Generalised Unitarity For Massive One-Loop Amplitudes

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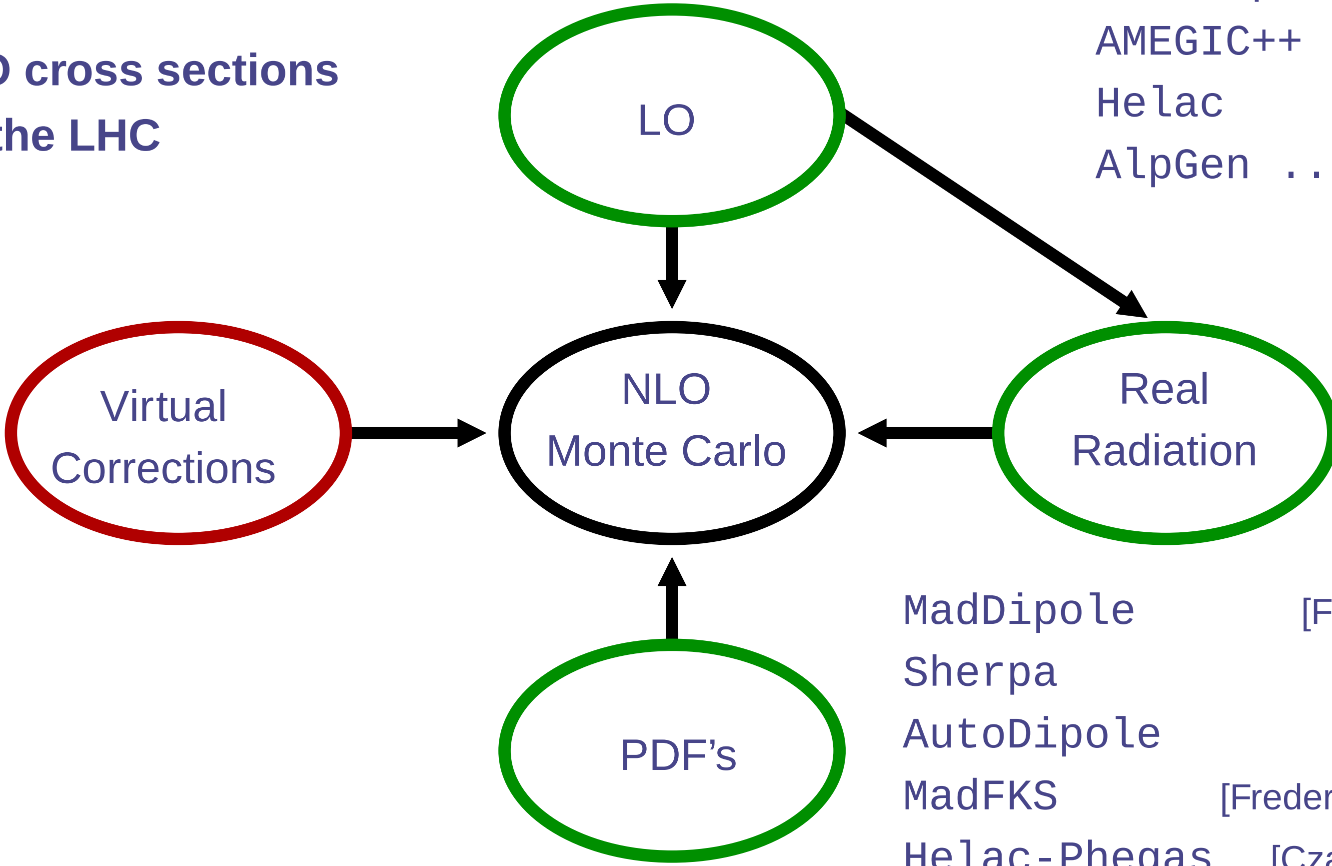
Radcor, Ascona, Switzerland

QCD for the LHC

NLO cross sections for the LHC



`MadGraph`
`AMEGIC++`
`Helac`
`AlpGen ...`

`MadDipole`	[Frederix,Gehrmann,Greiner]
`Sherpa`	[Gleisberg,Krauss]
`AutoDipole`	[Hasegawa,Moch,Uwer]
`MadFKS`	[Frederix,Frixione,Maltioni,Stelzer]
`Helac-Phegas`	[Czakon,Papadopoulos,Worek]

