4}{3}\bar{\Lambda}\left[1+\frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2-4L+\frac{9}{4}-\frac{5}{12}\pi^2\right)\right] && \text{dim.4}\\
&+\frac{(-it)^2}{2}\left(2\bar{\Lambda}^2\left[1+\frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2-\frac{16}{3}L+\frac{35}{9}-\frac{5}{12}\pi^2\right)\right]\right. && \text{dim.5}\\
&+\frac{2}{3}\lambda_E^2(\mu)\left[1+\frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2-2L+\frac{2}{3}-\frac{5}{12}\pi^2\right)+C_G\left(\frac{3}{4}L-\frac{1}{2}\right)\right\}\right]\\
&\left.+\frac{1}{3}\lambda_H^2(\mu)\left[1+\frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2-\frac{2}{3}-\frac{5}{12}\pi^2\right)+C_G\left(-\frac{1}{2}L+\frac{1}{2}\right)\right\}\right]\right) && \overline{\text{MS}}
\end{aligned}
$$

For quantitative evaluation: $t \lesssim \frac{1}{\mu} = 1\ \text{GeV}^{-1}$

$$\bar{\Lambda} = \bar{\Lambda}_{\text{DA}}(\mu)\left(1+\frac{7C_F\alpha_s(\mu)}{16\pi}\right) - \mu\frac{9C_F\alpha_s(\mu)}{8\pi}$$

[Lee, Neubert ('05)]

$\bar{\Lambda}_{\text{DA}}(1\ \text{GeV}) \simeq 0.52\ \text{GeV}$ from $B \to X_s\gamma,\ B \to X_u \ell \nu$

$$\lambda_E^2(1\ \text{GeV}) = 0.11 \pm 0.06\ \text{GeV}^2,\ \lambda_H^2(1\ \text{GeV}) = 0.18 \pm 0.07\ \text{GeV}^2$$

from QCD sum rules [Grozin, Neubert ('97)]

$$
\begin{aligned}
\tilde{\phi}_+(-i\tau,\mu) \;=\;& 1+\frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2-2L-\frac{5}{12}\pi^2\right) \\
& -\tau\frac{4}{3}\bar{\Lambda}\left[1+\frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2-4L+\frac{9}{4}-\frac{5}{12}\pi^2\right)\right] \\
& +\frac{\tau^2}{2}\Bigg(2\bar{\Lambda}^2\left[1+\frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2-\frac{16}{3}L+\frac{35}{9}-\frac{5}{12}\pi^2\right)\right] \\
& +\frac{2}{3}\lambda_E^2(\mu)\left[1+\frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2-2L+\frac{2}{3}-\frac{5}{12}\pi^2\right)+C_G\left(\frac{3}{4}L-\frac{1}{2}\right)\right\}\right] \\
& +\frac{1}{3}\lambda_H^2(\mu)\left[1+\frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2-\frac{2}{3}-\frac{5}{12}\pi^2\right)+C_G\left(-\frac{1}{2}L+\frac{1}{2}\right)\right\}\right]\Bigg)
\end{aligned}
$$

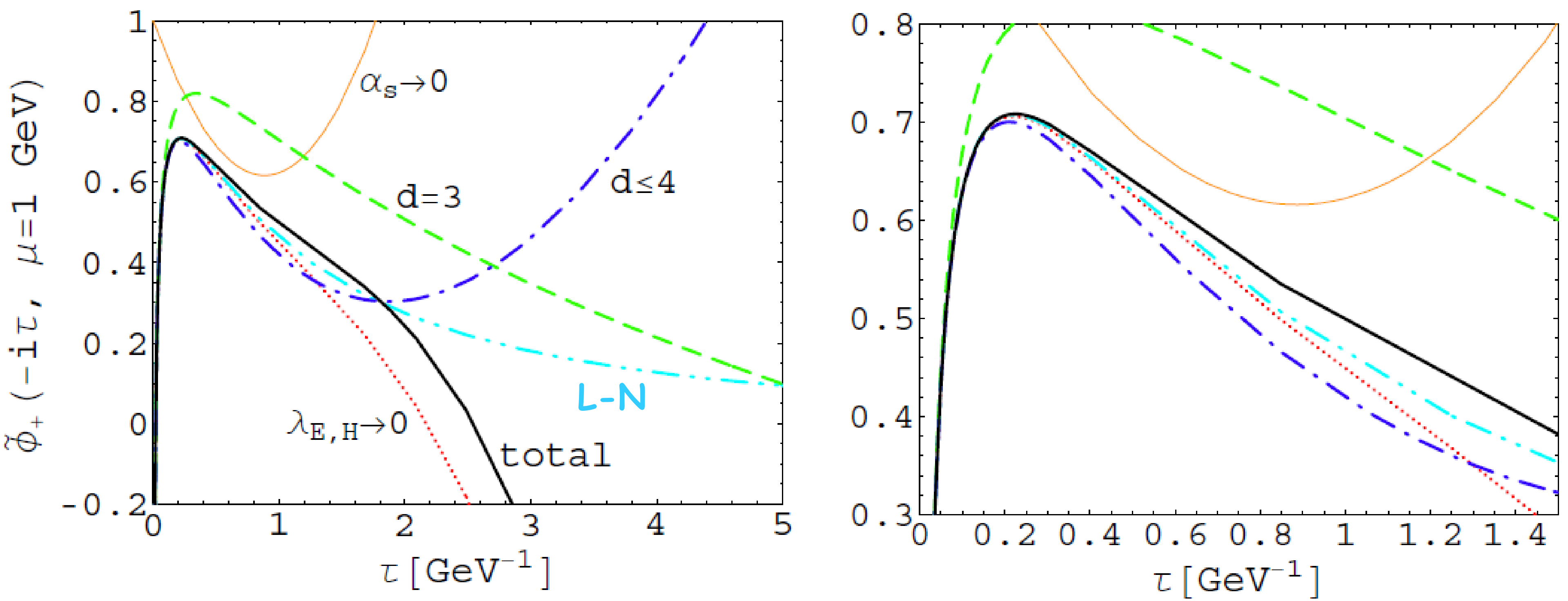
dim.3

dim.4

dim.5

$(t \to -i\tau)$

$L \equiv \log(\tau\mu e^{\gamma_E})$



- NLO perturbative corrections are very large for $\tau \to 0$ and 10-30% level for moderate τ
- Nonperturbative corrections from dim. 5 as well as dim. 4 operators are important (20-30% level)
- Effects from λ_E, λ_H are significant in dim. 5 contributions.

