Non-quasipartonic operators in QCD

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based on

V.M. Braun, A.N. Manashov, J. Rohrwild, Nucl. Phys. B807:89-137,2009.
V.M. Braun, A.N. Manashov, J. Rohrwild, arXiv:0908.1684

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Quasipartonic and Non–quasipartonic operators

Higher–twist operators = quasipartonic + nonquasipartonic

- **Quasipartonic operators:**

Bukhvostov, Frolov, Lipatov, Kuraev, 1985 (BFLK)

— multiparticle operators built of "plus" field components

$\Leftarrow$ set closed under renormalization

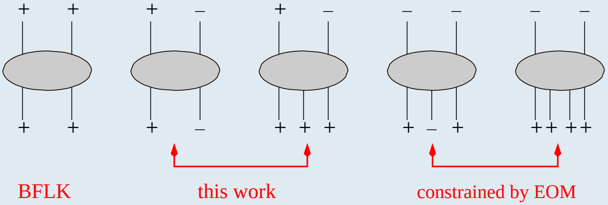
$\Leftarrow$ Two-particle structure of renormalization in one loop

- **Nonquasipartonic operators:**

— all others

$\Leftarrow$ mix with quasipartonic operators

$\Leftarrow$ appear starting twist four, e.g. $\psi_+ F_{+-} \psi_+$



motivation for this study came from recent developments in $N = 4$ SUSY:

Beisert, 2004; Beisert, Ferretti, Heise, Zarembo, 2005

- **Methods:**
  - Conformal operator basis for arbitrary twist [manifest $SL(2)$ invariance]
  - "plus-minus" $2 \to 2$ kernels by embedding $SL(2,\mathbb{R})$ in $SO(4,2)$
  - $2 \to 3$ kernels by Lorentz transformation of the BFLK kernels

- **For QCD practitioneer:**
  - Complete results for operator renormalization up to twist four
  - Can be extended to arbitrary twist and maybe beyond LO

Light–ray operators

Example: leading twist

$$\mathcal{O}(z_1, z_2) = \bar{q}(z_1 n)[z_1 n, z_2 n] \not{n}\, q(z_2 n)\,, \qquad n^2 = 0$$

RG-equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi}\mathbb{H}\right)[\mathcal{O}(z_1, z_2)]_R = 0$$

where $\mathbb{H}$ is the integral operator Balitsky, Braun, 1989

$$[\mathbb{H}\cdot\mathcal{O}](z_1, z_2) = 2C_F\left\{\int_0^1 \frac{d\alpha}{\alpha}\left[2\mathcal{O}(z_1, z_2) - \bar\alpha\mathcal{O}(z_{12}^\alpha, z_2) - \bar\alpha\mathcal{O}(z_1, z_{21}^\alpha)\right] - \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\, \mathcal{O}(z_{12}^\alpha, z_{21}^\beta) - \frac{3}{2}\mathcal{O}(z_1, z_2)\right\}$$

where $z_{12}^\alpha = z_1(1-\alpha) + z_2\alpha$, $\bar\alpha = 1-\alpha$.

- $\mathbb{H}$ is invariant under $SL(2,\mathbb{R})$ transformations of the light-ray, $z \longrightarrow \dfrac{az+b}{cz+d}$.

  $\Rightarrow$ DGLAP, ERBL, GPD

$$\varphi_{AB}(z_1, z_2) = \langle A|\mathcal{O}(z_1, z_2)|B\rangle$$

Spinor Representation

Coordinates:

$$x_{\alpha\dot\alpha} = x_\mu \sigma^\mu_{\alpha\dot\alpha} = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} = \begin{pmatrix} x_+ & w \\ \bar w & x_- \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}, \vec\sigma)$$