`POWHEG`	[Nason et al]
`MC@NLO`	[Frixione,Webber]

Computations of Virtual Corrections

- A lot of recent progress in computational methods for virtual corrections:

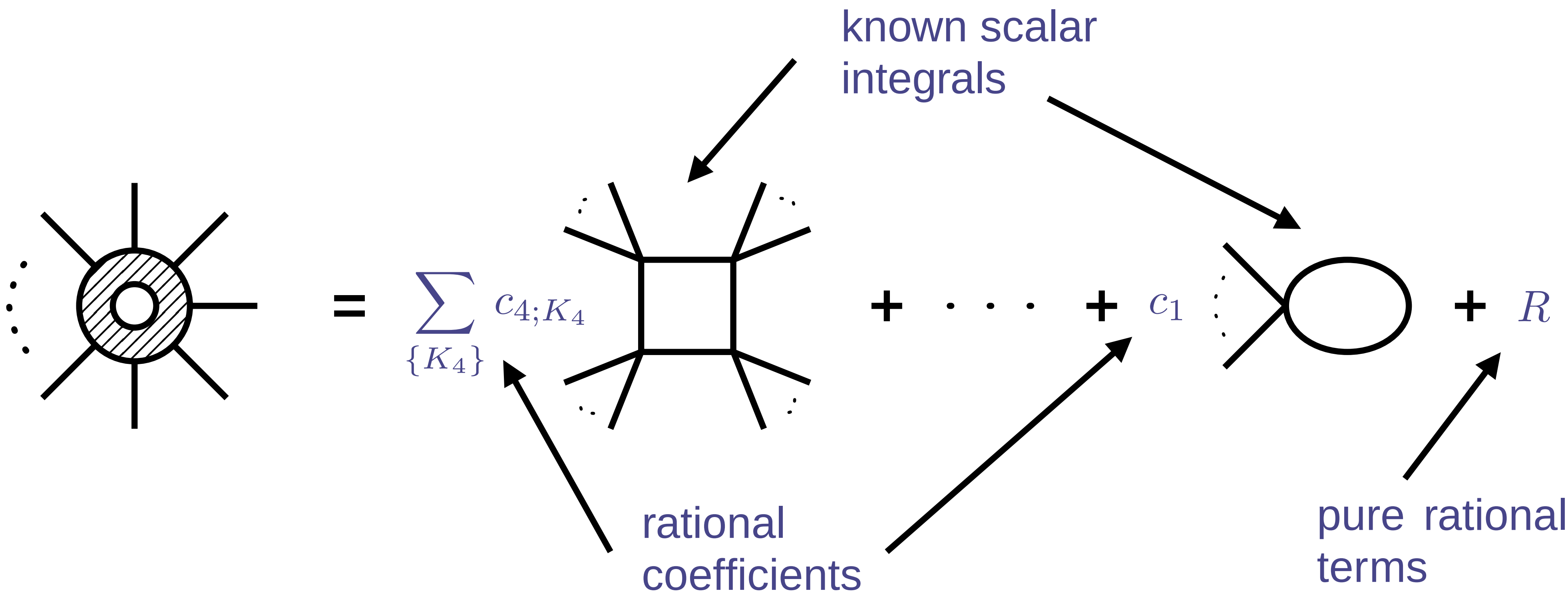
  Bern,Dixon,Dunbar,Kosower,Britto,Cachazo,Feng,Mastrolia, Ossola,Papadopoulos,Pittau,Ellis,Giele,Kunszt,Melnikov,Forde,. . .

- Automated numerical approaches:

  [`BlackHat`, `Rocket`, `CutTools/Helac-1loop`, `GOLEM`, Denner et al., . . .]

