All-order results for infrared and collinear singularities

Lorenzo Magnea

Dipartimento di Fisica Teorica, Università di Torino and INFN, Torino

RADCOR 2009 - October 26 , 2009

Outline

Introduction

Ancient History

JULY 15, 1937 PHYSICAL REVIEW VOLUME 52

Note on the Radiation Field of the Electron

F. BLOCH AND A. NORDSIECK*
Stanford University, California

(Received May 14, 1937)

Previous methods of treating radiative corrections in non-stationary processes such as the scattering of an electron in an atomic field or the emission of a β-ray, by an expansion in powers of $e^2/\hbar c$, are defective in that they predict infinite low frequency corrections to the transition probabilities. This difficulty can be avoided by a method developed here which is based on the alternative assumption that $e^2\omega/mc^3$, $\hbar\omega/mc^2$ and $\hbar\omega/c\Delta p$ (ω = angular frequency of radiation, Δp = change in momentum of electron) are small compared to unity. In contrast to the expansion in powers of $e^2/\hbar c$, this permits the transition to the classical limit $\hbar = 0$. External perturbations on the electron are treated in the Born approximation. It is shown that for frequencies such that the above three parameters are negligible the quantum mechanical calculation yields just the directly reinterpreted results of the classical formulae, namely that the total probability of a given change in the motion of the electron is unaffected by the interaction with radiation, and that the mean number of emitted quanta is infinite in such a way that the mean radiated energy is equal to the energy radiated classically in the corresponding trajectory.

A remarkable achievement, before quantum field theory was born.

Modern History

Factorization

$$\mathcal{M}_{\{r_i\}}\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right)=\sum_L \mathcal{M}_L\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right)(c_L)_{\{r_i\}}$$

$$\mathcal{M}_L\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right) = \overline{\mathcal{S}}_{LK}\left(\rho_{ij},\alpha_s(\mu^2),\epsilon\right) H_K\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right) \times \prod_{i=1}^{n} J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

Progress

- Exponentiation applies to non-abelian gauge theories.
- Exponentiation extends to collinear divergences.
- Exponentiation is performed at the amplitude level.
- An optimal regularization scheme is used.

Tools: dimensional regularization

Nonabelian exponentiation of IR/C poles requires d-dimensional evolution equations. The running coupling in $d = 4 - 2\epsilon$ obeys

$$\mu \frac{\partial \overline{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \overline{\alpha}) = -2\,\epsilon\,\overline{\alpha} + \hat{\beta}(\overline{\alpha}) \ , \quad \hat{\beta}(\overline{\alpha}) = -\frac{\overline{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\overline{\alpha}}{\pi} \right)^n \ .$$

The one-loop solution is

$$\overline{\alpha}\left(\mu^2, \epsilon\right) = \alpha_s(\mu_0^2) \left[\left(\frac{\mu^2}{\mu_0^2} \right)^{\epsilon} - \frac{1}{\epsilon} \left(1 - \left(\frac{\mu^2}{\mu_0^2} \right)^{\epsilon} \right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} \ .$$

The β function develops an IR free fixed point, so that $\overline{\alpha}(0, \epsilon) = 0$ for $\epsilon < 0$. The Landau pole is at

$$\mu^2 = \Lambda^2 \equiv Q^2 \left(1 + \frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)} \right)^{-1/\epsilon} \ .$$

- Integrations over the scale of the coupling can be analytically performed.
- All infrared and collinear poles arise by integration of $\alpha_s(\mu^2, \epsilon)$.

Tools: factorization

All factorizations separating dynamics at different energy scales lead to resummation of logarithms of the ratio of scales.

- Renormalization group logarithms.

  Renormalization factorizes cutoff dependence

$$G_0^{(n)}\left(p_i, \Lambda, g_0\right) = \prod_{i=1}^{n} Z_i^{1/2}\left(\Lambda/\mu, g(\mu)\right)\, G_R^{(n)}\left(p_i, \mu, g(\mu)\right)\ ,$$

$$\frac{dG_0^{(n)}}{d\mu} = 0 \ \rightarrow\ \frac{d\log G_R^{(n)}}{d\log\mu} = -\sum_{i=1}^{n}\gamma_i\left(g(\mu)\right)\ .$$

- RG evolution resums $\alpha_s^n(\mu^2)\log^n\left(Q^2/\mu^2\right)$ into $\alpha_s(Q^2)$.

