Towards p p $\rightarrow$ W W at NNLO

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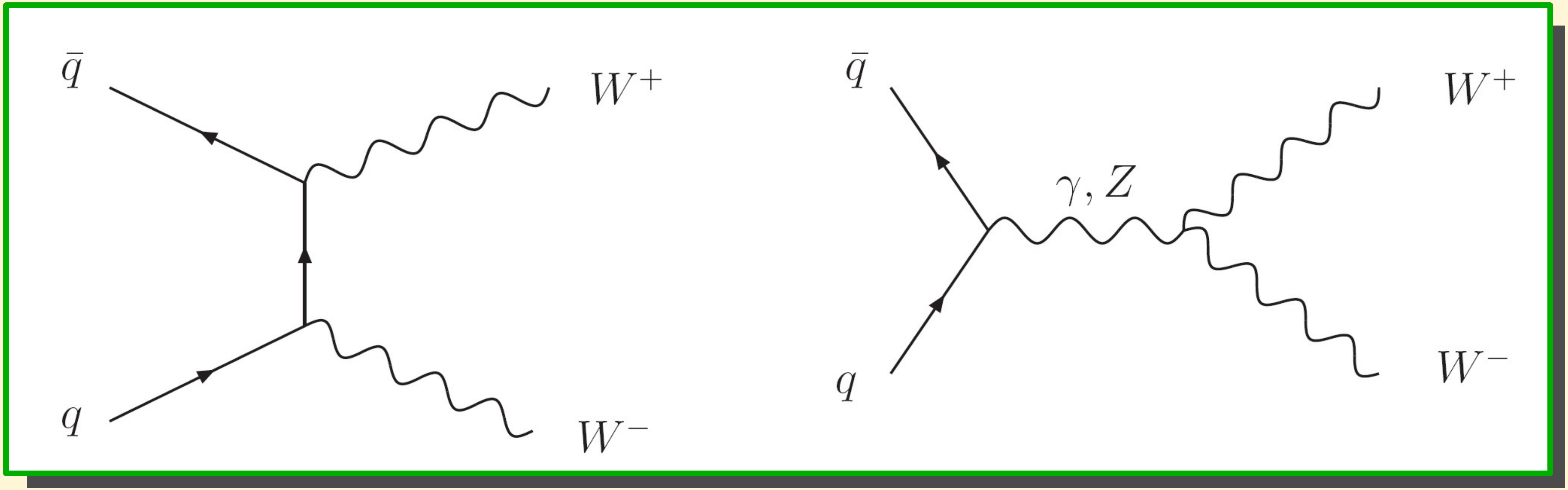
in collaboration with M. Czakon and D. Eiras

RADCOR, Ascona, 29 October 2009

The process

W Pair Production in the quark-anti-quark annihilation channel – Tree level diagrams



It is a $2 \rightarrow 2$ process with **massive** particles

Goal: Calculate this process at NNLO (two-loop) in QCD

One is bound to ask:

Do we really need to go up to NNLO?

NNLO is needed when

- NLO corrections are large
- For benchmark measurements

Theoretical uncertainty smaller or matching the experimental errors

State of the art (for LHC):

Relative Order	$2 \to 1$	$2 \to 2$	$2 \to 3$	$2 \to 4$	$2 \to 5$	$2 \to 6$
1	LO					
α_s	NLO	LO				
α_s^2	NNLO	NLO	LO			
α_s^3		NNLO	NLO	LO		
α_s^4				NLO	LO	
α_s^5					NLO	LO

Glover

Outline

- Focus on hadronic W pair production:
  Virtual two-loop and one-loop squared amplitudes
- Motivation for studying $q\ q \rightarrow W\ W$ accurately
- **Results:** NNLO Virtual Corrections in the High Energy Limit, Power Corrections (11 terms)
- Conclusions - Outlook

Full mass dependence

Motivation I

W pair production important as a **signal** in searches for **New Physics.** Testing ground for non-abelian structure of SM, triple gauge couplings, γWW, ZWW

$$\sigma(p\bar{p} \rightarrow W^+W^-) =$$

$$14.6^{+5.8}_{-5.1}(\text{stat})\ ^{+1.8}_{-3.0}(\text{syst}) \pm 0.9(\text{lum})\ \text{pb}$$ CDF

$$13.8^{+4.3}_{-3.8}(\text{stat})\ ^{+1.2}_{-0.9}(\text{syst}) \pm 0.9(\text{lum})\ \text{pb}$$ DØ

$\sqrt{s} = 2$ TeV ($p\bar{p}$)	W^+W^-	
	MRS98	CTEQ5
Born [pb]	10.0	10.3
Full [pb]	13.0	13.5

Tevatron

$\sqrt{s} = 14$ TeV (pp)	W^+W^-	
	MRS98	CTEQ5
Born [pb]	81.8	86.7
Full [pb]	120.6	127.8

?