- Efficiency:
  - Numerical stablility
  - Fast numerical evaluation
  - On-shell simplifications
- Rest of this talk: Concentrate on analytic approach to generalised unitarity

Structure of One-Loop Amplitudes



- General gauge theory amplitudes reduced to box topologies or simpler

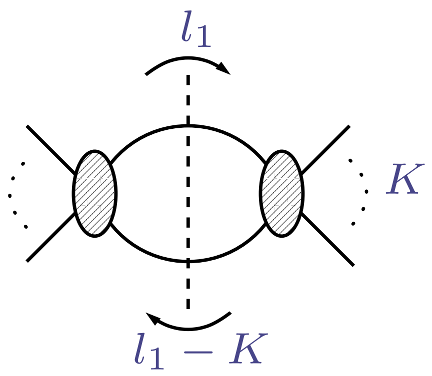
  [Passarino,Veltman;Melrose]

- Isolate logarithms with cuts $\rightarrow$ exploit on-shell simplifications
- General cutting principle:
  - apply δ-functions to L and R sides
  - generate and solve the linear system for the coefficients

Unitarity

- Cutkosky Rules to compute $\mathrm{Disc}_p(A^{1-\mathrm{loop}})$

$$\frac{1}{p^2 - m^2 + iO^+} \rightarrow i\delta^{(+)}(p^2 - m^2)$$



- Construction of amplitudes from double cuts [Bern,Dixon,Dunbar,Kosower (1994)]
- Improved fitting with triple cuts [Bern,Dixon,Kosower (1997)]
- Generalised cuts with complex momenta [Britto,Cachazo,Feng (2004)]
- D-dimensional cuts can also access rational contributions

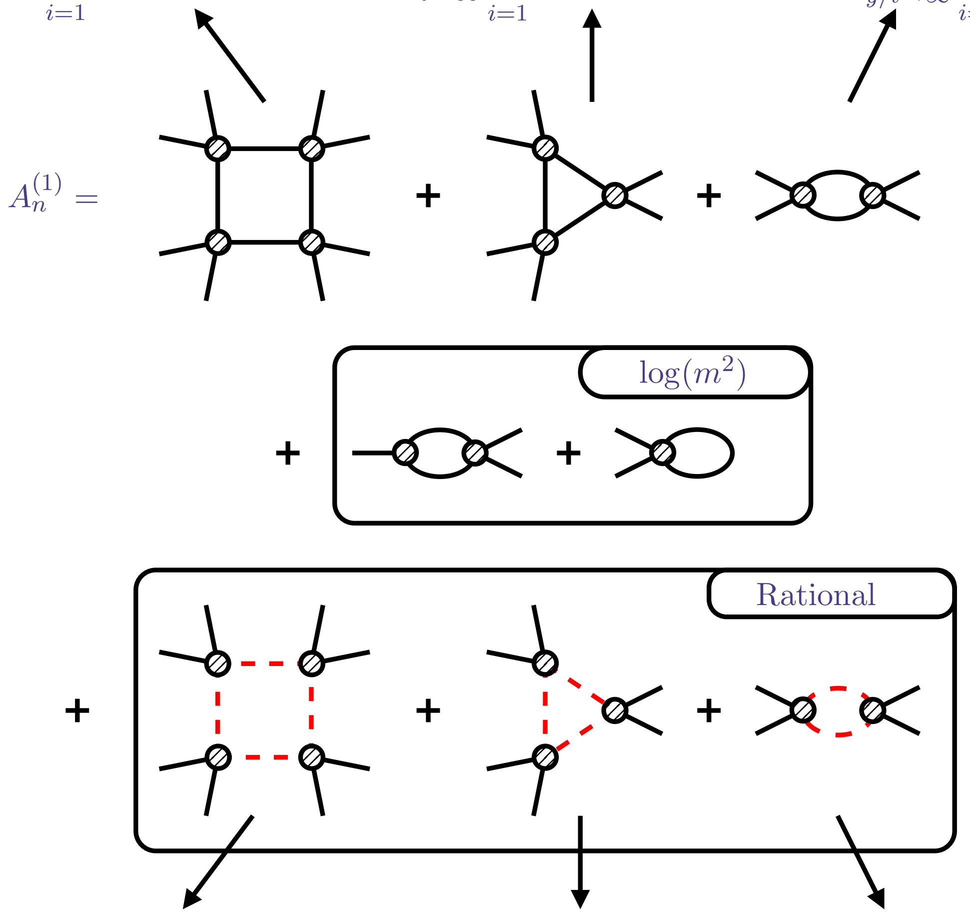
[Bern,Morgan (1995)]

[Bern,Dixon,Dunbar,Kosower (1997)]

[Giele,Kunszt,Melnikov;Anastasiou,Britto,Feng,Mastrolia,Kunszt;SB]