Lee-Neubert's two-component ansatz ('05)

Lee, Neubert, PRD72 ('05) 094028

$$\phi_+^{LN}(\omega,\mu) = N\frac{\omega}{\omega_0^2}e^{-\omega/\omega_0} + \theta(\omega-\omega_t)\frac{C_F\alpha_s}{\pi\omega}\left[\left(\frac{1}{2}-\ln\frac{\omega}{\mu}\right)+\frac{4\bar{\Lambda}_{\rm DA}}{3\omega}\left(2-\ln\frac{\omega}{\mu}\right)\right]$$

$$\omega_t = 2.33\ \mathrm{GeV},\ N = 0.963,\ \omega_0 = 0.438\ \mathrm{GeV} \quad (\mu = 1\ \mathrm{GeV})$$

the first term produces also particular contributions
associated with the operators of dimension d > 4

OPE for cut-moments:

$$\int_0^{\Lambda_{\rm UV}} d\omega\,\omega^N \phi_+(\omega,\mu) = \sum_i \mathscr{C}_i^{(N)}(\Lambda_{\rm UV},\mu)\langle 0|O_i(\mu)|\bar{B}(v)\rangle$$

$\mathscr{C}_i^{(N=0,1)}$: up to $O(\alpha_s)$ $\qquad$ O_i : up to dim.4

$\bar{q}\Gamma h_v \qquad \bar{q}D\Gamma h_v$

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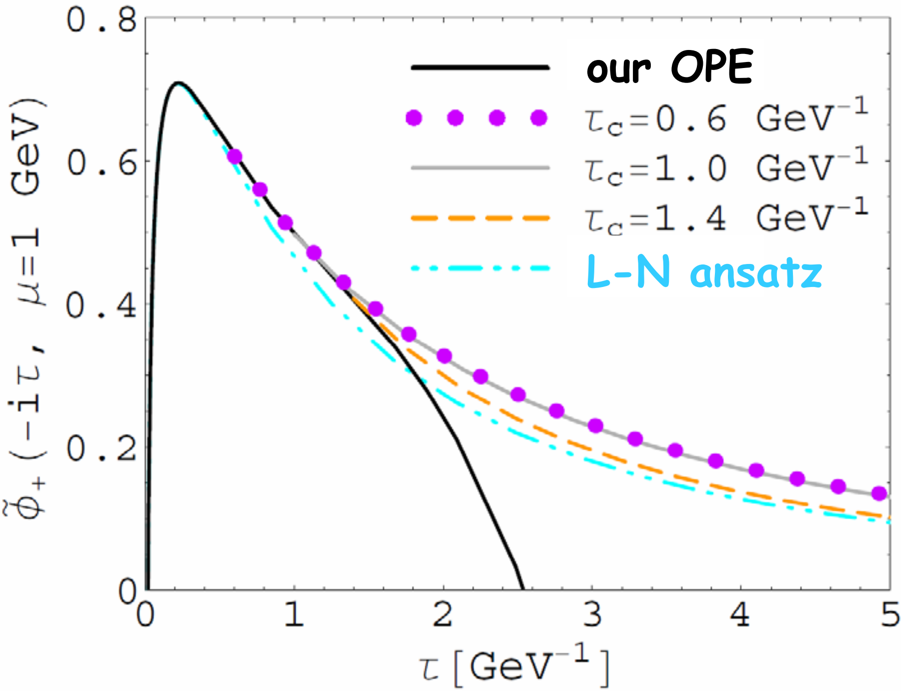


$$\frac{N}{\omega_0^2} = \frac{9}{4\bar{\Lambda}_{DA}^2}\left\{1+\tau_c\bar{\Lambda}_{DA}\left[\frac{\lambda_E^2}{\bar{\Lambda}_{DA}^2}+\frac{\lambda_H^2}{2\bar{\Lambda}_{DA}^2}-1\right]\right\}+\cdots$$

$$\lambda_B^{-1}(\mu) = \int_0^\infty d\omega \frac{\phi_+(\omega,\mu)}{\omega}$$

$$= \int_0^{\tau_c} d\tau \tilde{\phi}_+(-i\tau,\mu) + \int_{\tau_c}^\infty d\tau \tilde{\phi}_+(-i\tau,\mu)$$

continuity at $\tau = \tau_c$

$$\left\{ \tilde{\phi}_+(-i\tau), \quad \frac{\partial \tilde{\phi}_+(-i\tau)}{\partial \tau} \right.$$

OPE up to dim. 5 ops. τ_c

$$\int_0^\infty d\omega e^{-\omega\tau}\, N\frac{\omega}{\omega_0^2}e^{-\omega/\omega_0} = \frac{N}{(\tau\omega_0+1)^2}$$

τ_c [GeV^{-1}]	N	ω_0 [GeV]	$\lambda_B^{-1}(\mu = 1\text{ GeV})$ [GeV^{-1}]
0.4	0.816	0.257	3.11 (0.228 + 2.88)
0.6	0.850	0.306	2.70 (0.356 + 2.35)
0.8	0.852	0.308	2.69 (0.471 + 2.22)
1.0	0.858	0.313	2.66 (0.576 + 2.08)
1.2	0.910	0.349	2.51 (0.671 + 1.84)
1.4	1.09	0.456	2.22 (0.757 + 1.46)
1.6	1.81	0.777	1.87 (0.833 + 1.04)

"stable" behavior for $0.6\text{GeV}^{-1} \lesssim \tau_c \lesssim 1\text{GeV}^{-1}$

$$\lambda_B(\mu=1\text{GeV}) \simeq 0.37\text{ GeV}$$

$$\lambda_B^{LN}(\mu=1\text{GeV}) \simeq 0.48\text{ GeV}$$

$$\lambda_B(\mu = 1\ \text{GeV}) \qquad \left(\lambda_B^{-1}(\mu = 1\ \text{GeV}) = \int_0^\infty d\omega \frac{\phi_+(\omega, \mu = 1\ \text{GeV})}{\omega} = \int_0^\infty d\tau \tilde{\phi}_+(-i\tau, \mu = 1\ \text{GeV}) \right)$$