To maintain Lorentz–covariance, introduce two light-like vectors $n^2 = \tilde n^2 = 0$

$$n_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}, \qquad \tilde n_{\alpha\dot\alpha} = \mu_\alpha \bar\mu_{\dot\alpha}$$

with auxiliary spinors λ and μ

$$x_{\alpha\dot\alpha} = z\,\lambda_\alpha \bar\lambda_{\dot\alpha} + \tilde z\,\mu_\alpha \bar\mu_{\dot\alpha} + w\,\lambda_\alpha \bar\mu_{\dot\alpha} + \bar w\,\mu_\alpha \bar\lambda_{\dot\alpha}$$

Fields:

$$q = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix}, \qquad \bar q = (\chi^\beta, \bar\psi_{\dot\alpha}),$$

$$F_{\alpha\beta,\dot\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} - \epsilon_{\alpha\beta} \bar f_{\dot\alpha\dot\beta}\right)$$

$f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ transform according to $(1,0)$ and $(0,1)$ representations of Lorentz group

"Plus" and "Minus" components

$$
\begin{array}{lll}
\psi_+ = \lambda^\alpha \psi_\alpha\,, & \chi_+ = \lambda^\alpha \chi_\alpha\,, & f_{++} = \lambda^\alpha\lambda^\beta f_{\alpha\beta}\,,\\
\bar\psi_+ = \bar\lambda^{\dot\alpha} \bar\psi_{\dot\alpha}\,, & \bar\chi_+ = \bar\lambda^{\dot\alpha} \chi_{\dot\alpha}\,, & \bar f_{++} = \bar\lambda^{\dot\alpha}\bar\lambda^{\dot\beta} \bar f_{\dot\alpha\dot\beta}\,,\\
\psi_- = \mu^\alpha \psi_\alpha\,, & \bar\psi_- = \bar\mu^{\dot\alpha} \bar\psi_{\dot\alpha} & f_{+-} = \lambda^\alpha\mu^\beta f_{\alpha\beta}
\end{array}
$$

similar for derivatives $\partial_\mu \to \partial_{\alpha\dot\alpha}$

$$
\partial_{++} = 2\partial_z\,, \qquad \partial_{--} = 2\partial_{\bar z}\,, \qquad \partial_{+-} = 2\partial_w\,, \qquad \partial_{-+} = 2\partial_{\bar w}
$$

- ψ_+, χ_+, f_{++} and $\bar\psi_+, \bar\chi_+, \bar f_{++}$ are defined as quasipartonic

Operator basis for higher twists

- Operator basis containing fields and all possible derivatives is overcompleted
- In general fields with derivatives have "bad" $SL(2,\mathbb{R})$ transformation properties.

- under infinitesimal special conformal trafo in the light-cone direction: $x = \{z, \tilde{z}, w, \bar{w}\}$

$$\psi_-(x) \quad \rightarrow \quad \frac{1}{(1+z\epsilon)}\psi_-\left(\frac{z}{1+\epsilon z}, \tilde{z}, \frac{w}{1+\epsilon z}, \frac{\bar{w}}{1+\epsilon z}\right)$$

where from e.g.

$$[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-](z) \quad = \quad \frac{1}{(1+z\epsilon)^3}[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-]\left(\frac{z}{1+\epsilon z}\right)$$

$\Rightarrow$ $[D_w D_{\bar{w}} D_{\tilde{z}} \psi_-](z)$ is a "primary" field with $j = 3/2$

- but:

$$\psi_+(x) \quad \rightarrow \quad \frac{1}{(1+z\epsilon)^2}\left\{\psi_+\left(\frac{z}{1+\epsilon z}, \tilde{z}, \frac{w}{1+\epsilon z}, \frac{\bar{w}}{1+\epsilon z}\right) + \epsilon z \bar{w} \psi_-\left(\ldots\right)\right\}$$

$\Rightarrow$ e.g. $[D_{\bar{w}} \psi_+](z)$ does not transform homogeneously under $SL(2,\mathbb{R})$