Massive Integral Basis

$$C_4 = \prod_{i=1}^{4} A_i(l) \qquad C_3 = \lim_{t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0} \qquad C_2 \sim \lim_{y/t\to\infty} \prod_{i=1}^{2} A_i(l(y,t))$$



$$C_4^{[4]} = \lim_{\mu^2\to\infty} \prod_{i=1}^{4} A_i(l)|_{\mu^4} \qquad C_3^{[2]} = \lim_{\mu^2, t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0,\mu^2} \qquad C_2^{[2]} \sim \lim_{\mu^2, y/t\to\infty} \prod_{i=1}^{2} A_i(l(y,t))$$

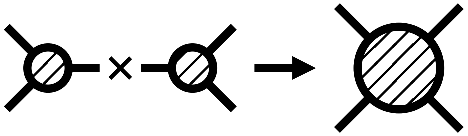
products of trees
$\longrightarrow$ integral coefficients

Tree-Level Amplitudes

- BCFW recursion generates compact analytic expressions

  [Britto,Cachazo,Feng,Witten (2005)]

  - Application of Cauchy's theorem and factorisation



$$0 = \oint A(z)/z = \sum_{\text{residues}} A(z)/z = \sum A_L \frac{1}{P^2} A_R + A(0)$$

- Simple generalisation to include internal masses [SB,Glover,Khoze,Svrček (2005)]
- Helicity basis for massive fermions for compact analytic formulae:

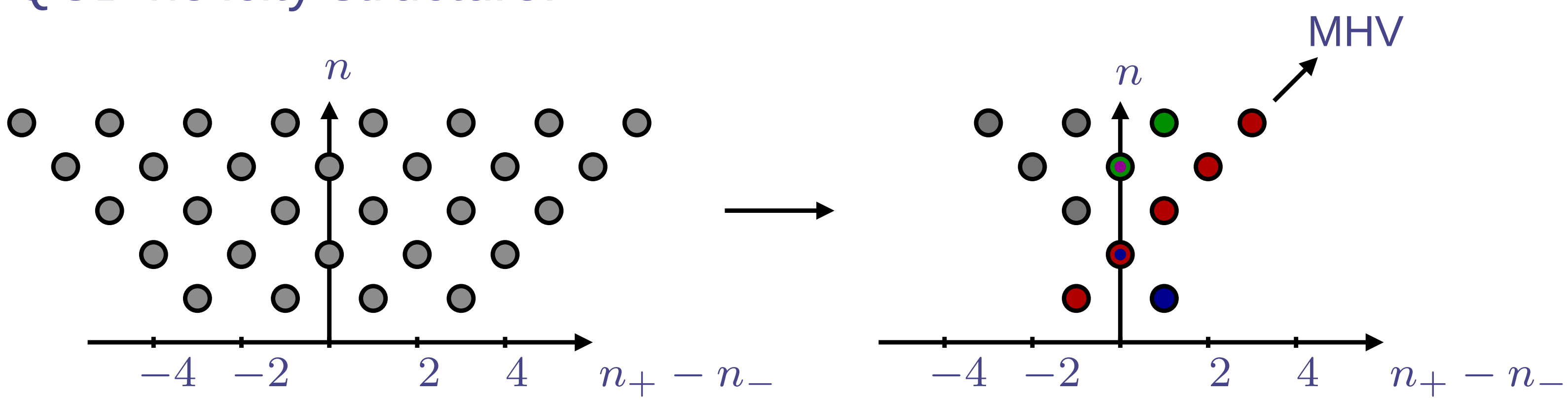
$$u_\pm(p,m;p^\flat,\eta) = \frac{(\not{p}+m)|\eta\mp\rangle}{\langle p^\flat \pm |\eta\mp\rangle}$$

[Kleiss,Stirling]