$\simeq 0.37$ GeV **this work**

$\simeq 0.6$ GeV QCD SR (LO) Ball, Kou ('03)

0.46 ± 0.11 GeV QCD SR (NLO + power corr.) Braun, Ivanov, Korchemsky ('04)

0.48 ± 0.15 GeV OPE up to dim.4 ops., combined with model ansatz Lee, Neubert ('05)

input for QCD factorization formula

0.35 ± 0.15 GeV Beneke, Buchalla, Neubert, Sachrajda ('99), Beneke, Jager ('07)

0.40 ± 0.15 GeV Bell, Pillip ('09)

$$A \Leftarrow H \otimes \tilde{\phi}_+ \Big|_{\mu_i = \sqrt{m_b \Lambda_{\text{QCD}}}} \propto \lambda_B^{-1}(\mu_i)$$

$$H = 1 + O(\alpha_s)$$

Need RG evolution to μ_i

LCDA from OPE

$$L \equiv \log(it\mu e^{\gamma_E})$$

$\overline{\text{MS}}$ scheme

$$\begin{aligned}\tilde{\phi}_+(t,\mu) &= 1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2 - 2L - \frac{5}{12}\pi^2\right) && \text{dim.3}\\
&+(-it)\frac{4}{3}\bar{\Lambda}\left[1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2 - 4L + \frac{9}{4} - \frac{5}{12}\pi^2\right)\right] && \text{dim.4}\\
&+\frac{(-it)^2}{2}\left(2\bar{\Lambda}^2\left[1 + \frac{\alpha_s(\mu)}{4\pi}C_F\left(-2L^2 - \frac{16}{3}L + \frac{35}{9} - \frac{5}{12}\pi^2\right)\right]\right. && \text{dim.5}\\
&+\frac{2}{3}\lambda_E^2(\mu)\left[1 + \frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2 - 2L + \frac{2}{3} - \frac{5}{12}\pi^2\right) + C_G\left(\frac{3}{4}L - \frac{1}{2}\right)\right\}\right]\\
&\left.+\frac{1}{3}\lambda_H^2(\mu)\left[1 + \frac{\alpha_s(\mu)}{4\pi}\left\{C_F\left(-2L^2 - \frac{2}{3} - \frac{5}{12}\pi^2\right) + C_G\left(-\frac{1}{2}L + \frac{1}{2}\right)\right\}\right]\right)\end{aligned}$$

$$\bar{\Lambda} = m_B - m_b, \quad \langle 0|\bar{q}\alpha\cdot g\mathbf{E}\gamma_5 h_v|\bar{B}(v)\rangle = F(\mu)\lambda_E^2(\mu)\ , \quad \langle 0|\bar{q}\sigma\cdot g\mathbf{H}\gamma_5 h_v|\bar{B}(v)\rangle = iF(\mu)\lambda_H^2(\mu)$$

$$\mu\frac{d\bar{\Lambda}}{d\mu} = 0, \qquad \mu\frac{d}{d\mu}\begin{pmatrix}\lambda_E^2(\mu)\\ \lambda_H^2(\mu)\end{pmatrix} = -\frac{\alpha_s}{4\pi}\begin{pmatrix}\frac{8}{3}C_F + \frac{3}{2}C_G & \frac{4}{3}C_F - \frac{3}{2}C_G\\ \frac{4}{3}C_F - \frac{3}{2}C_G & \frac{8}{3}C_F + \frac{5}{2}C_G\end{pmatrix}\begin{pmatrix}\lambda_E^2(\mu)\\ \lambda_H^2(\mu)\end{pmatrix}$$

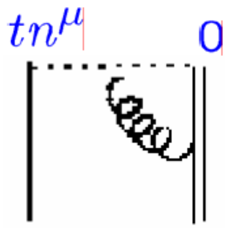
Grozin, Neubert ('97)

$$\mu\frac{d}{d\mu}\tilde{\phi}_+(t,\mu) = \frac{\alpha_s}{4\pi}\left[-\left(4C_F L + 2C_F\right)\tilde{\phi}_+(t,\mu) + \int_0^1 dz\, 4C_F\left(\frac{z}{1-z}\right)_+\tilde{\phi}_+(zt,\mu)\right]$$

$t \lesssim \frac{1}{\mu}$ ➡ valid also for $t \gtrsim \frac{1}{\mu}$

$$\int_0^1 dz\left(\frac{z}{1-z}\right)_+ f(z) \equiv \int_0^1 dz\,\frac{z\left[f(z) - f(1)\right]}{1-z}$$

$$\mu \frac{d}{d\mu}\tilde{\phi}_+(t,\mu) = -\left[\Gamma_{\rm cusp}(\alpha_s) L + \gamma_F(\alpha_s)\right]\tilde{\phi}_+(t,\mu) + \int_0^1 dz K(z,\alpha_s)\tilde{\phi}_+(zt,\mu)$$



$$\langle 0|\bar{q}\not{n}\gamma_5 h_v|\bar{B}(v)\rangle = iF(\mu)$$

$$\Gamma_{\rm cusp}(\alpha_s) = \Gamma^{(1)}_{\rm cusp}\frac{\alpha_s}{4\pi}$$

$$\Gamma^{(1)}_{\rm cusp} = 4C_F$$

$$\gamma_F(\alpha_s) = \gamma_F^{(1)}\frac{\alpha_s}{4\pi}$$

$$\gamma_F^{(1)} = 2C_F$$

$$K(z,\alpha_s) = K^{(1)}(z)\frac{\alpha_s}{4\pi}$$

$$K^{(1)}(z) = 4C_F\left(\frac{z}{1-z}\right)_+$$

"DGLAP"

"cusp"