LHC

Campbell, Ellis ('99)

Motivation II

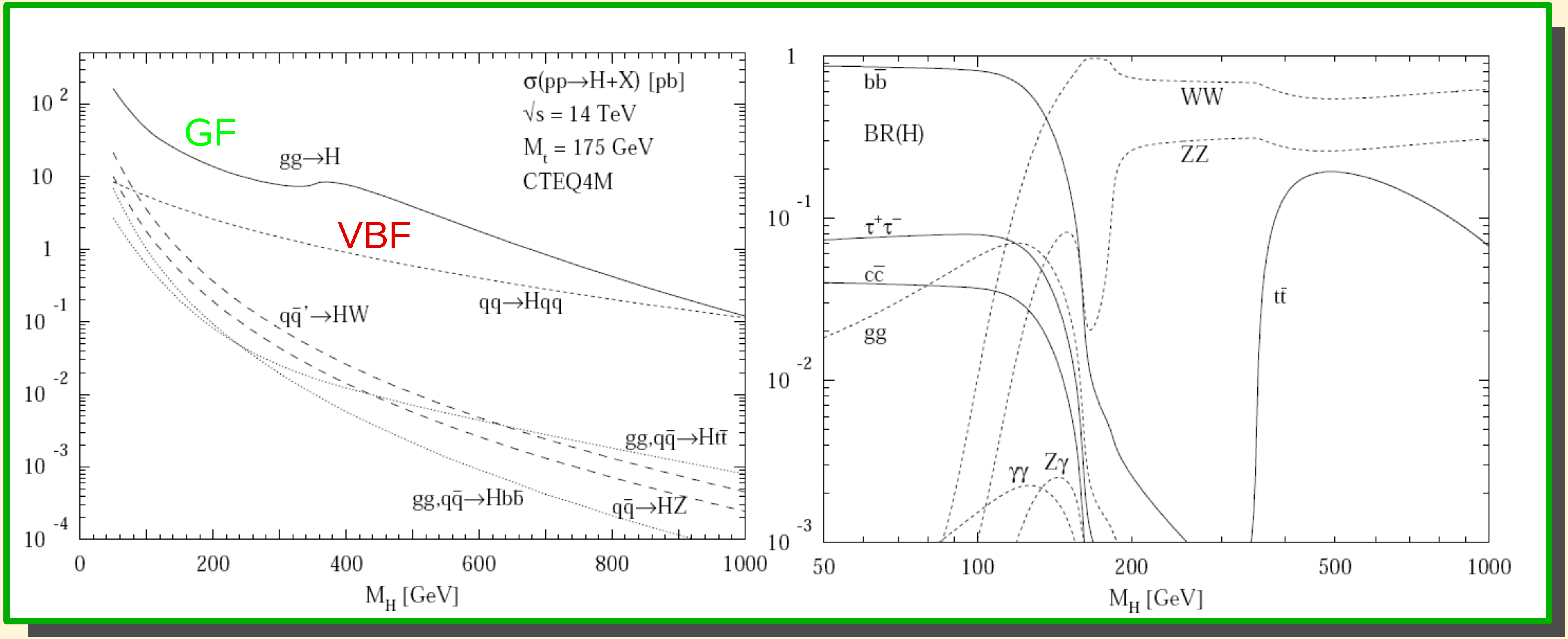
The 'elusive' Higgs boson

Higgs:

- Only constituent of the SM not experimentally observed yet.
- Electroweak symmetry breaking
- Description of particle masses

Discovery by itself is not enough!
Properties of the Higgs needed
to exclude or verify
alternative models

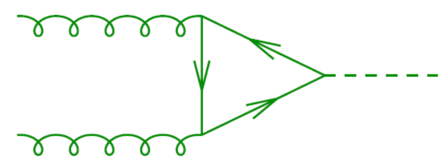
Motivation II



Spira '97

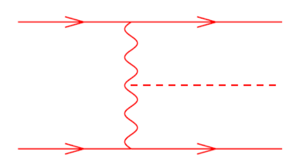
LHC Higgs production ...