Gauge theories defined from basis of three-point on-shell vertices

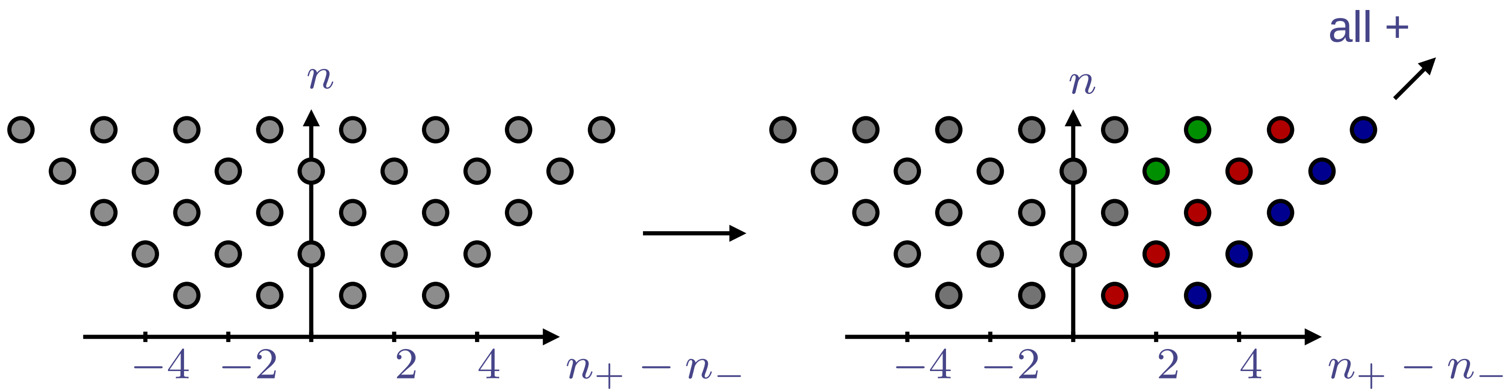
Massive Helicity Amplitudes

Massless QCD helicity structure:



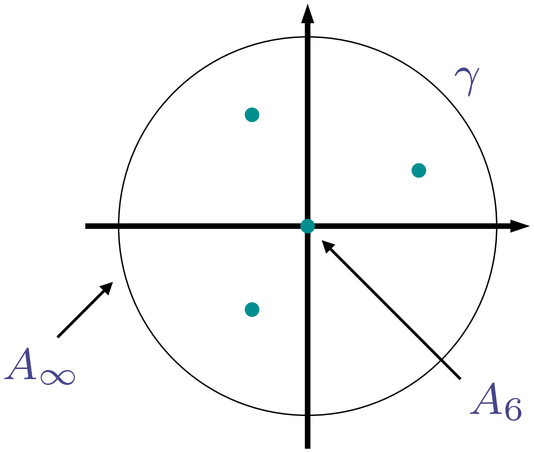
Symmetry relates massive quark helicity states:

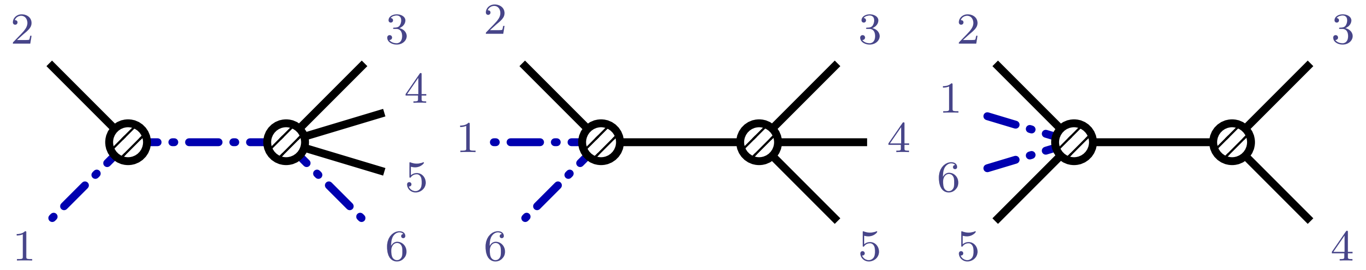
$$u_-(p, m; p^\flat, \eta) = \frac{\langle p^\flat \eta \rangle}{m} u_+(p, m; \eta, p^\flat)$$



Tree-level Amplitudes: $t\bar{t} + 4g$

- Complex continuation
  $(p_2, p_3) \to (p_2(z), p_3(z))$
- Sum over residues and internal helicities





$\langle ab \rangle \sim \sqrt{a.b}$

$$A_6(1_t^+, 2^+, 3^+, 4^+, 5^+, 6_{\bar{t}}^+) = \frac{i m^3 \langle \eta_6 \eta_1 \rangle [5|6 P_{45} P_{23} 1|2]}{\langle 1^\flat \eta_1 \rangle \langle 6^\flat \eta_6 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 2|1|2] \langle 5|6|5] (s_{123} - m^2)}$$