Solution: allow only

V. Braun, A. Manashov, J. Rohrwild, 2008

$$\psi_+(z,\tilde{z},w,0) = \sum_{n,k} \frac{\tilde{z}^k}{k!}\frac{w^n}{n!}[D_w^n D_{\tilde{z}}^k \psi_+](z)$$

$$\psi_-(z,\tilde{z},0,\bar{w}) = \sum_{n,k} \frac{\tilde{z}^k}{k!}\frac{\bar{w}^n}{n!}[D_{\bar{w}}^n D_{\tilde{z}}^k \psi_-](z)$$

and eliminate remaining "half" of transverse derivatives using EOM, e.g.

$$[D_{\bar{w}}\psi_+](z) \equiv [D_{-+}\psi_+](z) = [D_{++}\psi_-](z) + EOM = 2\partial_z\psi_-(z) + EOM$$

- similar for gluon fields

basis fields: $\quad (E =$ collinear twist, $j =$ conformal spin $)$

	$j = 1/2$	$j = 1$	$j = 3/2$	$j = 2$	$j = 5/2$
$E = 1$		ψ_+			
$E = 2$	ψ_-		$D_w\psi_+$		
$E = 3$		$D_{\bar{w}}\psi_-,\ D_{\bar{z}}\psi_+$		$D_w^2\psi_+$	
$E = 4$	$D_{\bar{z}}\psi_-$		$D_{\bar{w}}^2\psi_-,\ D_w D_{\bar{z}}\psi_+$		$D_w^3\psi_+$

building blocks for composite light-ray operators, e.g.

$$\mathbb{C}^{abc}\left\{[0,z_1]\bar{\psi}_+(z_1)\right\}^a\left\{[0,z_2]f_{++}(z_2)\right\}^b\left\{[0,z_3]D_w\psi_+(z_3)\right\}^c$$

Premium

Manifest $SL(2)$ symmetry of higher-twist evolution equations

$SL(2)$ Algebra

is generated by $\mathbf{P}_+$, $\mathbf{M}_{-+}$, $\mathbf{D}$ and $\mathbf{K}_-$

$$\begin{aligned} \mathbf{L}_+ &= \mathbf{L}_1 + i\mathbf{L}_2 = -i\mathbf{P}_+ \\ \mathbf{L}_- &= \mathbf{L}_1 - i\mathbf{L}_2 = (i/2)\mathbf{K}_- \\ \mathbf{L}_0 &= (i/2)(\mathbf{D} + \mathbf{M}_{-+}) \\ \mathbf{E} &= (i/2)(\mathbf{D} - \mathbf{M}_{-+}) \end{aligned}$$

can be traded for the algebra of differential operators acting on the field coordinates

$$\begin{aligned} [\mathbf{L}_+, \Phi(z)] &\equiv L_+\Phi(z) \\ [\mathbf{L}_-, \Phi(z)] &\equiv L_-\Phi(z) \\ [\mathbf{L}_0, \Phi(z)] &\equiv L_0\Phi(z) \end{aligned}$$

$$\begin{aligned} L_+ &= -\frac{d}{dz} \\ L_- &= \left(z^2\frac{d}{dz} + 2j\,z\right) \\ L_0 &= \left(z\frac{d}{dz} + j\right) \end{aligned}$$

They satisfy the $SL(2)$ commutation relations

$$\begin{aligned} [L_0, L_\mp] &= \pm L_\mp \\ [L_-, L_+] &= 2L_0 \end{aligned}$$

The remaining generator $\mathbf{E}$ counts the *twist* $t = \ell - s$ of the field Φ

$$[\mathbf{E}, \Phi(z)] = \frac{1}{2}(\ell - s)\Phi(z)$$

collinear twist = dimension - spin projection on the plus-direction

$SL(2)$ Algebra

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$$
\begin{aligned}
\mathbf{L}_+ &= \mathbf{L}_1 + i\mathbf{L}_2 = -i\mathbf{P}_+ \\
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\mathbf{L}_0 &= (i/2)(\mathbf{D} + \mathbf{M}_{-+}) \\
\mathbf{E} &= (i/2)(\mathbf{D} - \mathbf{M}_{-+})
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$$
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$$

