Peculiar features of $O(\alpha_s^4)$ estimates of the relation between pole and running heavy quak masses

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Based on the work done in collaboration with **Victor Kim (PNPI, Gatchina)**
Some comments on the work in collaboration with **Sergey Mikhailov (JINR, Dubna)** on the interesting features of perturbative series do be discussed at RADCOR-09 (Ascona, 25-30 October , 2009)

There are different definitions of heavy quark masses: pole and $\overline{\rm MS}$-scheme running. The letter ones are commonly used. BUT in characteristics of DIS processes (including parton distributions) pole masses of c and b-quark masses are used and t-quark mass is defined from experiment. QUESTION: what is there relation? ANSWER: known analytically at α_s^3 level. **Aim**: Study of structure of perturbative series for the relations between different definitions of masses up to α_s^4 estimated -term Kataev, Kim(09). The procedure : Chetyrkin-Kniehl-Sirlin (97) analog of effective charges motivated approach by Kataev-Starshenko (94-95)
1) Concrete peculiar features: existence of Minkowskian-type π^2-terms in α_s corrections (are they seen in analytical results?).
2) Results for mass relations of c, b, t-quarks at the α_s^4-level.
3) A.Kataev,S. Mikhailov, CERN-PH-TH/2009-203; 23.10. 09. New extended Crewther-type relation. Waiting for α_s^4 to D^{e+e-} and Bjorken SR :Dr. P.Baikov, Prof.K.Chetyrkin, Prof. H.Kuhn...

Part 1: Definitions:

$M_{(N_L+1)} = \overline{m}_{(N_L+1)}(\overline{m}_{(N_L+1)}) \sum_{n=0}^{4} t_n^M a_s^n(\overline{m}^2_{(N_L+1)})$ where

N_L-number of quarks, lighter, than considered heavy quark (c, b, t)

Chetyrkin-Kniehl-Sirlin dispersion relation- model:

$M_{(N_L+1)} = \frac{1}{2\pi i} \int_{-\overline{m}_{(N_L+1)}(\overline{m}_{(N_L+1)})-i\varepsilon}^{-\overline{m}_{(N_L+1)}(\overline{m}_{(N_L+1)})+i\varepsilon} ds' \int_0^\infty ds \frac{T(s)}{(s+s')^2}$

$T(s) = \overline{m}_{(N_L+1)}(s) \sum_{n=0}^{4} t_n^M a_s^n(\overline{m}^2_{(N_L+1)})$. Its Euclidian analog is

$F(Q^2) = \overline{m}_{(N_L+1)}(Q^2) \sum_{n=0}^{4} f_n^E a_s^n(Q^2)$

The coefficients are related as : $f_0^E = t_0, f_i^E = t_i^M + e_i$ $(1 \leq i \leq 4)$

$e_1 = 0$, $e_2 = \frac{\pi^2}{6} t_0 \gamma_0 (\beta_0 + \gamma_0)$ (β_i and γ_i are the coefficients of QCD β-function and mass anomalous dimension function).

$e_3 = \frac{\pi^2}{3} \left\{ t_1(\beta_0 + \gamma_0) \left(\beta_0 + \frac{\gamma_0}{2}\right) + t_0 \left[\frac{\beta_1\gamma_0}{2} + \gamma_1(\beta_0 + \gamma_0) \right] \right\}$

$e_4 = \pi^2 \left\{ t_2(\beta_0 + \frac{\gamma_0}{2}) + t_1 \left[\frac{\beta_1}{2}(\frac{5}{3}\beta_0 + \gamma_0) + \frac{\gamma_1}{3}(2\beta_0 + \gamma_0) \right] + t_0 \left[\frac{\beta_2\gamma_0}{6} + \frac{\gamma_1}{3}\left(\beta_1 + \frac{\gamma_1}{2}\right) + \gamma_2 \left(\frac{\beta_0}{2} + \frac{\gamma_0}{3}\right) \right] \right\} + \frac{7\pi^4}{60} t_0 \gamma_0 (\beta_0 + \gamma_0)(\beta_0 + \frac{\gamma_0}{2})(\beta_0 + \frac{\gamma_0}{3})$

Are π^2- analytical continuation terms observed in reality?

$$\frac{\overline{m}(M)}{M} = 1 - \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(N_L\left(\frac{71}{144} + \frac{\pi^2}{18}\right) - \frac{3019}{288} + \frac{1}{6}\zeta_3 - \frac{\pi^2}{9}\log 2 - \frac{\pi^2}{3}\right)$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \left(N_L^2\left(-\frac{2353}{23328} - \frac{7}{54}\zeta_3 - \frac{13}{324}\pi^2\right) + N_L\left(\frac{246643}{23328} + \frac{241}{72}\zeta_3 + \frac{11}{81}\pi^2\log 2 - \frac{2}{81}\pi^2\log^2 2\right.\right.$$
$$\left. + \frac{967}{648}\pi^2 - \frac{61}{1944}\pi^4 - \frac{1}{81}\log^4 2 - \frac{8}{27}a_4\right) - \frac{9478333}{93312} + \frac{1439}{432}\zeta_3\pi^2 - \frac{61}{27}\zeta_3 - \frac{1975}{216}\zeta_5$$
$$\left. + \frac{587}{162}\pi^2\log 2 + \frac{22}{81}\pi^2\log^2 2 - \frac{644201}{38880}\pi^2 + \frac{695}{7776}\pi^4 + \frac{55}{162}\log^4 2 + \frac{220}{27}a_4\right).$$