Tree-level Amplitudes: $t\bar{t} + n(g)$

For this "all-plus" configuration we can look at higher point functions:

$$A_7(1_t^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7_{\bar{t}}^+) =$$
$$\frac{im^3\langle\eta_7\eta_1\rangle\left((s_{1,3}-m^2)[6|7P_{56}P_{2,4}1|2]-m^2[6|7P_{56}43|2]\right)}{\langle 1^\flat\eta_1\rangle\langle 7^\flat\eta_7\rangle\prod_{\alpha=2}^{4}\langle\alpha\alpha+1\rangle(s_{1,\alpha}-m^2)}$$

$$A_8(1_t^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8_{\bar{t}}^+) = \frac{im^3\langle\eta_8\eta_1\rangle}{\langle 1^\flat\eta_1\rangle\langle 8^\flat\eta_8\rangle\prod_{\alpha=2}^{5}\langle\alpha\alpha+1\rangle(s_{1,\alpha}-m^2)}\times$$
$$\left((s_{1,4}-m^2)\left((s_{1,3}-m^2)[7|8P_{67}P_{2,5}1|2]+m^2[7|8P_{67}P_{45}3|2]\right)-m^2[7|8P_{67}54P_{23}1|2]\right)$$

- All-order formula possible

[Forde,Kosower]

[Schwinn,Weinzierl]

[Ferrario,Rodrigo,Talavera]

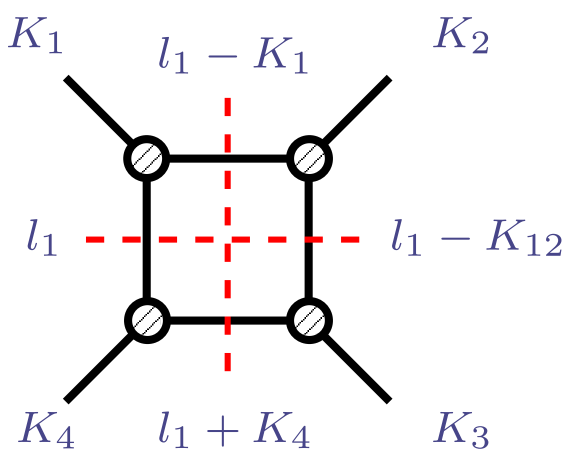
Analytic Solutions for Quadruple Cuts

- Quadruple cut freezes loop momentum in 4 dimensions

[BCF]

- Two complex solutions to on-shell constraints

$$\{(l_1 - K_{1,i-1})^2 = m_i^2\}$$



- Parametrise loop momentum with two projected massless vectors:

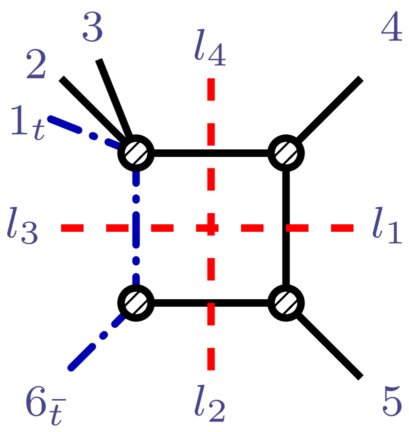
[Forde;Ossola,Papaodpoulos,Pittau]

$$l^\nu = a(K_1^\flat)^\nu + b(K_2^\flat)^\nu + c\langle K_1^\flat|\gamma^\nu|K_2^\flat] + \left(d - \frac{m_1^2}{2K_1^\flat\cdot K_2^\flat}\right)\langle K_2^\flat|\gamma^\nu|K_1^\flat]$$

- Sum over solutions to obtain integral coefficient

$$C_{4;K_1|K_2|K_3|K_4} = \frac{1}{2}\sum_\sigma A_1 A_2 A_2 A_4(l^\sigma)$$

Application to $t\bar{t} + 4g$



$$\{l_1^2 = 0, (l_1 - 5)^2 = 0, (l_1 - 5 - 6)^2 = m^2, (l + 4)^2 = 0\}$$

$$l_1^{(1)} = \frac{\langle 5|6|5]}{2\langle 4|6|5]}\langle 4|\gamma^\mu|5], \qquad l_1^{(2)} = \frac{\langle 5|6|5]}{2\langle 5|6|4]}\langle 5|\gamma^\mu|4]$$