collinear twist = dimension - spin projection on the plus-direction

$SL(2)$–invariant RG equations

$\mathcal{O}(z_1, z_2) = \bar{\psi}_+(z_1)\psi_+(z_2)$

Example: leading twist RG equation, $\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi}\mathbb{H}\right)[\mathcal{O}(z_1, z_2)]_R = 0$

$$[\mathbb{H}\cdot\mathcal{O}](z_1, z_2) = 2C_F\Big\{\int_0^1 \frac{d\alpha}{\alpha}\Big[2\mathcal{O}(z_1, z_2) - \bar{\alpha}\mathcal{O}(z_{12}^\alpha, z_2) - \bar{\alpha}\mathcal{O}(z_1, z_{21}^\alpha)\Big] - \int_0^1 d\alpha\int_0^{\bar{\alpha}} d\beta\, \mathcal{O}(z_{12}^\alpha, z_{21}^\beta) - \frac{3}{2}\mathcal{O}(z_1, z_2)\Big\}$$

- $SL(2,R)$–invariance $\quad [\mathbb{H}\cdot L_k\mathcal{O}](z_1, z_2) = L_k[H\cdot\mathcal{O}](z_1, z_2)$
- Two–particle representations are not degenerate $\quad T^{j_1}\otimes T^{j_2} = \sum_{n=0}^{\infty}\otimes T^{j_1+j_2+n}$

$\hookrightarrow$ $\mathbb{H}$ **can be written as a function of two-particle Casimir operator**

$$\mathbb{C}_2^{SL(2,R)} = -\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}(z_1 - z_2)^2 = \widehat{J}_{12}(\widehat{J}_{12} - 1)$$

$\hookrightarrow$

Invariant representation

$$\mathbb{H} = 2C_F\left[\psi(\widehat{J}_{12}+1) + \psi(\widehat{J}_{12}-1) - 2\psi(1) - \frac{3}{2}\right]$$

Footnote:

to obtain this result, notice that $\mathbb{H}$ and $\mathbb{C}_2^{SL(2,R)}$ share the same eigenfunctions:

$$\begin{aligned} \mathbb{H}\,\phi_n(z_1,z_2) &= h_n\,\phi_n(z_1,z_2) \\ \mathbb{C}_2^{SL(2,R)}\phi_n(z_1,z_2) &= j_n(j_n-1)\,\phi_n(z_1,z_2) \end{aligned}$$

further, it is easy to see that

$$\phi_n(z_1,z_2) = (z_1-z_2)^n, \qquad j_n = n+2$$

so one has to calculate action of $\mathbb{H}$ on these polynomials and express $h_n = h(j_n)$

Footnote to the footnote:

$\phi_n(z_1,z_2) = z_{12}^n$ become Gegenbauer polynomials $C_n^{3/2}$ in adjoint representation of $SL(2)$

$SL(2,\mathbb{R}) \to SO(4,2)$

Beisert, 2004, Beisert et al, 2005:

- For primary fields that we are using, the same two conditions are true with respect to the full conformal group $SO(4,2)$

$\hookrightarrow$ **For arbitrary operators $\mathbb{H}$ can be written as a function of $\mathbb{C}_2^{SO(4,2)}$**

$$SL(2,\mathbb{R}): \qquad \mathbb{C}_2^{SL(2,R)} = J(J-1)$$
$$SO(4,2): \qquad \mathbb{C}_2^{SO(4,2)} = \mathbf{J}(\mathbf{J}-1)$$

$$\mathbb{H}(J) \quad \longrightarrow \quad \mathbb{H}(\mathbf{J})$$

the same function !