Note: Factorization is the difficult step. It requires a diagrammatic analysis

- all-order power counting (UV, IR, collinear ...);
- implementation of gauge invariance via Ward identities.

Tools: factorization

- Collinear factorization logarithms.

  Mellin moments of partonic DIS structure functions factorize

$$\widetilde{F}_2\left(N, \frac{Q^2}{m^2}, \alpha_s\right) = \widetilde{C}\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \widetilde{f}\left(N, \frac{\mu_F^2}{m^2}, \alpha_s\right)$$

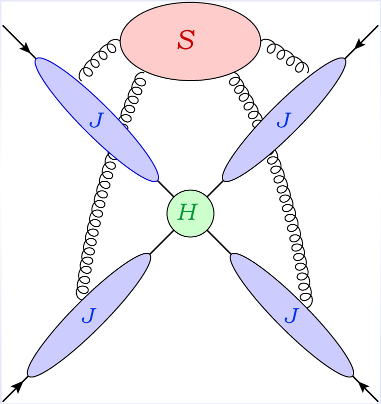
$$\frac{d\widetilde{F}_2}{d\mu_F} = 0 \quad \rightarrow \quad \frac{d\log \widetilde{f}}{d\log \mu_F} = \gamma_N\left(\alpha_s\right) \ .$$

- Altarelli-Parisi evolution resums collinear logarithms into evolved parton distributions (or fragmentation functions).

Note: Sudakov (double) logarithms are more difficult.

- A double factorization is required: hard vs. collinear vs. soft. Gauge invariance plays a key role in the decoupling.
- After identification of the relevant modes, effective field theory can be used (SCET).

Sudakov factorization



Leading regions for Sudakov factorization.

- Divergences arise in fixed-angle amplitudes from leading regions in loop momentum space.
- Soft gluons factorize both form hard (easy) and from collinear (intricate) virtual exchanges.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: $S \propto \mathbf{I}$.
- In the planar limit S can be reabsorbed defining jets J as square roots of elementary form factors.
- Beyond the planar limit S is determined by an anomalous dimension matrix Γ_S.
- Phenomenological applications to jet and heavy quark production at hadron colliders.

Form Factors and Planar Amplitudes

(with Lance Dixon and George Sterman)

Gauge theory form factors

Consider as an example the quark form factor

$$\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_\mu(0) | p_1, p_2 \rangle = \bar{v}(p_2) \gamma_\mu u(p_1) \, \Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) .$$

- The form factor obeys the evolution equation

$$Q^2 \frac{\partial}{\partial Q^2} \log\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] = \frac{1}{2}\left[K\left(\epsilon, \alpha_s(\mu^2)\right) + G\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] ,$$

- Renormalization group invariance requires

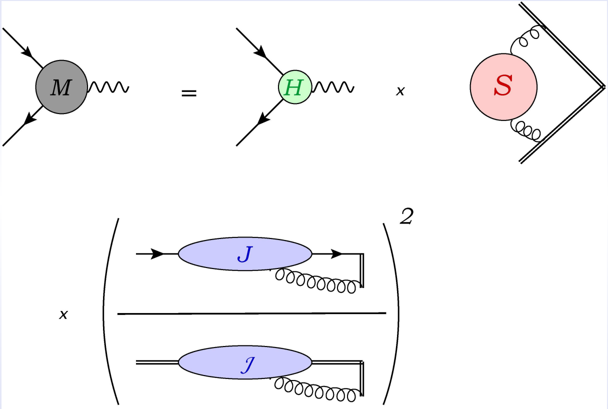
$$\mu \frac{dG}{d\mu} = -\mu \frac{dK}{d\mu} = \gamma_K\left(\alpha_s(\mu^2)\right) .$$

$\gamma_K(\alpha_s)$ is the cusp anomalous dimension (G. Korchemsky and A. Radyushkin; ...).

- Dimensional regularization provides a trivial initial condition for evolution if $\epsilon < 0$ (for IR regularization).

$$\overline{\alpha}(\mu^2 = 0, \epsilon < 0) = 0 \quad \rightarrow \Gamma\left(0, \alpha_s(\mu^2), \epsilon\right) = \Gamma\left(1, \overline{\alpha}\left(0, \epsilon\right), \epsilon\right) = 1 .$$

Detailed factorization



Operator factorization for the Sudakov form factor, with subtractions.