Gluon Fusion channel is the dominant production mechanism up to $M_H \sim 1$ TeV : $g\, g \rightarrow H$



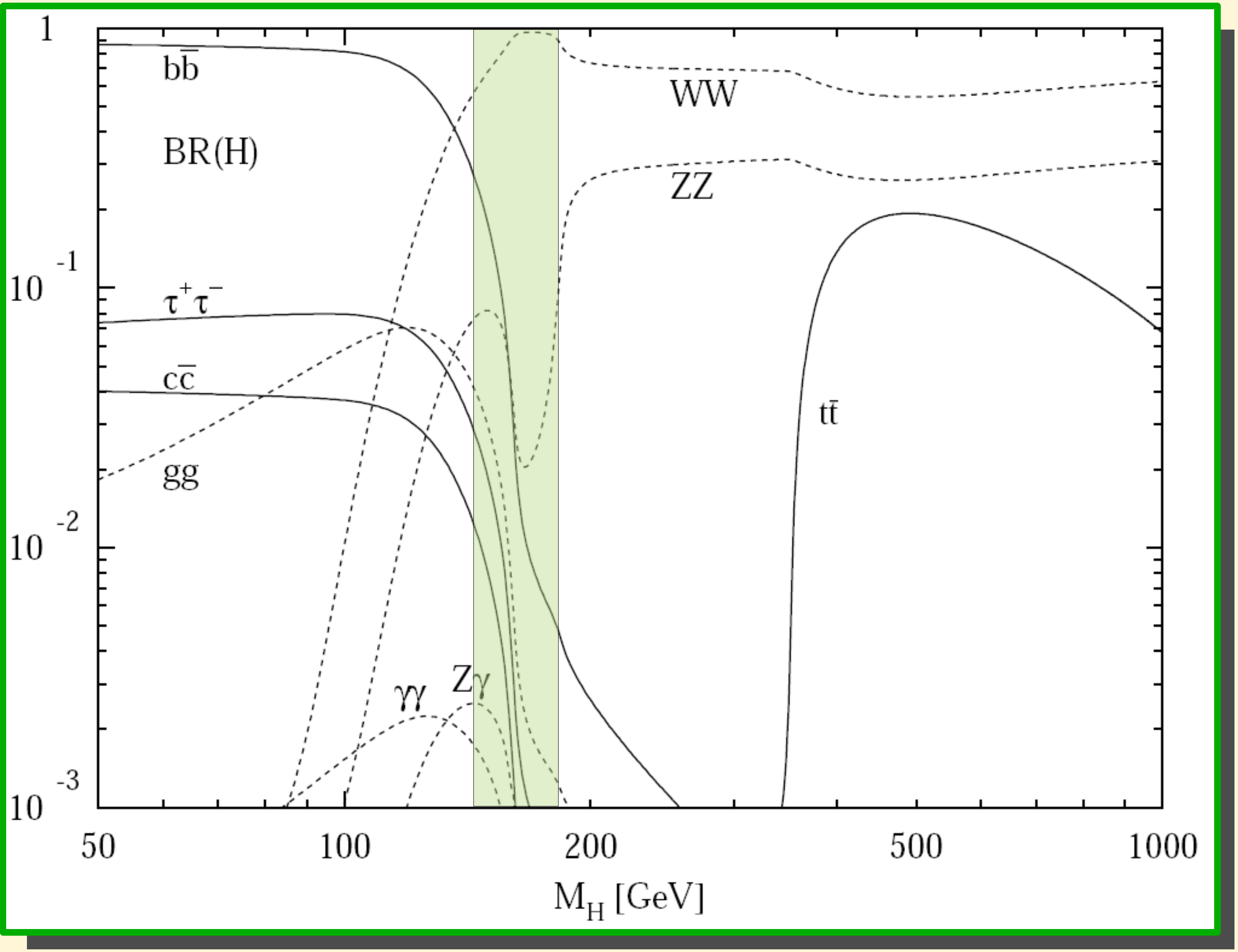
gluon fusion (GF)