- Have to work out two-dimensional (matrix) representations:

$$\mathbb{C}_2^{SO(4,2)} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix} = \begin{pmatrix} \mathbb{C}_{++} & \mathbb{C}_{+-} \\ \mathbb{C}_{-+} & \mathbb{C}_{--} \end{pmatrix} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix}$$
$$\mathbb{H} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix} = \begin{pmatrix} \mathbb{H}_{++} & \mathbb{H}_{+-} \\ \mathbb{H}_{-+} & \mathbb{H}_{--} \end{pmatrix} \begin{pmatrix} \psi_- \otimes \psi_+ \\ \psi_+ \otimes \psi_- \end{pmatrix}$$

V.M. Braun, A.N. Manashov, J. Rohrwild, arXiv:0908.1684

Results:

$z_{12} = z_1 - z_2$

$$\mathbb{C}_2^{SO(4,2)} = \widehat{\mathbf{J}}(\widehat{\mathbf{J}} - 1), \qquad \widehat{\mathbf{J}} = -\begin{pmatrix} 0 & \partial_2 z_{21} \\ \partial_1 z_{12} & 0 \end{pmatrix}$$

Eigenfunctions

$$\varphi_n^{\pm}(z_1, z_2) = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} z_{12}^n :$$

$$\mathbb{C}_2^{SO(4,2)} \varphi_n^+ = (n+2)(n+1)\varphi_n^+ \qquad J = n+2$$

$$\mathbb{C}_2^{SO(4,2)} \varphi_n^- = (n+1)n\varphi_n^- \qquad J = n+1$$

$\hookrightarrow$

Complete results for $2 \to 2$ RG kernels
! Not a single Feynman diagram calculated !

Example

$$\mathcal{O}^{ij}_{-+}(z_1,z_2) = \psi^i_-(z_1)\psi^j_+(z_2)\,, \qquad \mathcal{O}^{ij}_{+-}(z_1,z_2) = \psi^i_+(z_1)\psi^i_-(z_2)$$

$$\mathbb{H}\begin{pmatrix}\psi_-\otimes\psi_+\\ \psi_+\otimes\psi_-\end{pmatrix} = \begin{pmatrix}\mathbb{H}_{11} & \mathbb{H}_{12}\\ \mathbb{H}_{21} & \mathbb{H}_{22}\end{pmatrix}\begin{pmatrix}\psi_-\otimes\psi_+\\ \psi_+\otimes\psi_-\end{pmatrix}$$

consider

$$\begin{aligned}\mathbb{H}\begin{pmatrix}a\\ b\end{pmatrix} z_{12}^n &= \mathbb{H}\left[\frac{a+b}{2}\begin{pmatrix}1\\ 1\end{pmatrix} z_{12}^n + \frac{a-b}{2}\begin{pmatrix}1\\ -1\end{pmatrix} z_{12}^n\right] = \frac{a+b}{2}\mathbb{H}\varphi_n^+ + \frac{a-b}{2}\mathbb{H}\varphi_n^- \\ &= \frac{a+b}{2}E(n)\varphi_n^+ + \frac{a-b}{2}E(n-1)\varphi_n^- = \begin{pmatrix}h_{11}(n) & h_{12}(n)\\ h_{21}(n) & h_{22}(n)\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} z_{12}^n\end{aligned}$$

$E(n)$ is the same function as in $++$ operators:

$$h_{11}(n) = \psi(n+2)+\psi(n+1)-2\psi(1)\,, \qquad h_{12}(n) = \frac{1}{n+1}$$

obtain

$$\begin{aligned}[\mathbb{H}\,\mathcal{O}^{ij}_{-+}](z_1,z_2) = -2t^a_{ii'}t^a_{jj'}\Big\{\int_0^1\frac{d\alpha}{\alpha}\Big[2\mathcal{O}^{i'j'}_{-+}(z_1,z_2) - \mathcal{O}^{i'j'}_{-+}(z_{12}^\alpha,z_2) - \bar\alpha\mathcal{O}^{i'j'}_{-+}(z_1,z_{21}^\alpha)\Big] \\ + \int_0^1 d\alpha\,\mathcal{O}^{i'j'}_{+-}(z_{12}^\alpha,z_2)\Big\}\end{aligned}$$

Does the Lorentz symmetry fix $2 \to 3$ kernels?