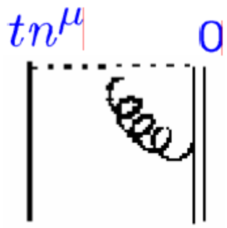
$$L = \log(it\mu e^{\gamma_E}) \sim \log\frac{\mu}{\Lambda_{\rm QCD}} \sim \frac{1}{\alpha_s}$$

$$t \sim \frac{1}{\Lambda_{\rm QCD}}$$

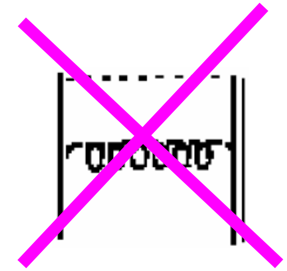
$$P\exp\left[ig\int_{-\infty}^{0} ds\, v_\mu A^\mu(sv)\right] h_v(-\infty v)$$

$$\tilde{\phi}_+(t,\mu) = \frac{1}{iF(\mu)}\langle 0|[\bar{q}(tn)\mathcal{P}\exp\left(ig\int_0^t d\lambda n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)]_\mu|\bar{B}(v)\rangle$$

$$\mu \frac{d}{d\mu}\tilde{\phi}_+(t,\mu) = -\left[\Gamma_{\rm cusp}(\alpha_s) L + \gamma_F(\alpha_s)\right]\tilde{\phi}_+(t,\mu) + \int_0^1 dz K(z,\alpha_s)\tilde{\phi}_+(zt,\mu)$$



$$\langle 0|\bar{q}\not{n}\gamma_5 h_v|\bar{B}(v)\rangle = iF(\mu)$$



$$\Gamma_{\rm cusp}(\alpha_s) = \Gamma^{(1)}_{\rm cusp}\frac{\alpha_s}{4\pi} + \Gamma^{(2)}_{\rm cusp}\left(\frac{\alpha_s}{4\pi}\right)^2$$

$$\gamma_F(\alpha_s) = \gamma_F^{(1)}\frac{\alpha_s}{4\pi}$$

$$K(z,\alpha_s) = K^{(1)}(z)\frac{\alpha_s}{4\pi}$$

$$\Gamma^{(1)}_{\rm cusp} = 4C_F$$

$$\gamma_F^{(1)} = 2C_F$$

$$K^{(1)}(z) = 4C_F\left(\frac{z}{1-z}\right)_+$$

$$\Gamma^{(2)}_{\rm cusp} = 4C_F\left[\left(\frac{67}{9}-\frac{\pi^2}{3}\right)C_G - \frac{10}{9}N_f\right]$$

"DGLAP"

"cusp"

$$L = \log(it\mu e^{\gamma_E}) \sim \log\frac{\mu}{\Lambda_{\rm QCD}} \sim \frac{1}{\alpha_s}$$

$$t \sim \frac{1}{\Lambda_{\rm QCD}}$$

$$\mathrm{P}\exp\left(ig\int_{-\infty}^{0} ds v_\mu A^\mu(sv)\right) h_v(-\infty v)$$

$$\tilde{\phi}_+(t,\mu) = \frac{1}{iF(\mu)}\langle 0|[\bar{q}(tn)\mathcal{P}\exp\left(ig\int_0^t d\lambda n_\mu A^\mu(\lambda n)\right)\not{n}\gamma_5 h_v(0)]_\mu|\bar{B}(v)\rangle$$

$$\mu\frac{d}{d\mu}\tilde{\phi}_+(t,\mu) = -\left[\Gamma_{\text{cusp}}(\alpha_s)L + \gamma_F(\alpha_s)\right]\tilde{\phi}_+(t,\mu) + \int_0^1 dz K(z,\alpha_s)\tilde{\phi}_+(zt,\mu)$$

$$\Gamma_{\text{cusp}}(\alpha_s) = \Gamma^{(1)}_{\text{cusp}}\frac{\alpha_s}{4\pi} + \Gamma^{(2)}_{\text{cusp}}\left(\frac{\alpha_s}{4\pi}\right)^2$$

$$\Gamma^{(1)}_{\text{cusp}} = 4C_F$$

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"DGLAP"

$$L = \log(it\mu e^{\gamma_E})$$

Evolution kernel is "quasilocal" in the coordinate space!

Braun, Ivanov, Korchemsky ('04)

Momentum space: $\phi_+(\omega,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{i\omega t}\tilde{\phi}_+(t,\mu)$ Lange, Neubert ('03); Lee, Neubert ('05)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{i\omega t} L\tilde{\phi}_+(t,\mu) = -\phi_+(\omega,\mu)\log\frac{\omega}{\mu} + \int_0^{\omega} d\omega' \frac{\phi_+(\omega',\mu)-\phi_+(\omega,\mu)}{\omega'-\omega}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{i\omega t}\int_0^1 dz\left(\frac{z}{1-z}\right)_+ \tilde{\phi}_+(zt,\mu) = \phi_+(\omega,\mu) + \int_{\omega}^{\infty} d\omega' \frac{\omega}{\omega'}\frac{\phi_+(\omega',\mu)-\phi_+(\omega,\mu)}{\omega'-\omega}$$

Moment space: Taylor expansion about $t=0$ suffered by additional divergence

$t \to -i\tau$

$L = \log(\tau\mu e^{\gamma_E})$

$$\mu \frac{d}{d\mu}\tilde{\phi}_+(-i\tau,\mu) = -\left[\Gamma_{\rm cusp}(\alpha_s) L + \gamma_F(\alpha_s)\right]\tilde{\phi}_+(-i\tau,\mu) + \int_0^1 dz K(z,\alpha_s)\tilde{\phi}_+(-iz\tau,\mu)$$

DGLAP-type splitting function can be diagonalized in the moment space:

$$\mathcal{K}(j,\alpha_s) = \int_0^1 dz z^j K(z,\alpha_s) = \mathcal{K}^{(1)}(j)\frac{\alpha_s}{4\pi} + \cdots,$$

$$\mathcal{K}^{(1)}(j) = 4C_F \int_0^1 dz z^j \left(\frac{z}{1-z}\right)_+ = -4C_F[\psi(j+2) + \gamma_E - 1]$$

$$\psi(x) = \frac{d\log\Gamma(x)}{dx}$$

$\Gamma_{\rm cusp}(\alpha_s) = 0:$ usual inverse Mellin transformation

$$\tilde{\phi}_+(-i\tau,\mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dj \left(\tau\mu_0 e^{\gamma_E}\right)^j \varphi(j,\mu)$$