Operator definitions

The functional form of this graphical factorization is

$$\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = H\left(\frac{Q^2}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)\times\mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right)$$
$$\times\prod_{i=1}^{2}\left[\frac{J\left(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)}{\mathcal{J}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\epsilon\right)}\right].$$

We introduced factorization vectors n_i^μ, with $n_i^2\neq 0$, to define the jets,

$$J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)u(p) = \langle 0\,|\Phi_n(\infty,0)\,\psi(0)\,|p\rangle\,.$$

where Φ_n is the Wilson line operator along the direction n^μ.

$$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[\mathrm{i}g\int_{\lambda_1}^{\lambda_2}d\lambda\,n\cdot A(\lambda n)\right].$$

The jet J has collinear divergences only along p.

Operator definitions

The soft function $\mathcal{S}$ is the eikonal limit of the massless form factor

$$\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) = \langle 0 | \Phi_{\beta_2}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, | 0 \rangle \; .$$

Soft-collinear regions are subtracted dividing by eikonal jets $\mathcal{J}$.

$$\mathcal{J}\left(\frac{(\beta_1 \cdot n_1)^2}{n_1^2}, \alpha_s(\mu^2), \epsilon\right) \; = \; \langle 0 | \Phi_{n_1}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, | 0 \rangle \; ,$$

- $\mathcal{S}$ and $\mathcal{J}$ are pure counterterms in dimensional regularization.
- β_i-dependence of $\mathcal{S}$ and $\mathcal{J}$ violates rescaling invariance of Wilson lines.
  $\Rightarrow$ It arises from double poles, associated with γ_K.
- A single pole function where the cusp anomaly cancels is

$$\overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right) \equiv \frac{\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{2} \mathcal{J}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}$$

It can only depend on the scaling variable

$$\rho_{12} \equiv \frac{(\beta_1 \cdot \beta_2)^2 \; n_1^2 \; n_2^2}{(\beta_1 \cdot n_1)^2 \, (\beta_2 \cdot n_2)^2} \; .$$

Jet evolution

The full form factor does not depend on the factorization vectors n_i^μ.
Defining $x_i \equiv (\beta_i \cdot n_i)^2 / n_i^2$,

$$x_i \frac{\partial}{\partial x_i} \log \Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = 0 \, .$$

This dictates the evolution of the jet J, through a '$K + G$' equation

$$\begin{aligned} x_i \frac{\partial}{\partial x_i} \log J_i &= - x_i \frac{\partial}{\partial x_i} \log H + x_i \frac{\partial}{\partial x_i} \log \mathcal{J}_i \\ &\equiv \frac{1}{2} \left[\mathcal{G}_i \Big(x_i, \alpha_s(\mu^2), \epsilon \Big) + \mathcal{K} \Big(\alpha_s(\mu^2), \epsilon \Big) \right] , \end{aligned}$$

Imposing RG invariance of the form factor

$$\gamma_{\overline{\mathcal{S}}} \left(\rho_{12}, \alpha_s \right) + \gamma_H \left(\rho_{12}, \alpha_s \right) + 2 \gamma_J \left(\alpha_s \right) = 0 \, .$$

leads to the final evolution equation

$$Q \frac{\partial}{\partial Q} \log \Gamma = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log H - \gamma_{\bar{\mathcal{S}}} - 2 \, \gamma_J + \sum_{i=1}^{2} \left(\mathcal{G}_i + \mathcal{K} \right) \, .$$

Results for Sudakov form factors

- The counterterm function K is determined by γ_K.

$$\mu \frac{d}{d\mu} K(\epsilon, \alpha_s) = -\gamma_K(\alpha_s) \quad \Longrightarrow \quad K\left(\epsilon, \alpha_s(\mu^2)\right) = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K\left(\bar{\alpha}(\lambda^2, \epsilon)\right) .$$

- The form factor can be written in terms of just G and γ_K,

$$\Gamma\left(Q^2, \epsilon\right) = \exp\left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[G\Big(-1, \overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\Big) - \frac{1}{2} \gamma_K\left(\overline{\alpha}\left(\xi^2, \epsilon\right)\right) \log\left(\frac{-Q^2}{\xi^2}\right) \right] \right\} .$$

$\Rightarrow$ In general, poles up to $\alpha_s^n/\epsilon^{n+1}$ appear in the exponent.