Result of Melnikov, van Ritbergen (2000) in agreement with Chetyrkin, Steinhauser (2000). We were not able to separate effects of analytical continuation from π^2-terms typical to OS calculations. Further studies ?

Part 2 . Study of the asymptotic structure of perturbative relations. We are interested in the following series

$$M_{(N_L+1)} = \overline{m}_{(N_L+1)}(M_{(N_L+1)})\left[1 + \sum_{i=1}^{4} v_i^M a_s^i(M_{(N_L+1)})\right]$$

$$M_{(N_L+1)} = \overline{m}_{(N_L+1)}(\overline{m}_{(N_L+1)})\left[1 + \sum_{i=1}^{4} t_i^M a_s^i(\overline{m}_{(N_L+1)})\right]$$

The coefficients $v_i^M(N_L)$, $u_i^M(N_L)$ and $t_i^M(N_L)$ $(1 \geq i \geq 3)$ are known analytically (and numerically). How to estimate v_4^M, t_4^M- ???

Effective charges motivated approach (Kataev-Starshnko (94-95)): using $f_1^E = t_1^M$, $f_2^E(N_L) = t_2^M(N_L) + e_2(N_L)$, $\beta_0(N_L)$, $\beta_1(N_L)$ estimate $f_3^{est}(N_L)$ for fixed $N_L = 3, 4, 5$, compare with $f_3(N_L) = t_3(N_L) + e_3(N_L)$. In case agreement is go- go one step further - add $f_3^E(N_L)$ and $\beta_2(N_L)$ and get $f_4^E(M) = t_4^{est}(N_L) + e_4(N_L)$ and thus t_4^{est} and then v_4^{est}.

Chetyrkin-Kniehl- Sirlin (97) used this approach for the quantities, which are proportional to running mass. In the case we are interested in they got :

$$f_3^{est}(N_L) = f_2(N_L)\left(\frac{f_2(N_L)}{f_1} + \frac{\beta_1(N_L)}{\beta_0(N_L)}\right)$$

So: $f_3^{est}(N_L = 3, 4, 5) \approx \left[\ 201 \quad ; \quad 166 \quad 133.97\ \right]$. Compare with explicit results: $\left[\ \mathbf{193.45} \quad , \quad \mathbf{163.03} \quad , \quad \mathbf{133.97}\ \right]$. For t-quark the agreement is fantastic! $:-)$ One step further to $f_4^{est}(N_L)$:

$$f_4^{est}(N_L) = f_2(N_L)\left(3\frac{f_3(N_L)}{f_1} - 2\frac{f_2^2(N_L)}{f_1^2} - \frac{f_2\beta_1(N_L)}{2f_1\beta_0(N_L)} + \frac{\beta_2(N_L)}{\beta_0(N_L)}\right)$$

Results: $f_4^{est}(N_L = 3, 4, 5) \approx \left[\ \mathbf{2502}\ ,\ \mathbf{1943}\ ,\ \mathbf{1434}\ \right]$ and

$t_4^{est}(N_L) = f_4^{est}(N_L) - e_4(N_L) \approx \left[\ \mathbf{1264}\ ,\ \mathbf{976}\ ,\ \mathbf{704}\ \right]$

Lower order coefficients:

$f_3^{exact}(N_L = 3, 4, 5) \approx \left[\ \mathbf{193}\ ,\ \mathbf{163}\ ,\ \mathbf{134}\ \right]$ and

$t_3(N_L = 3, 4, 5) = f_3^{exact}(N_L) - e_3(N_L) \approx \left[\ \mathbf{116}\ ,\ \mathbf{94}\ ,\ \mathbf{74}\ \right]$

$f_2(N_L = 3, 4, 5) \approx \left[\ \mathbf{15}\ ,\ \mathbf{14}\ ,\ \mathbf{13}\ \right]$ and

$t_2(N_L = 3, 4, 5) = f_2(N_L) - e_2(N_L) \approx \left[\ \mathbf{10}\ ,\ \mathbf{9}\ ,\ \mathbf{8}\ \right]$

The effects of analytical continuation, contained in the the values of e_2, e_3, e_4 numerically are **important**. Obviously f_i^E are growing fast! Their ratio f_4/f_3 should be compared with IR renormalon predictions. (Indeed, t_i are defined for the Minkowskian-type PT series).