$\Gamma_{\rm cusp}(\alpha_s) \neq 0:$ the power j will evolve under the variation of μ

$$\tilde{\phi}_+(-i\tau,\mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dj \left(\tau\mu_0 e^{\gamma_E}\right)^{j-\xi(\mu,\mu_0)} \varphi(j,\mu)$$

$$t \longrightarrow -i\tau \qquad L = \log(\tau\mu e^{\gamma_E})$$

$$\mu\frac{d}{d\mu}\tilde{\phi}_+(-i\tau,\mu) = -\left[\Gamma_{\rm cusp}(\alpha_s)L + \gamma_F(\alpha_s)\right]\tilde{\phi}_+(-i\tau,\mu) + \int_0^1 dz K(z,\alpha_s)\tilde{\phi}_+(-iz\tau,\mu)$$

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$$\mathscr{K}^{(1)}(j) = 4C_F\int_0^1 dz z^j\left(\frac{z}{1-z}\right)_+ = -4C_F[\psi(j+2)+\gamma_E-1]$$

$$\tilde{\phi}_+(-i\tau,\mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dj \left(\tau\mu_0 e^{\gamma_E}\right)^{j-\xi(\mu,\mu_0)}\varphi(j,\mu)$$

$$\psi(x) = \frac{d\log\Gamma(x)}{dx}$$

$$\begin{cases} \mu\dfrac{d\xi(\mu,\mu_0)}{d\mu} = \Gamma_{\rm cusp}(\alpha_s) \\ \mu\dfrac{d}{d\mu}\varphi(j,\mu) = \left[-\Gamma_{\rm cusp}(\alpha_s)\ln\dfrac{\mu}{\mu_0} - \gamma_F(\alpha_s) + \mathscr{K}\left(j-\xi(\mu,\mu_0),\alpha_s\right)\right]\varphi(j,\mu) \end{cases}$$

$t \to -i\tau$

$L = \log(\tau\mu e^{\gamma_E})$

$$\mu \frac{d}{d\mu}\tilde{\phi}_+(-i\tau,\mu) = -\left[\Gamma_{\rm cusp}(\alpha_s)L + \gamma_F(\alpha_s)\right]\tilde{\phi}_+(-i\tau,\mu) + \int_0^1 dz K(z,\alpha_s)\tilde{\phi}_+(-iz\tau,\mu)$$

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$$\mathcal{K}(j,\alpha_s) = \int_0^1 dz z^j K(z,\alpha_s) = \mathcal{K}^{(1)}(j)\frac{\alpha_s}{4\pi} + \cdots,$$

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$$\psi(x) = \frac{d\log\Gamma(x)}{dx}$$

$$\begin{cases} \mu \dfrac{d\xi(\mu,\mu_0)}{d\mu} = \Gamma_{\rm cusp}(\alpha_s) \\ \mu \dfrac{d}{d\mu}\varphi(j,\mu) = \left[-\Gamma_{\rm cusp}(\alpha_s)\ln\dfrac{\mu}{\mu_0} - \gamma_F(\alpha_s) + \mathcal{K}\left(j - \xi(\mu,\mu_0), \alpha_s\right)\right]\varphi(j,\mu) \end{cases}$$

$$\xi(\mu,\mu_0) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)} d\alpha$$

$$\varphi(j,\mu) = \exp\left[V(\mu,\mu_0) + W(\mu,\mu_0,j)\right]\varphi(j,\mu_0)$$

$$V(\mu,\mu_0) = -\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\left[\Gamma_{\rm cusp}(\alpha)\int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} + \gamma_F(\alpha)\right]$$

$$W(\mu,\mu_0,j) = \int_{\mu_0}^{\mu} \frac{d\lambda}{\lambda}\mathcal{K}\left(j - \xi\left(\alpha_s(\lambda),\alpha_s(\mu_0)\right), \alpha_s(\lambda)\right)$$

$$\tilde{\phi}_+(-i\tau,\mu)=e^{V(\mu,\mu_0)}(\tau\mu_0 e^{\gamma_E})^{-\xi}\int_0^\infty \frac{d\tau'}{\tau'}\tilde{\phi}_+(-i\tau',\mu_0)\int_{c-i\infty}^{c+i\infty}\frac{dj}{2\pi i}\left(\frac{\tau}{\tau'}\right)^j e^{W(\mu,\mu_0,j)}$$

$$\xi\equiv\xi(\mu,\mu_0)=\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}d\alpha$$

$$V(\mu,\mu_0)=-\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha}{\beta(\alpha)}\left[\Gamma_{\text{cusp}}(\alpha)\int_{\alpha_s(\mu_0)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')}+\gamma_F(\alpha)\right]$$

at 1-loop level

$$e^{(1-\gamma_E)\xi}\frac{\Gamma(j+2-\xi)}{\Gamma(j+2)}$$

$$\tilde{\phi}_+(-i\tau,\mu)=e^{V(\mu,\mu_0)}\left(\tau\mu_0 e^{\gamma_E}\right)^{-\xi}\times\frac{e^{(1-\gamma_E)\xi}}{\Gamma(\xi)}\int_0^1 dz\left(\frac{z}{1-z}\right)^{1-\xi}\tilde{\phi}_+(-iz\tau,\mu_0)$$

$$\exp\left(-\int_{\mu_0}^{\mu}\frac{d\lambda}{\lambda}\left[\Gamma_{\text{cusp}}(\alpha_s(\lambda))\ln\left(\tau\lambda e^{\gamma_E}\right)+\gamma_F(\alpha_s(\lambda))\right]\right)$$

RG improvement of $\left(\frac{z}{1-z}\right)_+$

$$\xi(\mu,\mu_0)=\frac{\Gamma^{(1)}_{\text{cusp}}}{2\beta_0}\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}+\cdots$$

$$V(\mu,\mu_0)=\frac{\Gamma^{(1)}_{\text{cusp}}}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s(\mu_0)}\left(1+\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)-\frac{4\pi}{\alpha_s(\mu)}\right\}$$

$$-\frac{\gamma_F^{(1)}}{2\beta_0}\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}+\frac{\Gamma^{(1)}_{\text{cusp}}}{4\beta_0^2}\left\{\frac{\beta_1}{2\beta_0}\ln^2\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}+\left(\frac{\Gamma^{(2)}_{\text{cusp}}}{\Gamma^{(1)}_{\text{cusp}}}-\frac{\beta_1}{\beta_0}\right)\left(\frac{\alpha_s(\mu_0)-\alpha_s(\mu)}{\alpha_s(\mu_0)}-\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)\right\}+\cdots$$