- The ratio of the timelike to the spacelike form factor is

$$\log\left[\frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)}\right] = \mathrm{i}\frac{\pi}{2} K(\epsilon) + \frac{\mathrm{i}}{2} \int_0^{\pi} \left[G\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\theta} Q^2\right), \epsilon\right) - \frac{\mathrm{i}}{2} \int_0^{\theta} d\phi \, \gamma_K\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\phi} Q^2\right)\right) \right]$$

$\Rightarrow$ Infinities are confined to a phase given by γ_K.

$\Rightarrow$ The modulus of the ratio is finite, and physically relevant.

Form factors in $\mathcal{N} = 4$ SYM

- In $d = 4 - 2\epsilon$ conformal invariance is broken and $\beta(\alpha_s) = -2\,\epsilon\,\alpha_s$.
- All integrations are trivial. The exponent has only double and single poles to all orders (Z. Bern, L. Dixon, A. Smirnov).

$$\log\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \left(\frac{\mu^2}{-Q^2}\right)^{n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right]$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \mathrm{e}^{-\mathrm{i}\pi n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right],$$

- In the planar limit this captures all singularities of fixed-angle amplitudes in $\mathcal{N} = 4$ SYM. The structure remains valid at strong coupling, in the planar limit (F. Alday, J. Maldacena).
- The analytic continuation yields a finite result in four dimensions, arguably exact.

$$\left|\frac{\Gamma(Q^2)}{\Gamma(-Q^2)}\right|^2 = \exp\left[\frac{\pi^2}{4}\,\gamma_K\left(\alpha_s(Q^2)\right)\right].$$

Characterizing $G(\alpha_s, \epsilon)$

The single-pole function $G(\alpha_s, \epsilon)$ is a sum of anomalous dimensions

$$G(\alpha_s, \epsilon) = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log H - \gamma_{\bar{\mathcal{S}}} - 2\gamma_J + \sum_{i=1}^{2} \mathcal{G}_i \,,$$

In $d = 4 - 2\epsilon$ finite remainders can be neatly exponentiated

$$C\left(\alpha_s(Q^2), \epsilon\right) = \exp\left[\int_0^{Q^2} \frac{d\xi^2}{\xi^2} \left\{ \frac{d \, \log C\left(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\right)}{d \ln \xi^2} \right\}\right] \equiv \exp\left[\frac{1}{2} \int_0^{Q^2} \frac{d\xi^2}{\xi^2} G_C\left(\overline{\alpha}\left(\xi^2, \epsilon\right), \epsilon\right)\right]$$

The soft function exponentiates like the full form factor

$$\mathcal{S}\left(\alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \left[G_{\rm eik}\left(\overline{\alpha}\left(\xi^2, \epsilon\right)\right) - \frac{1}{2} \gamma_K\left(\overline{\alpha}\left(\xi^2, \epsilon\right)\right) \log\left(\frac{\mu^2}{\xi^2}\right)\right]\right\} .$$

$G(\alpha_s, \epsilon)$ is then simply related to collinear splitting functions and to the eikonal approximation

$$G(\alpha_s, \epsilon) = 2\, B_\delta\left(\alpha_s\right) + G_{\rm eik}\left(\alpha_s\right) + G_{\overline{H}}\left(\alpha_s, \epsilon\right) \,,$$

$\Rightarrow$ $G_{\overline{H}}$ does not generate poles; it vanishes in $\mathcal{N} = 4$ SYM.

$\Rightarrow$ Checked at strong coupling, in the planar limit (F. Alday).

Beyond the Planar Limit

(with Einan Gardi)

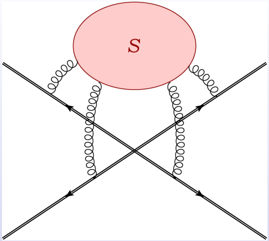
Factorization at fixed angle

Fixed-angle scattering amplitudes in any massless gauge theory can also be factorized into hard, jet and soft functions.

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right],$$

The soft function is now a matrix, mixing the available color tensors.

$$(c_L)_{\{\alpha_k\}} \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)$$
$$= \sum_{\{\eta_k\}} \langle 0| \prod_{i=1}^{n}\left[\Phi_{\beta_i}(\infty, 0)_{\alpha_k, \eta_k}\right] |0\rangle \, (c_K)_{\{\eta_k\}},$$



Soft exchanges mix color structures.