Next step: the comparison with the values of v_i in
$M_{(N_L+1)} = \overline{m}_{(N_L+1)}\left[1 + \sum_{i=1}^{4} v_i a_s^i(M_{(N_L+1)})\right]$
The coefficients v_i can be calculated using their relations with t_i^M and β_i, γ_i-terms of RG-functions
$M_c \approx \overline{m}_c(M_c)\left[1 + \frac{4}{3}a_s(M_c) + 13a_s^2(M_c) + 156a_s^3(M_c) + 1836a_s^4(M_c)\right]$
$M_b \approx \overline{m}_b(M_b)\left[1 + \frac{4}{3}a_s(M_b) + 12a_s^2(M_b) + 131a_s^3(M_b) + 1450a_s^4(M_b)\right]$
$M_t \approx \overline{m}_t(M_t)\left[1 + \frac{4}{3}a_s(M_t) + 11a_s^2(M_t) + 107a_s^3(M_t) + 1086a_s^4(M_t)\right]$
In these series coefficients are growing up similar to those in the case. $M_{(N_L+1)} = \overline{m}_{(N_L+1)}(\overline{m}_{(N_L+1)})\left[1 + \sum_{i=1}^{4} t_i^M a_s^i(\overline{m}_{(N_L+1)})\right]$

Part 3. Structure of perturbative expansions for the D-function and Bjorken sum rule at α_s^4: **Broadhurst-Kataev (93)** : generalization of Crewther relation $D^{NS}K_{Bjp}=1$

$$D^{NS}(Q^2)K_{Bjp}(Q^2)=$$
$$1+\frac{\beta^{(2)}(a_s)}{a_s}\left[S_1C_Fa_s+\left(S_2T_fN_f+S_3C_A+S_4C_F\right)a_s^2+O(a_s^3)\right],$$
$$S_1=-\frac{21}{8}+3\zeta_3,\ S_2=\frac{163}{24}-\frac{19}{3}\zeta_3,\ S_3=-\frac{629}{32}+\frac{221}{12}\zeta_3$$
$$S_4=\frac{397}{96}+\frac{17}{2}\zeta_3-15\zeta_5$$

Crewther (97) - proof that β-function is factorized indeed in all orders (in x-space). The consideration in p-space - Gabadadze, Kataev (95); Essential point in both considerations- study of OPE application to AVV triangle diagram. Attempt to use generalized Crewther in phenomenolgy-Brodsky, Gabadadze, Kataev, Lu (96). Calculations of α_s^4 may add new insight to study of Crewther relation.

However, even beffore these studies we (KM) are proposing **NEW extended Crewther-type relation**:

$$
\begin{aligned}
D^{NS}\cdot K &= 1+\sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n \mathcal{P}_n(a_s) && (1)\\
,\mathcal{P}_1(a_s) &= a_s 3C_F\left\{\left(\frac{7}{2}-4\zeta_3\right)+a_s\left[\left(\frac{43}{9}-\frac{16}{3}\zeta_3\right)C_A-\left(\frac{397}{18}+\frac{136}{3}\zeta_3-80\zeta_5\right)C_F\right]+O(a_s^3)\right\}\\
\mathcal{P}_2(a_s) &= a_s 3C_F\left(\frac{163}{6}-\frac{76}{3}\zeta_3\right)+O(a_s^2) && (2)\\
\mathcal{P}_3(a_s) &= 0 && (3)
\end{aligned}
$$

At α_s^4 we expect:

$$
\begin{aligned}
\mathcal{P}_1^{(3)}(a_s) &= a_s^3 C_F\left[C_F^2 \mathrm{as}_1 + C_F C_A \mathrm{as}_2 + C_A^2 \mathrm{as}_3\right] && (4)\\
\mathcal{P}_2^{(2)}(a_s) &= a_s^2 C_F\left[C_F \mathrm{as}_4 + C_A \mathrm{as}_5\right] && (5)\\
\mathcal{P}_3^{(1)}(a_s) &= a_s C_F \mathrm{as}_6 && (6)
\end{aligned}
$$

as_i may be fixed from BChK α_s^4 with Casimir's. Waiting the **Talk**.

CONCLUSION

It is the pleasure to participate at **RADCOR-2009** and to discuss the NEW results, closely related to the talk at **RADCOR-1994- 15 Years After-** still not **20 !** From **Conclusion** of this talk: "QCD is in good condition. However a lot of problems are still waiting for a solution" Among 9 mentioned were 3, discussed today:

- More rigorous and precise determinations of the values of light and heavy quark masses
- The breakthrough in the understanding of theoretical structure of perturbative series in QCD and gauge models
- Future more detailed studies of the Crewther relation and its different generalization might give us the opportunity of comparing theoretical results for the DIS SR's with the ones for the annihilation process on the new level

QCD still provide new intriguing results ! Waiting for the **Talks**.