Sub-dominant production process is Vector Boson Fusion: $q\, q \rightarrow V\, V \rightarrow q\, q\, H$



vector boson fusion (VBF)

Motivation II



... and decay

Once the Higgs is produced it will eventually decay into different particles depending on its mass. In the Higgs mass range $140 - 180$ GeV the main decay mode is into W pairs

Motivation II

Going after the Higgs: Main discovery Channels

M_H : 114 – 140 GeV
$H \rightarrow \gamma\, \gamma$

M_H : 180 – 600 GeV
$H \rightarrow Z\, Z \rightarrow 4\, l$

M_H: 140 – 180 GeV
$H \rightarrow W\, W \rightarrow 2\, l$ + missing Energy E_T

- Pick up the signal process
- Avoid or suppress the usually large **background**
- Accurate theoretical predictions for both signal and background

→ Main background (irreducible): W pair production

Mini Summary

W Pair Production is important at the LHC:

- Searches for New Physics
- Irreducible background to Higgs production

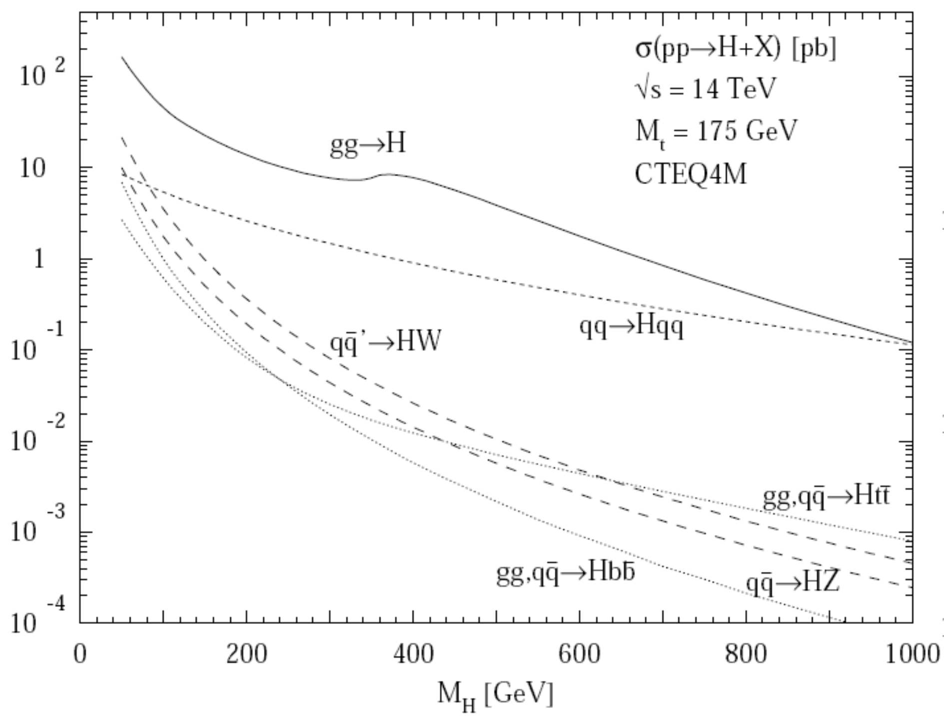
All these are nice but still ...