Soft anomalous dimensions

The soft function $\mathcal{S}$ obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) = - \Gamma^{\mathcal{S}}_{IJ} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) \mathcal{S}_{JK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) ,$$

- Note: $\Gamma^{\mathcal{S}}$ is singular due to overlapping UV and collinear poles.

As before, $\mathcal{S}$ is a pure counterterm. In dimensional regularization, then

$$\mathcal{S} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) = P \exp \left[-\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon \right) \right] .$$

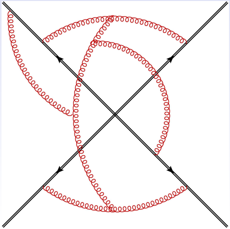
Double poles cancel in the reduced soft function

$$\overline{\mathcal{S}}_{LK} \left(\rho_{ij}, \alpha_s(\mu^2), \epsilon \right) = \frac{\mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right)}{\prod_{i=1}^{n} \mathcal{J}_i \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}$$

- $\overline{\mathcal{S}}$ must depend on rescaling invariant variables, $\rho_{ij} \equiv \frac{n_i^2 \, n_j^2 \, (\beta_i \cdot \beta_j)^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2}$.
- The anomalous dimension $\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$ for the evolution of $\overline{\mathcal{S}}$ is finite.

Surprising simplicity

- $\Gamma^{\mathcal{S}}$ can be computed from UV poles of $\mathcal{S}$
- Non-abelian eikonal exponentiation selects the relevant diagrams: webs
- $\Gamma^{\mathcal{S}}$ appears highly complex at high orders.



A web contributing to $\Gamma^{\mathcal{S}}$.

The two-loop calculation (M. Aybat, L. Dixon, G. Sterman) leads to a surprising result: for any number of light-like eikonal lines

$$\Gamma^{(2)}_{\mathcal{S}} = \frac{\kappa}{2}\,\Gamma^{(1)}_{\mathcal{S}} \qquad \kappa = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F C_F \,.$$

- No new kinematic dependence; no new matrix structure.
- κ is the two-loop coefficient of γ_K, rescaled by the appropriate Casimir,

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)}\left[2\,\frac{\alpha_s}{\pi} + \kappa\left(\frac{\alpha_s}{\pi}\right)^2\right] + \mathcal{O}\left(\alpha_s^3\right) \,.$$

Factorization constraints

- The classical rescaling symmetry of Wilson line correlators under $\beta_i \to \kappa \beta_i$ is violated only through the cusp anomaly.

  $\Rightarrow$ For eikonal jets, no β_i dependence is possible at all except through the cusp.

- In the reduced soft function $\overline{\mathcal{S}}$ the cusp anomaly cancels.

  $\Rightarrow$ $\overline{\mathcal{S}}$ must depend on β_i only through rescaling-invarant combinations such as ρ_{ij}, or, for $n \geq 4$ legs, the cross ratios $\rho_{ijkl} \equiv (\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)/(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)$

Consider then the anomalous dimension for the reduced soft function

$$\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s(\mu^2)\right) = \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}\left(\frac{(\beta_k \cdot n_k)^2}{n_k^2}, \alpha_s(\mu^2), \epsilon\right) .$$

This poses strong constraints on the soft matrix. Indeed

- Singular terms in $\Gamma^{\mathcal{S}}$ must be diagonal and proportional to γ_K.
- Finite diagonal terms must conspire to construct ρ_{ij}'s combining $\beta_i \cdot \beta_j$ with x_i.
- Off-diagonal terms in $\Gamma^{\mathcal{S}}$ must be finite, and must depend only on the cross-ratios ρ_{ijkl}.