Sudakov resummation up to the NLL accuracy $\alpha_s^n\log^{2n}\frac{\mu^2}{\mu_0^2},\quad \alpha_s^n\log^{2n-1}\frac{\mu^2}{\mu_0^2},\quad \alpha_s^n\log^{2n-2}\frac{\mu^2}{\mu_0^2}$

New analytic solution exact up to the NLL accuracy

$$\tilde{\phi}_+(-i\tau,\mu) = e^{V(\mu,\mu_0)}\left(\tau\mu_0 e^{\gamma_E}\right)^{-\xi}\frac{e^{(1-\gamma_E)\xi}}{\Gamma(\xi)}\int_0^1 dz\left(\frac{z}{1-z}\right)^{1-\xi}\tilde{\phi}_+(-iz\tau,\mu_0)$$

$$\xi = \frac{\Gamma_{\rm cusp}^{(1)}}{2\beta_0}\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)},\quad V(\mu,\mu_0) = \frac{\Gamma_{\rm cusp}^{(1)}}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s(\mu_0)}\left(1+\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)-\frac{4\pi}{\alpha_s(\mu)}\right\}-\frac{\gamma_F^{(1)}}{2\beta_0}\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}$$

$$+\frac{\Gamma_{\rm cusp}^{(1)}}{4\beta_0^2}\left\{\frac{\beta_1}{2\beta_0}\ln^2\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}+\left(\frac{\Gamma_{\rm cusp}^{(2)}}{\Gamma_{\rm cusp}^{(1)}}-\frac{\beta_1}{\beta_0}\right)\left(\frac{\alpha_s(\mu_0)-\alpha_s(\mu)}{\alpha_s(\mu_0)}-\ln\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)\right\}$$

($\Gamma_{\rm cusp}^{(2)}\to 0,\ \beta_1\to 0$: reduces to RG improved perturbation theory at 1-loop level)

$$\phi_+(\omega,\mu) = e^{V(\mu,\mu_0)+(1-2\gamma_E)\xi}\frac{\Gamma(2-\xi)}{\Gamma(\xi)}\int_0^\infty\frac{d\omega'}{\omega'}\left(\frac{\omega_>}{\mu_0}\right)^{\xi}\frac{\omega_<}{\omega_>}\,{}_2F_1\left(1-\xi,2-\xi;2;\frac{\omega_<}{\omega_>}\right)\phi_+(\omega',\mu_0)$$

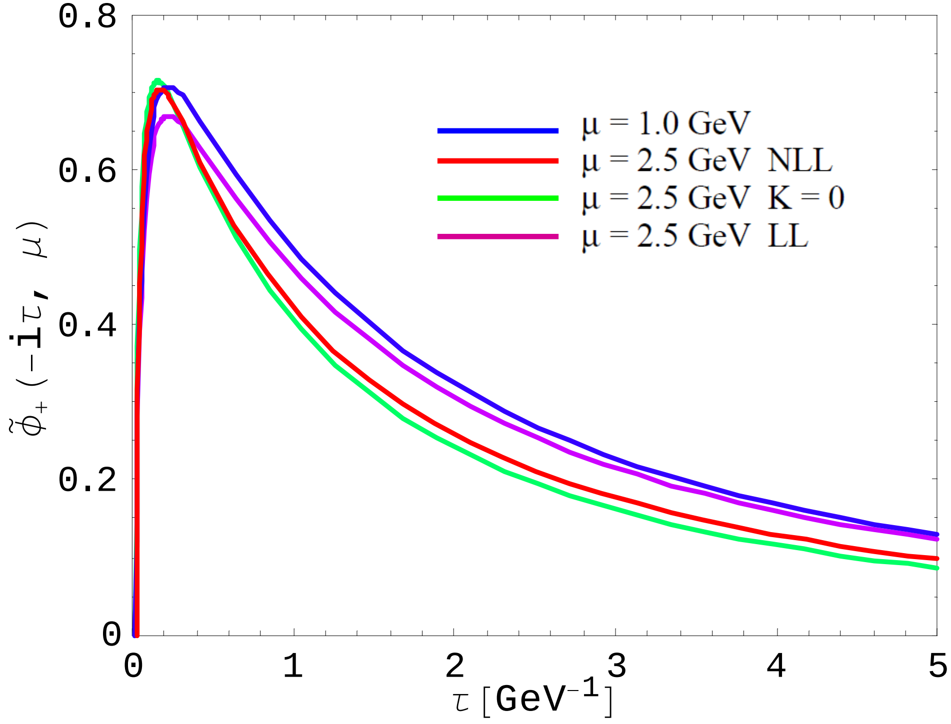
$$\omega_> \equiv \max(\omega,\omega'),\ \ \omega_< \equiv \min(\omega,\omega')$$

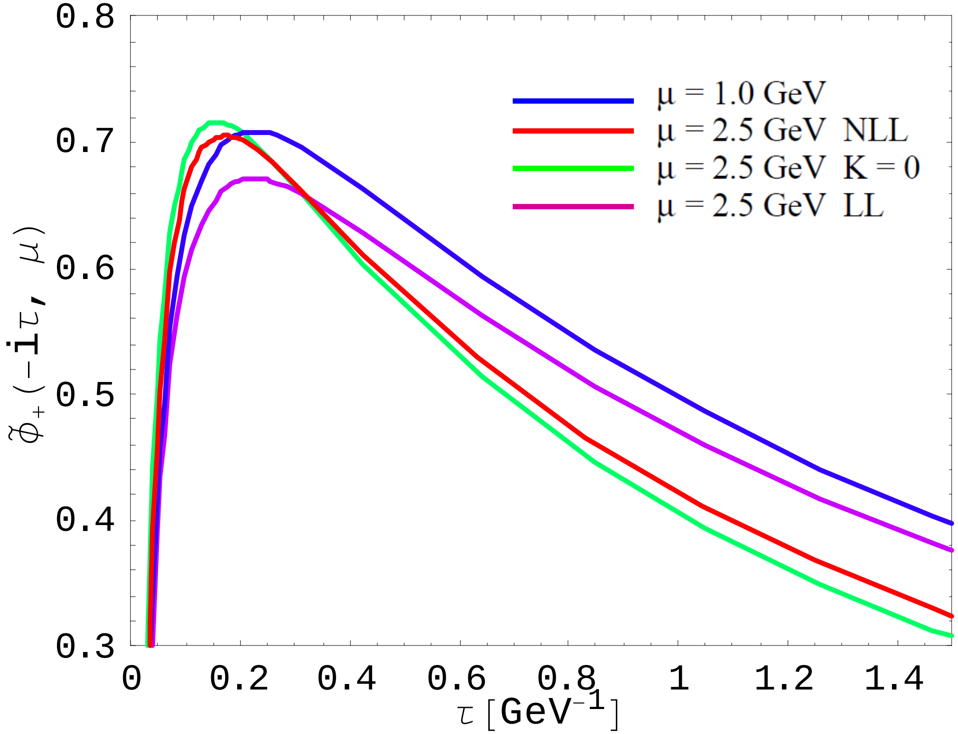
reproduces the momentum-space solution by Lange, Neubert ('03); Lee, Neubert ('05)