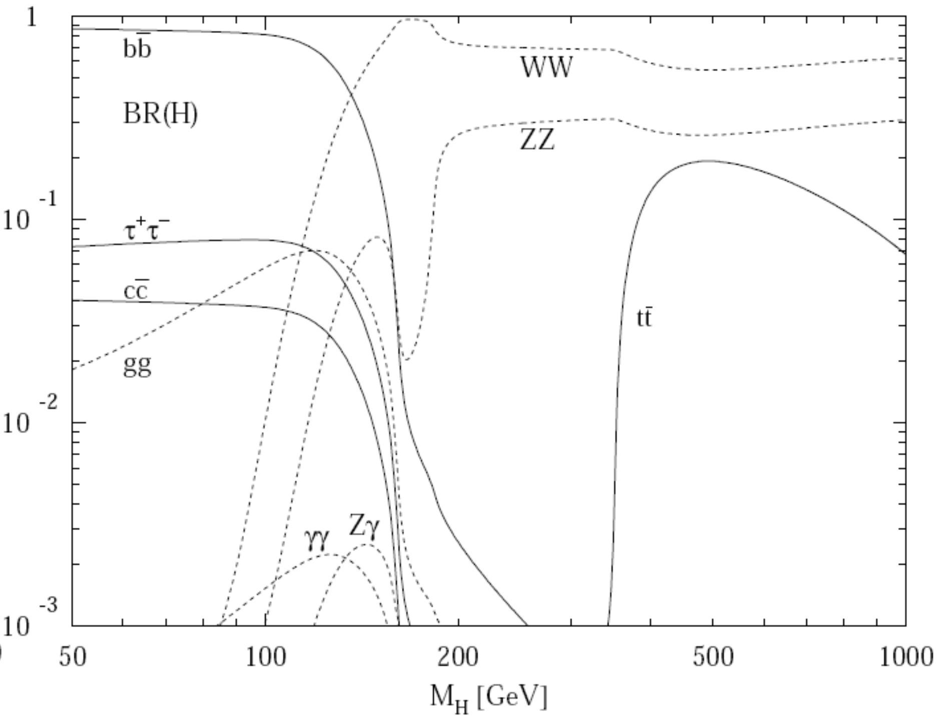
... do we really need to go up to NNLO?

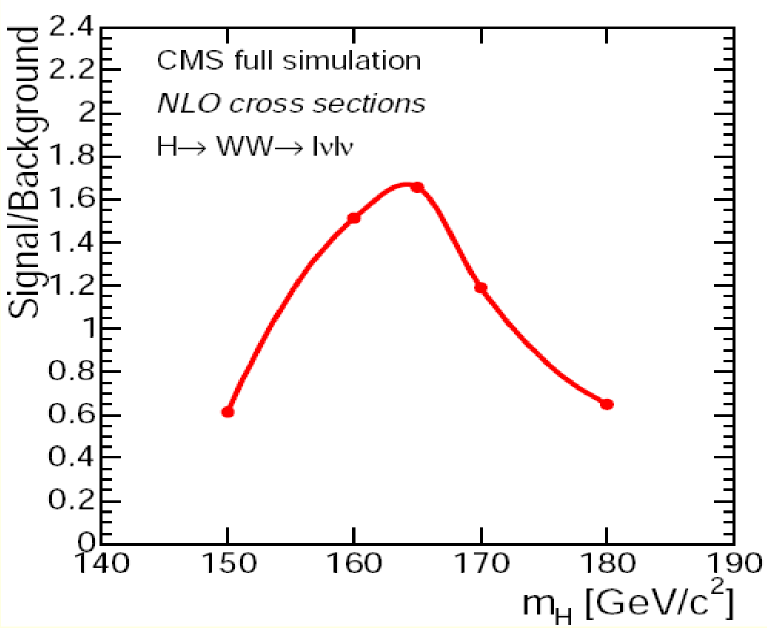
The answer is

YES!

Signal/background







Signal – background ratio: of order unity in the mass range 150 – 180 GeV

Signal known to NNLO

QCD corrections to :

$$\underline{g\, g \rightarrow H}$$

NLO: Contribute ~ 70%

Dawson ('91); Djouadi, Graudenz, Spira, Zerwas ('95)

NNLO: Contribute an additional 20% for LHC

Harlander, Kilgore ('02); Anastasiou, Melnikov ('02)

Ravindran, Smith, van Neerven ('03)

With a Jet veto at NNLO: corrections ~ 85%

Catani, de Florian, Grazzini ('02)

Davatz, Dissertori, Dittmar, Grazzini, Pauss ('04)

Anastasiou, Melnikov, Petrielo ('04)

NNLO $\underline{H \rightarrow W\, W \rightarrow l\, \nu\, l\, \nu}$

Anastasiou, Dissertori, Stöckli, Webber ('08)

Grazinni ('08)

Background

- qq→WW

  70% enhancement at NLO. With a jet veto the enhancements fall to 20-30%

  Dixon, Kunszt, Signer ('98, '99)

- loop induced gg→WW

  Contributes to the quark annihilation channel at $\mathcal{O}(\alpha_s^2)$.
  Enhanced by the **large gluon flux.** After Higgs search cuts it increases the background by 30%, with no cuts by 5%