What to do with $\mathbb{H}^{(2\to 3)}$?? E.g. $\psi_-\psi_+,\ \psi_+\psi_- \to \psi_+\psi_+\bar{f}_{++}$

Idea:

- Infinitesimal translation in transverse plane $P_{\mu\bar\lambda}$

$$i[\mathbf{P}_{\mu\bar\lambda}, \psi_+] \quad = \quad 2\partial_z\psi_- + igA_{\mu\bar\lambda}\psi_+ + \text{EOM}\,,$$

- Lorentz Rotation $M_{\mu\mu}$

$$i[\mathbf{M}_{\mu\mu}, \psi_+] \quad \sim \quad (z\partial_z + 1)\psi_- + \frac{1}{2} igzA_{\mu\bar\lambda}\psi_+ + \text{EOM}\,,$$

$\hookrightarrow$ Exact relations between renormalized operators containing "plus" and "minus" fields
$\hookrightarrow$ The counterterms on the LHS and RHS must coincide

! It works and proves to be very efficient !

Translation

Notation

$$\mathcal{O}^{ij}_{-+}(z_1,z_2) = \psi^i_-(z_1)\otimes\psi^j_+(z_2) \qquad \mathcal{O}^{ij}_{+-}(z_1,z_2) = \psi^i_+(z_1)\otimes\psi^j_-(z_2)$$

$$\mathcal{O}^{ij}_{++}(z_1,z_2) = \psi^i_+(z_1)\otimes\psi^j_+(z_2) \qquad \mathcal{O}^{ija}_{f}(z_1,z_2,z_3) = \psi^i_+(z_1)\otimes\psi^j_+(z_2)\otimes\bar{f}^a_{++}(z_3)$$

We are looking for three-particle counterterms

$$[\mathcal{O}^{ij}_{\pm\mp}(z_1,z_2)]'_R \sim \frac{1}{\epsilon}[\mathbb{H}^{(\pm\mp)}_{\to f}\,\mathcal{O}_f]^{ij}(z_1,z_2)$$

- Apply transverse derivative to leading-twist $\mathcal{O}_{++}$ operator

$$\begin{aligned}\partial_{\mu\bar\lambda}[\mathcal{O}^{ij}_{++}(z_1,z_2)]'_R &= 2\partial_{z_1}[\mathcal{O}^{ij}_{-+}(z_1,z_2)]'_R + 2\partial_{z_2}[\mathcal{O}^{ij}_{+-}(z_1,z_2)]'_R \\ &\quad +ig[A_{\mu\bar\lambda}(z_1)\psi_+(z_1)\otimes\psi_+(z_2)]'_R \\ &\quad +ig[\psi_+(z_1)\otimes A_{\mu\bar\lambda}(z_2)\psi_+(z_2)]'_R + \text{EOM}\end{aligned}$$

- Convert $A_{\mu\bar\lambda}$ into $\bar{f}_{++}$

$$A^b_{\mu\bar\lambda}(z_1) - A^b_{\mu\bar\lambda}(z_2) = -z_{12}(\mu\lambda)\int_0^1 d\tau\,\bar{f}^b_{++}(z^\tau_{12})$$

- rewrite the expression on the LHS:

$$\partial_{\mu\bar\lambda}[\mathcal{O}^{ij}_{++}(z_1,z_2)]'_R \quad = \quad \partial_{\mu\bar\lambda}\frac{1}{\epsilon}[\mathbb{H}^{++}\cdot\mathcal{O}_{++}]^{ij}(z_1,z_2)$$

$\longrightarrow$ contains two-particle and three-particle counterterms

- after a little algebra:

$$\boxed{\mathbb{LHS} \equiv \partial_1\mathbb{H}^{(-+)}_{\to f} + \partial_2\mathbb{H}^{(+-)}_{\to f} \quad = \quad \mathbb{RHS} \quad \text{(known expression)}}$$

- This equation is not $SL(2,R)$ invariant!