quasilocal evolution in the coordinate space

Model-independence for $\tau \le \tau_c \simeq \dfrac{1}{\mu_0} = 1\ {\rm GeV}^{-1}$ **is preserved under evolution**

logarithmic expansion up to NLL accuracy treating as $\beta_0\dfrac{\alpha_s(\mu)}{4\pi}\ln\dfrac{\mu^2}{\mu_0^2}\sim 1$



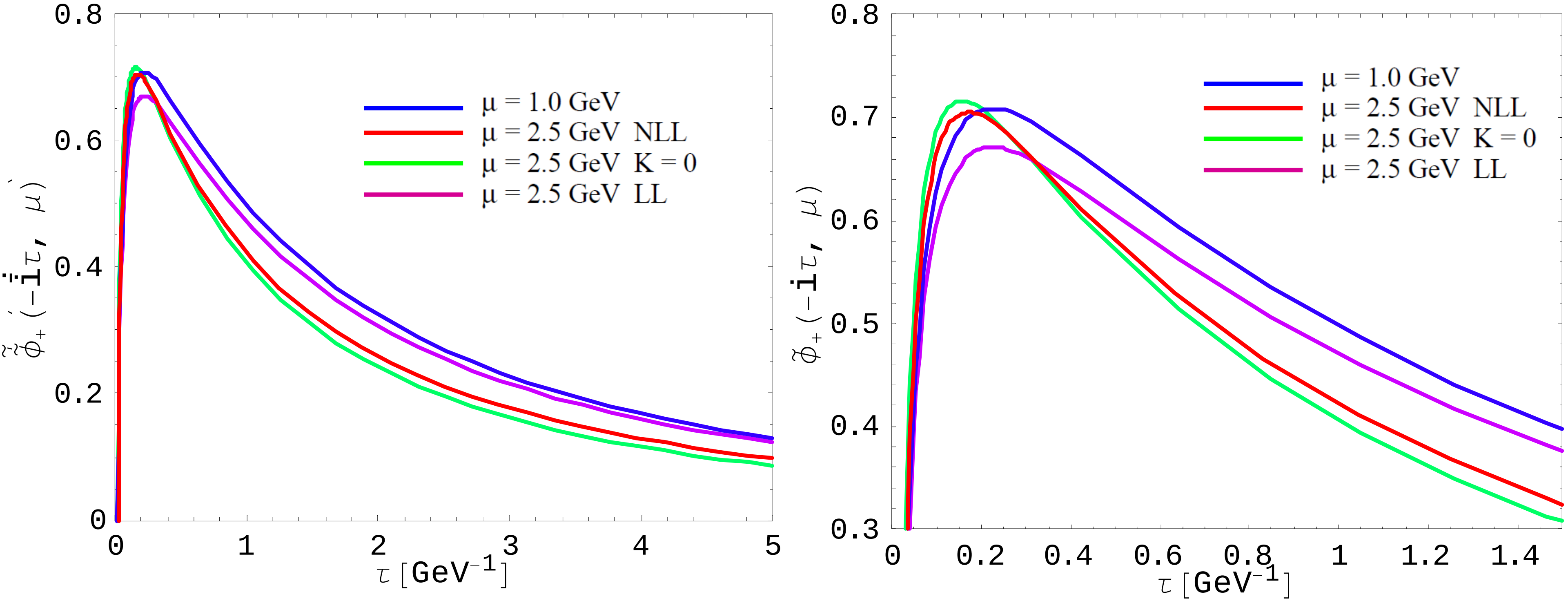


$$\lambda_B^{-1}(\mu_i) = \int_0^\infty d\tau \tilde\phi_+(-i\tau,\mu_i) = \int_0^{\tau_c} d\tau \tilde\phi_+(-i\tau,\mu_i) + \int_{\tau_c}^\infty d\tau \tilde\phi_+(-i\tau,\mu_i)$$

$$= \frac{e^{V(\mu_i,\mu_0)+(1-2\gamma_E)\xi}}{\mu_0^\xi \Gamma(1+\xi)} \int_0^\infty d\tau \frac{\tilde\phi_+(-i\tau,\mu_0)}{\tau^\xi}$$

NEW relations!

$$\lambda_B^{-1}(\mu_i) = \int_0^\infty d\omega \frac{\phi_+(\omega,\mu_i)}{\omega} = \frac{e^{V(\mu_i,\mu_0)+(1-2\gamma_E)\xi}\Gamma(1-\xi)}{\mu_0^\xi \Gamma(1+\xi)} \int_0^\infty d\omega \frac{\phi_+(\omega,\mu_0)}{\omega^{1-\xi}}$$



μ [GeV^{-1}]	$\lambda_B^{-1}(\mu)$ [GeV^{-1}]	Lee, Neubert ('05)	Braun et al. ('04)
1.0	2.7 (=0.58+2.08)	2.1	2.2
1.5	2.4 (=0.59+1.80)	1.9	
2.0	2.2 (=0.59+1.64)	1.7	
2.5	2.1 (=0.58+1.52)	1.6	

Summary

B-meson LCDA for exclusive B decays

OPE of the bilocal operator for B-meson LCDA

up to dim.5 local operators; NLO corrections for Wilson coefficients

$\sim \log^2(i\mu t)$ **terms from cusp singularity**

Model-independent behavior of B-meson LCDA from the OPE $t\mu \lesssim 1$

large NLO pert. effects; significant nonpert. effects by $\bar{\Lambda},\ \lambda_E^2,\ \lambda_H^2$