  Glover, van der Bij ('89); Kao, Dicus ('91)
  Binoth, Ciccolini, Kauer, Krämer ('05)
  Duhrssen, Jackobs, v. d. Bij, Marquard ('05)

- EW corrections Accomando, Denner,Kaiser('05)

Necessity of NNLO calculation for a few % level accuracy

W Pair Production

1980 | 1990 | 2000 | Present

LO

Brown, Mikaelian ('79)

NLO

Ohnemus ('91); Frixione ('93);
Ohnemus('94);
Dixon, Kunszt, Signer ('98,'99);
Campbell, K. Ellis ('99)

NNLO massless

Anastasiou, Glover, Tajeda-Yeomans ('02)

resummation

Grazzini ('06)

NNLO

The p p $\rightarrow$ W W story till recently...

We would now like to have...

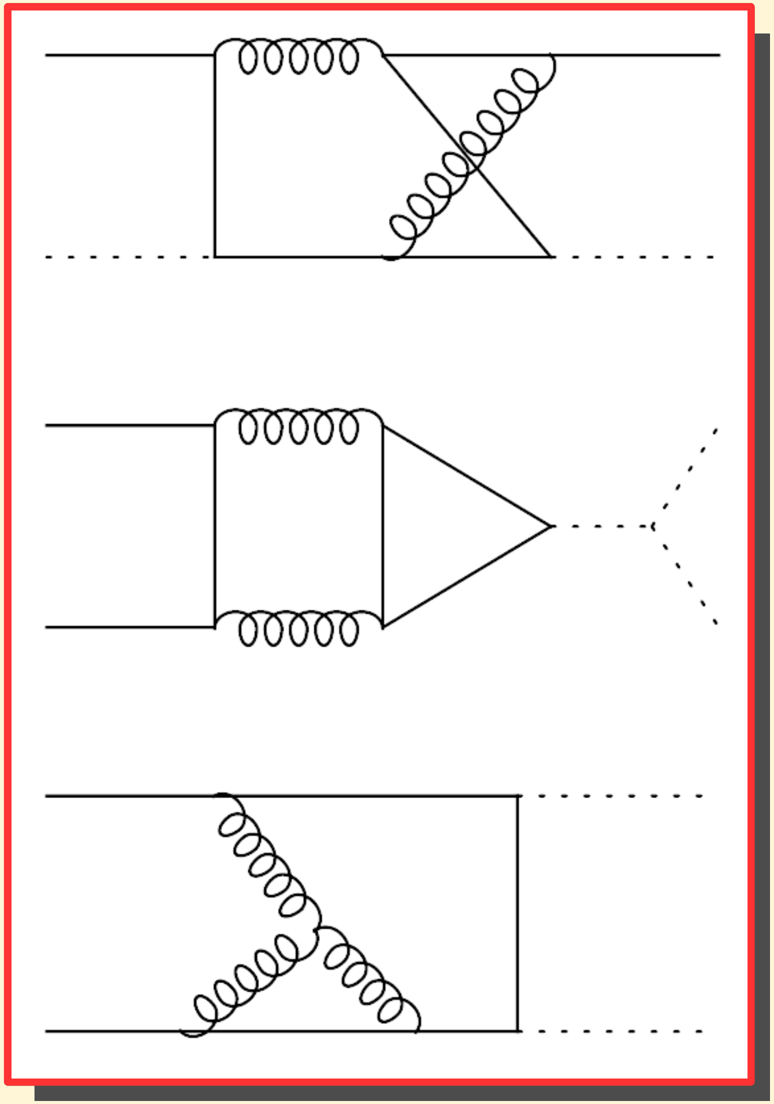
- ... Cross sections for W Pair production at NNLO