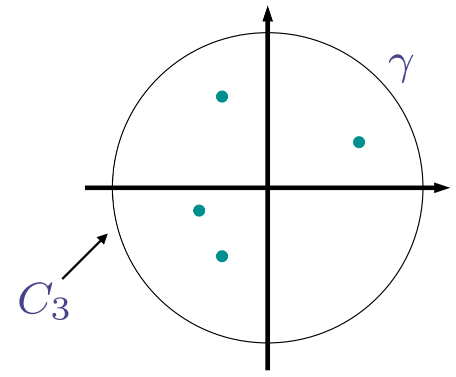
$$C^{[L]}_{4;4|5|6|123}(1_t^+, 2^+, 3^+, 4^+, 5^+, 6_{\bar{t}}^+) =$$

$$\frac{im^3\langle \eta_6\eta_1\rangle[54]}{\langle \eta_6 6^\flat\rangle\langle 1^\flat \eta_1\rangle\langle 23\rangle\langle 3P_{45}65\rangle\langle 2|1|2]}\left(\frac{s_{45}[4|P_{23}1|2]m^2}{(s_{123} - m^2)} + \frac{\langle 5|6|5]\langle 3|P_{45}P_{16}1|2]}{2\langle 34\rangle}\right)$$

Other configurations straightforward and relatively compact

Automation : BCFW to box expressions (`FORM`, `maple`)

Triple and Double Cuts



- Lower point coefficients have unfixed integrations
- Analytic extraction via Laurent Expansion
  - Complex analysis again: Residue theorem

$$C_3 = \sum_\sigma \mathrm{Inf}_t[A_1 A_2 A_3(l^\sigma(t))]\Big|_{t^0}$$

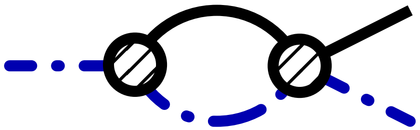
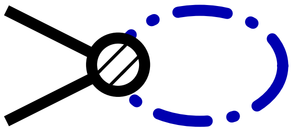
- Bubble coefficients also reduced to algebraic procedure [Forde]

$$C_2 \sim \mathrm{Inf}_{y,t}[A_1 A_2(l(y,t))] + \sum_{y_\pm} \mathrm{Inf}_t[A_1 A_2 A_3(l(y_\pm, t))]$$

  - Analysis of double cut: Stokes' Theorem [Mastrolia]
- Straightforward generalisation to massive case [Kilgore; SB]

Automation : closed analytic expressions (`FORM`,`maple`)

Wave-function Renormalisation and Tadpoles

$A^{(1)} \supset \quad C_{2;m^2}$  **+** C_1 

- More remaining logarithms - $\log(m^2)$

$$A^{(1)} \supset \frac{C_{1/\epsilon}}{\epsilon}\left(\frac{\mu_R^2}{m^2}\right)^{\epsilon} + \mathcal{O}(\epsilon^0)$$

- Double cut contains mass renormalisation divergence [Ellis,Giele,Kunszt,Melnikov]
- $C_{1/\epsilon}$ fixed from universal IR behaviour,e.g. [Catani,Dittmaier,Trócsányi] [Mitov,Moch]

$$C_{1/\epsilon}^{[L]}(t\bar{t} + n(g)) = \frac{1}{4}A_n^{(0)} - \sum_{\{t,\bar{t}\}\in K_2} C_{2:K_2}^{[L]}$$

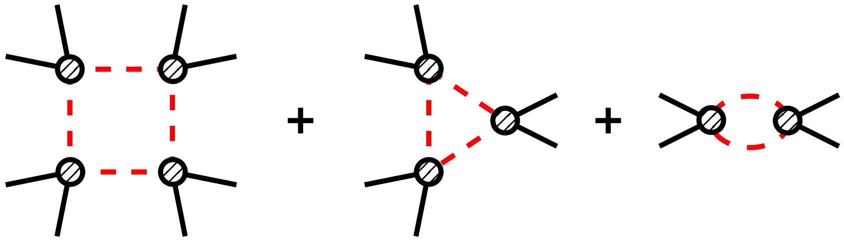
- Direct unitarity solution for tadpole (rational only) [Kilgore;Britto,Feng]