Connecting smoothly to an ansatz for the long-distance behavior:

$$\lambda_B^{-1}(\mu) = \int_0^\infty d\omega \frac{\phi_+(\omega,\mu)}{\omega} = \int_0^{\tau_c} d\tau \tilde{\phi}_+(-i\tau,\mu) + \int_{\tau_c}^\infty d\tau \tilde{\phi}_+(-i\tau,\mu)$$

$\lambda_B(\mu=1\,\mathrm{GeV}) \simeq 0.37\,\mathrm{GeV}$ decreases for increasing values of $\lambda_E^2,\ \lambda_H^2$

$(0.2 \sim 0.5\ \mathrm{GeV})$

Evolution in the coodinate space from $\mu_0 \simeq 1\,\mathrm{GeV}$ to $\mu_i \simeq \sqrt{m_b \Lambda_{\mathrm{QCD}}}$

quasilocal structure; resummation at NLL level 2-loop cusp anomalous dim.

Sudakov suppression at moderate and large τ

Shift to larger τ by DGLAP-type effect

$$\lambda_B(\mu=2.5\,\mathrm{GeV}) \simeq 0.48\,\mathrm{GeV}$$

cf. Lee, Neubert('05)

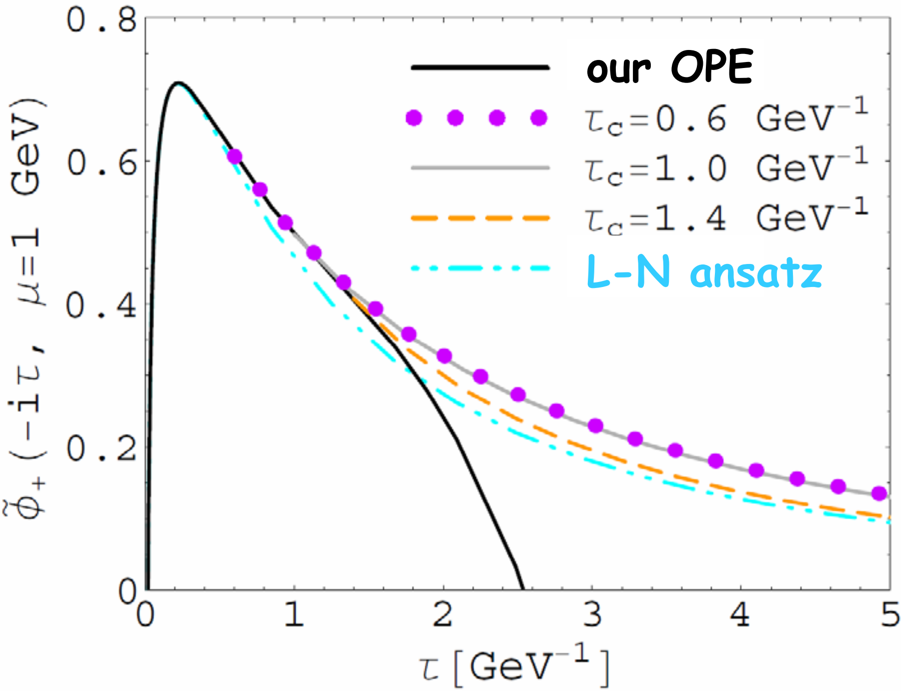
OPE up to dim.4 ops.

0.48 GeV

↓

0.62 GeV

Need {
- precise nonperturbative estimate of λ_E^2, λ_H^2
- functional form of long-distance behavior



$$\frac{N}{\omega_0^2} = \frac{9}{4\bar{\Lambda}_{DA}^2}\left\{1+\tau_c\bar{\Lambda}_{DA}\left[\frac{\lambda_E^2}{\bar{\Lambda}_{DA}^2}+\frac{\lambda_H^2}{2\bar{\Lambda}_{DA}^2}-1\right]\right\}+\cdots$$

$$\lambda_B^{-1}(\mu) = \int_0^\infty d\omega \frac{\phi_+(\omega,\mu)}{\omega}$$

$$= \int_0^{\tau_c} d\tau \tilde{\phi}_+(-i\tau,\mu) + \int_{\tau_c}^\infty d\tau \tilde{\phi}_+(-i\tau,\mu)$$

with μ =1 GeV

continuity at $\tau = \tau_c$

$$\left\{ \tilde{\phi}_+(-i\tau),\quad \frac{\partial \tilde{\phi}_+(-i\tau)}{\partial \tau} \right.$$

OPE up to dim. 5 ops. τ_c

$$\int_0^\infty d\omega e^{-\omega\tau}\, N\frac{\omega}{\omega_0^2} e^{-\omega/\omega_0} = \frac{N}{(\tau\omega_0+1)^2}$$

	$\lambda_E^2 = 0.11\ \mathrm{GeV}^2,\ \ \lambda_H^2 = 0.18\ \mathrm{GeV}^2$			$\lambda_E^2 = \lambda_H^2 = 0$		
τ_c [GeV^{-1}]	N	ω_0 [GeV]	λ_B^{-1} [GeV^{-1}]	N	ω_0 [GeV]	λ_B^{-1} [GeV^{-1}]
0.4	0.816	0.257	3.11 (0.23 + 2.88)	0.832	0.301	2.69 (0.23 + 2.46)
0.6	0.850	0.306	2.70 (0.35 + 2.35)	0.899	0.394	2.19 (0.35 + 1.84)
0.8	0.852	0.308	2.69 (0.47 + 2.22)	0.966	0.461	1.99 (0.46 + 1.53)
1.0	0.858	0.313	2.66 (0.58 + 2.08)	1.11	0.572	1.79 (0.56 + 1.23)
1.2	0.910	0.349	2.51 (0.67 + 1.84)	1.55	0.839	1.56 (0.64 + 0.92)
1.4	1.09	0.456	2.22 (0.76 + 1.46)	4.43	1.95	1.32 (0.71 + 0.61)
1.6	1.81	0.777	1.87 (0.83 + 1.04)	9.82	−4.55	1.11 (0.77 + 0.34)