Factorization constraints

The constraints can be formalized simply by using the chain rule.
$\Gamma^{\overline{\mathcal{S}}}$ depends on x_i in a simple way.

$$x_i \frac{\partial}{\partial x_i} \Gamma^{\overline{\mathcal{S}}}_{IJ} \left(\rho_{ij}, \alpha_s\right) = - \delta_{IJ} \; x_i \frac{\partial}{\partial x_i} \gamma_{\mathcal{J}} \left(x_i, \alpha_s, \epsilon\right) \; = \; - \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s\right) \delta_{IJ} \, .$$

This leads to a linear equation for the dependence of $\Gamma^{\overline{\mathcal{S}}}$ on ρ_{ij}

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{MN} \left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s\right) \delta_{MN} \qquad \forall i \, ,$$

- The equation relates $\Gamma^{\overline{\mathcal{S}}}$ to γ_K to all orders in perturbation theory
  $\Rightarrow$ and should remain true at strong coupling as well.
- It correlates color and kinematics for any number of hard partons.
- It admits a unique solution for amplitudes with up to three hard partons.
  $\Rightarrow$ For $n \geq 4$ hard partons, functions of ρ_{ijkl} solve the homogeneous equation.

Minimal solution

The cusp anomalous dimension exhibits Casimir scaling up to three loops.

- $\gamma_K^{(i)}(\alpha_s) = C_i\,\widehat{\gamma}_K(\alpha_s)$ with C_i the quadratic Casimir and $\widehat{\gamma}_K(\alpha_s)$ universal.

Denoting with $\widetilde{\gamma}_K^{(i)}$ possible terms violating Casimir scaling, we write

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\left[C_i\,\widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)\right] \qquad \forall i\,,$$

By linearity, using the color generator notation, the scaling term yields

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{S}}_{\text{Q.C.}}(\rho_{ij}, \alpha_s) = \frac{1}{4}\,\mathrm{T}_i \cdot \mathrm{T}_i\;\widehat{\gamma}_K(\alpha_s)\,, \qquad \forall i$$

An all-order solution is the dipole formula (E. Gardi, LM; T. Becher, M. Neubert)

$$\Gamma^{\overline{S}}_{\text{dip}}(\rho_{ij}, \alpha_s) = -\frac{1}{8}\,\widehat{\gamma}_K(\alpha_s)\sum_{j\neq i} \ln(\rho_{ij})\;\mathrm{T}_i \cdot \mathrm{T}_j + \frac{1}{2}\,\widehat{\delta}_{\overline{S}}(\alpha_s)\sum_i \mathrm{T}_i \cdot \mathrm{T}_i\,,$$

as easily checked using color conservation, $\sum_i T_i = 0$.

Note: all known results for massless gauge theories are of this form.

The full amplitude

It is possible to construct a dipole formula for the full amplitude enforcing the cancellation of the dependence on the factorization vectors n_i through

$$\ln\left(\frac{(2p_i\cdot n_i)^2}{n_i^2}\right)+\ln\left(\frac{(2p_j\cdot n_j)^2}{n_j^2}\right)+\ln\left(\frac{(-\beta_i\cdot\beta_j)^2\,n_i^2 n_j^2}{2(\beta_i\cdot n_i)^2\,2(\beta_j\cdot n_j)^2}\right)=2\ln\left(-2p_i\cdot p_j\right)\,.$$

Soft and collinear singularities can then be collected in a matrix Z

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\epsilon\right)=Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right)\,\mathcal{H}\left(\frac{p_i}{\mu},\frac{\mu_f}{\mu},\alpha_s(\mu^2),\epsilon\right)\,,$$

satisfying a matrix evolution equation

$$\frac{d}{d\ln\mu_f}Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right)\;=\;-\,\Gamma\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2)\right)Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\epsilon\right)\,.$$

The dipole structure of $\Gamma^{\overline{S}}$ is inherited by Γ, which is given by

$$\Gamma_{\rm dip}\left(\frac{p_i}{\lambda},\alpha_s(\lambda^2)\right)=-\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right)\sum_{j\neq i}\ln\left(\frac{-2\,p_i\cdot p_j}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j+\sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s(\lambda^2)\right)\,.$$

Beyond the dipole formula

(with Lance Dixon and Einan Gardi)

Beyond the minimal solution

- The cusp anomalous dimension may violate Casimir scaling starting at four loops. This would add a contribution $\Gamma^{\mathcal{S}}_{\rm H.C.}$ satisfying

$$\sum_{j,\, j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\mathcal{S}}_{\rm H.C.}\left(\rho_{ij}, \alpha_s\right) = \frac{1}{4}\, \widetilde{\gamma}_K^{(i)}\left(\alpha_s\right) , \qquad \forall i \, .$$