Rational Terms

- Simpler example: six-gluon amplitudes
- Analytic D-dimensional cuts
  - $4-2\epsilon$-dim. cuts $\rightarrow$ massive cuts $\int d^{4-2\epsilon} l = \int d^{-2\epsilon}\mu \int d^4 \bar{l}$

[Giele,Kunszt,Melnikov]

$R =$



  - Extraction of rational terms from Taylor expansion in μ^2 [SB]

$$R = -\frac{1}{6}\sum_{\{K_4\}} \mathrm{Inf}_{\mu^2}\left[\prod_{i=1}^{4} A_i\right]\Big|_{\mu^4} - \frac{1}{2}\sum_{\{K_3\}} \mathrm{Inf}_{\mu^2;t}\left[\prod_{i=3}^{4} A_i\right]\Big|_{\mu^2,t^0}$$

$$-\sum_{\{K_2\}} \frac{K_2 - 3(m_1^2+m_2^2)}{6}\left(\mathrm{Inf}_{\mu^2;y;t}\left[\prod_{i=2}^{4} A_i\right]\Big|_{\mu^2,t^0,Y_i} + \mathrm{Inf}_{\mu^2;t}\left[\prod_{i=3}^{4} A_i(y(t))\right]\Big|_{\mu^2,T_i,Y_i}\right)$$

Rational Terms

Example six-gluon all-plus

$$R_6^g(1^+,2^+,3^+,4^+,5^+,6^+) = \sum_{\sigma\in S_n} \Bigg(\frac{2i\left(s_{45}\langle 6|1+2|3][51][64]^2 - s_{46}\langle 5|1+2|3][54]^2[61]\right)[56]}{\langle 12\rangle\langle 23\rangle \mathrm{tr}_5(5,4,6,1)\mathrm{tr}_5(5,4,6,3)} + \frac{2i\langle 5|1+2|6]\langle 6|1+2|5][12][43][65]^2}{\langle 12\rangle\langle 34\rangle \mathrm{tr}_5(5,2,6,1)\mathrm{tr}_5(5,4,6,3)} + \frac{2i\left(\langle 3|1+2|3]\langle 6|1+2|6] - s_{36}s_{12}\right)[12][54][63]^2}{\langle 12\rangle\langle 45\rangle \mathrm{tr}_5(2,3,6,1)\mathrm{tr}_5(5,3,6,4)} \Bigg)$$

- Closed form expressions suitable numerical evaluation
- All helicity configurations available
- Compact? Compare with on-shell recursion [Berger,Bern,Dixon,Forde,Kosower]

Conclusions and Outlook

- Efficient constructions of one-loop amplitudes with generalised unitarity
- Simple extension to massive case [Britto,Feng,Mastrolia;Kilgore]
- Computation of scalar integral coefficients purely algebraic
- Complexity scales well with additional external legs
- Automated numerical tools

  `BlackHat` , `Rocket` , `CutTools/Helac-1loop` ,
  `Lazopoulos` , `Giele+Winter` , `Melnikov+Schulze,...`

- Analytic expressions: Speed, Stability,. . . needs further investigation
- Massive amplitudes contain more scales! Need to employ efficient tools
  - SM backgrounds to BSM signal @ LHC

A Bit of Numerical Analysis

- Test case: evaluation of $H + 2j$ virtual amplitudes: [Ciaran Williams' Talk]
  - Computation in $m_t \to \infty$ limit: Effective Higgs-gluon interaction

    [739 Feynman diagrams: 12 boxes, 24 triangles, 8 bubbles]

    [Existing semi-numerical evaluation: Ellis,Giele,Zanderighi (2006)]
  - Compact expressions for complete set of helicity amplitudes

    $gggg$ channels: [Berger,Del-Duca,Dixon;SB,Glover,Risager,Mastrolia,Williams]

    $q\bar{q}gg$ channels: [Dixon,Sofianatos;SB,Campbell,Ellis,Williams]

    $q\bar{q}Q\bar{Q}$ channels: [Dixon,Sofianatos;Ellis,Giele,Zanderighi]
- Basic C++ implementation of spinor products and scalar integrals

amplitude	time
$- - --$	60μs
$- - ++$	170μs
$- + -+$	180μs
$+ - --$	200μs
$\sum_{\text{col.,hel.}} \text{Re}(A^1 A^0)$	10 ms

[CPU: 64 bit 3GHz Intel Core Duo]