$$L^{+,(j_1j_2j_3)}_{123} = L^{+,j_1}_1 + S^{+,j_2}_2 + L^{+,j_3}_3 = \sum_{k=1}^{3} z_k^2\partial_k + 2j_kz_k$$

$$(\mathbb{LHS}-\mathbb{RHS})L^{+,(1,1,3/2)}_{123} \quad = \quad L^{+,(1,1)}_{12}(\mathbb{LHS}-\mathbb{RHS}) + (\widetilde{\mathbb{LHS}}-\widetilde{\mathbb{RHS}})$$

- This means that we have two equations

$$\begin{aligned} \mathbb{LHS} \quad &= \quad \partial_1\mathbb{H}^{(-+)}_{\to f} + \partial_2\mathbb{H}^{(+-)}_{\to f} = \mathbb{RHS}\,, \\ \widetilde{\mathbb{LHS}} \quad &= \quad \partial_1 z_1\mathbb{H}^{(-+)}_{\to f} + \partial_2 z_2\mathbb{H}^{(+-)}_{\to f} = \widetilde{\mathbb{RHS}} \end{aligned}$$

- Final set of equations

$$\partial_1 z_{12} \mathbb{H}^{(-+)}_{\to f} + \mathbb{H}^{(+-)}_{\to f} = \sum_{i=1}^{3} C_i \mathcal{A}_i$$

$$\partial_2 z_{21} \mathbb{H}^{(+-)}_{\to f} + \mathbb{H}^{(-+)}_{\to f} = \sum_{i=1}^{3} C_i \widetilde{\mathcal{A}}_i$$

$$[\mathcal{A}_1\varphi](z_1,z_2) = z_{12}^2 \left(\int_0^1 d\beta \bar\beta\, \varphi(z_1,z_2,z_{12}^\beta) - \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \beta\, \varphi(z_1, z_{21}^\alpha, z_{12}^\beta) \right)$$

$$[\mathcal{A}_2\varphi](z_1,z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar\alpha}^1 d\beta \frac{\bar\alpha\bar\beta}{\alpha}\, \varphi(z_1, z_{21}^\alpha, z_{12}^\beta)$$

$$[\mathcal{A}_3\varphi](z_1,z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar\alpha}^1 d\beta \frac{\bar\alpha}{\alpha^2} (\bar\alpha - \beta)\, \varphi(z_{12}^\alpha, z_2, z_{21}^\beta)$$

C_i are the color structures:

$$C_1 = f^{bcd}(t^b \otimes t^c)\,, \qquad C_2 = i(t^b \otimes t^d t^b)\,, \qquad C_3 = -i(t^d t^b \otimes t^b)$$

- have unique solution (proven)

Result

$$
[\mathbb{H}^{(-+)}_{\to f}\,\mathcal{O}_f](z_1,z_2) = z_{12}^2\Big\{ f^{abc} t^b \otimes t^c \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\beta\, \mathcal{O}_f(z_{12}^\alpha, z_2, z_{21}^\beta)
+ i(t^a t^b)\otimes t^b \int_0^1 d\alpha \int_{\bar\alpha}^1 d\beta \frac{\bar\alpha\bar\beta}{\alpha}\,\mathcal{O}_f(z_{12}^\alpha, z_2, z_{21}^\beta)\Big\}
$$

72/(Parity × Charge Conjugation) $\mapsto$ 18 independent kernels

see arXiv:0908.1684 for full list and technical details (many)

Summary

- Lorentz symmetry uniquely determines renormalization properties of operators involving higher-twist field components in terms of partonic ones
    - Probably true to all orders
    - Efficient technique at least to LO
    - Conformal symmetry is not necessary but simplifies the analysis dramatically

We are able to show that the same results can be obtained from Lorentz symmetry alone, by applying translations and rotations to the leading-twist kernels

This involves subtleties, since light-cone gauge condition is not Lorentz-invariant, but treatement of the corresponding corrections is simple because of a certain Ward identity

- Complete results for renormalization of arbitrary twist-four operators