- For $n \geq 4$ the constraints do not uniquely determine $\Gamma^{\mathcal{S}}$: one may write

$$\Gamma^{\mathcal{S}}\left(\rho_{ij}, \alpha_s\right) = \Gamma^{\mathcal{S}}_{\rm dip}\left(\rho_{ij}, \alpha_s\right) + \Delta^{\mathcal{S}}\left(\rho_{ij}, \alpha_s\right) ,$$

  where $\Delta^{\mathcal{S}}$ solves the homogeneous equation

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Delta^{\mathcal{S}}\left(\rho_{ij}, \alpha_s\right) = 0 \qquad \Leftrightarrow \qquad \Delta^{\mathcal{S}} = \Delta^{\mathcal{S}}\left(\rho_{ijkl}, \alpha_s\right) \, .$$

- By eikonal exponentiation $\Delta^{\mathcal{S}}$ must directly correlate four partons.
  - A nontrivial function of ρ_{ijkl} cannot appear in $\Gamma^{\mathcal{S}}$ at two loops.

$$\widetilde{\mathbf{H}}_{[\rm f]} = \sum_{j,k,l} \sum_{a,b,c} {\rm i} f_{abc}\, {\rm T}_j^a {\rm T}_k^b {\rm T}_l^c \, \ln\left(\rho_{ijkl}\right) \, \ln\left(\rho_{iklj}\right) \, \ln\left(\rho_{iljk}\right) \, .$$

  - The minimal solution holds for ‘matter loop’ diagrams at three loops (L. Dixon).

Collinear constraints

Factorization of fixed-angle amplitudes breaks down in collinear limits, as $p_i \cdot p_j \to 0$. New singularities are captured by a universal splitting function

$$\mathcal{M}_n \left(p_1, p_2, p_j; \mu, \epsilon\right) \xrightarrow{1\|2} \mathbf{Sp} \left(p_1, p_2; \mu, \epsilon\right) \mathcal{M}_{n-1} \left(P, p_j; \mu, \epsilon\right) .$$

Infrared poles of the splitting function are generated by a splitting anomalous dimension

$$\mathbf{Sp}(p_1, p_2; \mu, \epsilon) = \mathbf{Sp}_{\mathcal{H}}^{(0)}(p_1, p_2; \mu, \epsilon) \exp \left[-\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) \right] ,$$

related to the soft anomalous dimensions of the two amplitudes:

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu_f) \equiv \Gamma_n \left(p_1, p_2, p_j; \mu_f\right) - \Gamma_{n-1} \left(P, p_j; \mu_f\right) .$$

If the dipole formula receives corrections, so does the splitting amplitude

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \lambda) = \Gamma_{\mathbf{Sp},\,\mathrm{dip}}(p_1, p_2; \lambda) + \Delta_n \left(\rho_{ijkl}; \lambda\right) - \Delta_{n-1} \left(\rho_{ijkl}; \lambda\right) .$$

Universality of $\Gamma_{\mathbf{Sp}}$ constrains $\Delta_n - \Delta_{n-1}$: it must depend only on the collinear parton pair (T. Becher, M. Neubert).

Bose symmetry, transcendentality

Contributions to $\Delta_n(\rho_{ijkl})$ arise from gluon subdiagrams of eikonal correlators. They must be Bose symmetric. With four hard partons,

$$\Delta_4(\rho_{ijkl}) = \sum_i h^{(i)}_{abcd}\, \mathbf{T}_i^a\, \mathbf{T}_j^b\, \mathbf{T}_k^c\, \mathbf{T}_l^d\; \Delta^{(i)}_{4,\,\mathrm{kin}}(\rho_{ijkl})\,,$$

the symmetries of $\Delta^{(i)}_{4,\,\mathrm{kin}}$ must match those of $h^{(i)}_{abcd}$. For polynomials in $L_{ijkl} \equiv \log \rho_{ijkl}$ one easily matches symmetries of available color tensors

$$\Delta_4(\rho_{ijkl}) = \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f_{ade}\, f_{cb}{}^{e}\, L_{1234}^{h_1} \left(L_{1423}^{h_2}\, L_{1342}^{h_3} - (-1)^{h_1+h_2+h_3}\, L_{1342}^{h_2}\, L_{1423}^{h_3} \right) + \mathrm{cycl.} \right],$$

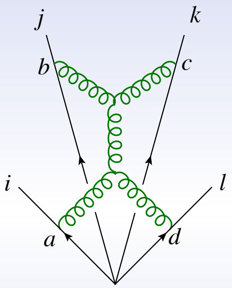
- Transcendentality constrains the powers of the logarithms. At L loops

$$h_{\mathrm{tot}} \equiv h_1 + h_2 + h_3 \leq \tau \leq 2L - 1$$

- For $\mathcal{N} = 4$ SYM, and for any massless gauge theory at three loops the bound is expected to be saturated.
- Collinear consistency requires $h_i \geq 1$ in any monomial.

Three loops

- Δ_n can first appear at three loops.
- A general Δ_n is a 'sum over quadrupoles'.
- Relevant webs are the same in $\mathcal{N} = 4$ SYM.
- The only available color tensors are $f_{ade}\, f_{cb}{}^{e}$
- Polynomials in L_{ijkl} are severely constrained.
- Using Jacobi identities for color and $L_{1234} + L_{1423} + L_{1342} = 0$ for kinematics, only one structure polynomial in L_{ijkl} survives.



Three-loop web contributing to $\Gamma^{\mathcal{S}}$.

h_1	h_2	h_3	$h_{\rm tot}$	comment	
1	1	1	3	vanishes identically by Jacobi identity	
2	1	1	4	kinematic factor vanishes identically	
1	1	2	4	allowed by symmetry, excluded by transcendentality	
1	2	2	5	viable possibility	all coincide
3	1	1	5	viable possibility	all coincide
2	1	2	5	viable possibility	all coincide
1	1	3	5	viable possibility	all coincide

Survivors

Just one maximal transcendentality, Bose symmetric, collinear safe polynomial in the logarithms survives.

$$\Delta_4^{(122)}(\rho_{ijkl}) = \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f_{ade}\, f_{cb}{}^{e}\; L_{1234}\; (L_{1423}\, L_{1342})^2 + f_{cae}\, f_{db}{}^{e}\; L_{1423}\; (L_{1234}\, L_{1342})^2 + f_{bae}\, f_{cd}{}^{e}\; L_{1342}\; (L_{1423}\, L_{1234})^2 \Big] .$$

Allowing for polylogarithms, structures mimicking the simple symmetries of L_{ijkl} must be constructed. Two examples are

$$\Delta_4^{(122,\,\mathrm{Li}_2)}(\rho_{ijkl}) = \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f_{ade}\, f_{cb}{}^{e}\; L_{1234}\; \Big(\mathrm{Li}_2(1-\rho_{1342}) - \mathrm{Li}_2(1-1/\rho_{1342})\Big) \times \Big(\mathrm{Li}_2(1-\rho_{1423}) - \mathrm{Li}_2(1-1/\rho_{1423})\Big) + \mathrm{cycl.} \Big] .$$

$$\Delta_4^{(311,\,\mathrm{Li}_3)}(\rho_{ijkl}) = \mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c\mathbf{T}_4^d \Big[f_{ade}\, f_{cb}{}^{e}\; \Big(\mathrm{Li}_3(1-\rho_{1342}) - \mathrm{Li}_3(1-1/\rho_{1342})\Big)\; L_{1423}\, L_{1342} + \mathrm{cycl.} \Big] .$$

Higher-order polylogarithms are ruled out by their trancendentality combined with collinear constraints.

Perspective

- After $\mathcal{O}(10^2)$ years, soft and collinear singularities in massless gauge theories are still a fertile field of study.
  - $\Rightarrow$ We are probing the all-order structure of the nonabelian exponent.
  - $\Rightarrow$ All-order results constrain and test fixed order calculations.
  - $\Rightarrow$ Understanding singularities has phenomenological applications through resummation.
- Factorization theorems $\Rightarrow$ Evolution equations $\Rightarrow$ Exponentiation.
- Dimensional continuation is the simplest and most elegant regulator.
  - $\Rightarrow$ Transparent mapping UV $\leftrightarrow$ IR for 'pure counterterm' functions.
- Remarkable simplifications in $\mathcal{N} = 4$ SYM point to exact results.
- Only three functions, γ_K, $G_{\rm eik}$ and B_δ determine all singularities in the planar limit, and possibly beyond.
- Factorization and classical rescaling invariance severely constrain soft anomalous dimensions to all orders and for any number of legs.
- A simple dipole formula may encode all infrared singularites for any massless gauge theory.
- The study of possible corrections to the dipole formula is under way.