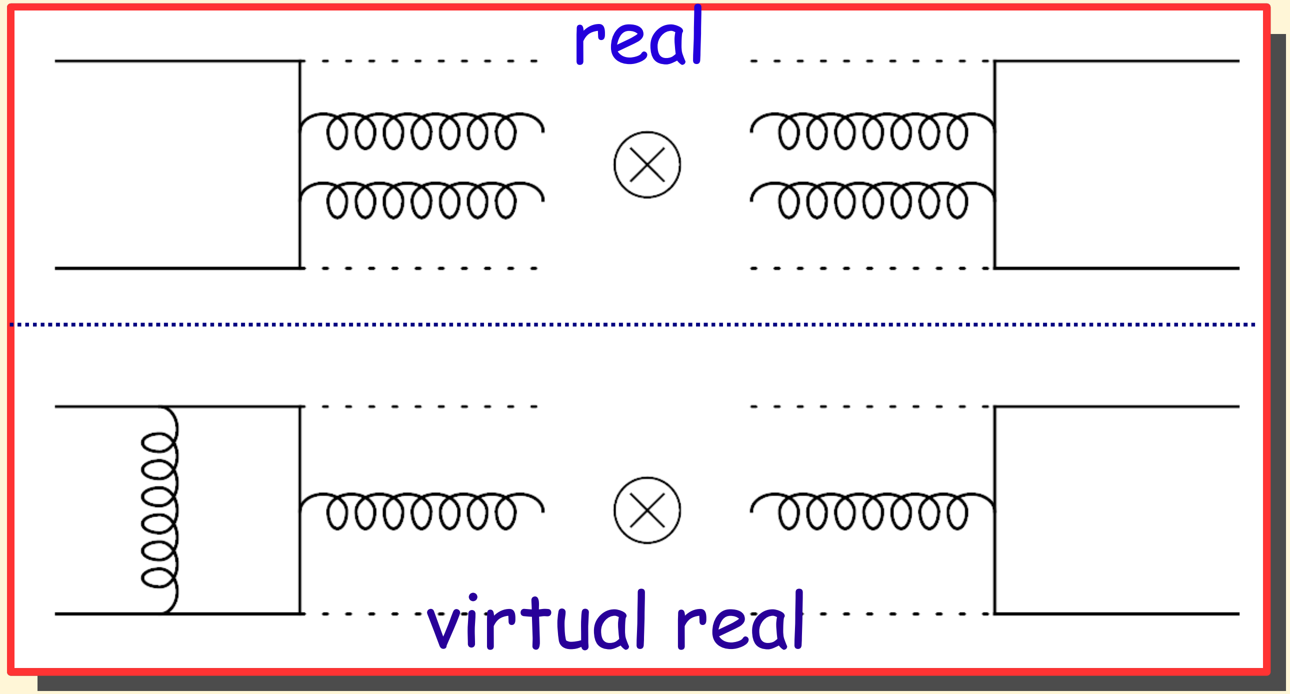
  with full mass dependence...
- ... Then start with the amplitudes...
- ... The difficult part on the amplitude level is

  the virtual corrections, in particular the two-loop

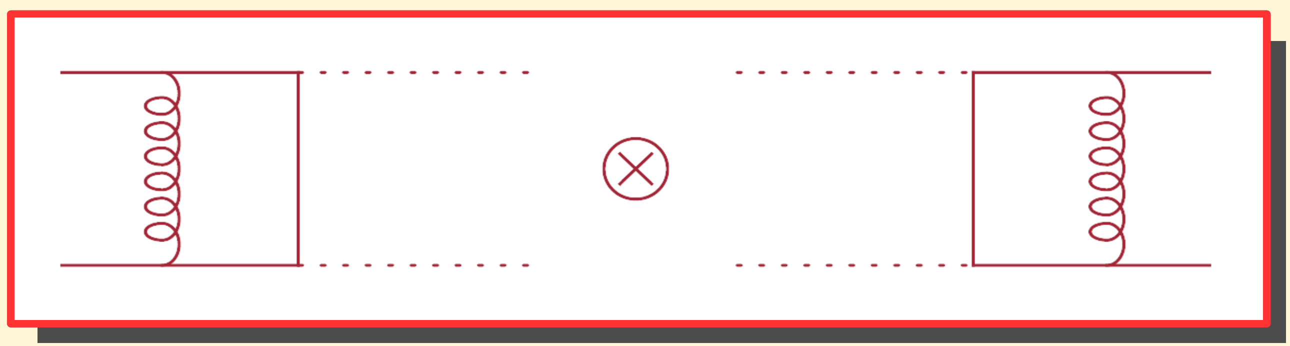
  diagrams contracted with the Born ones

Well, that means... many diagrams





(one loop) ⊗ (one loop)*



2 loop virtual

So what is at stake?

- A NNLO (4 legs, 2 loops) calculation of a process with massive particles (similarities to "heavy quark production")

  Czakon, Mitov, Moch('07), Czakon ('08)
- Color and spin averaged amplitudes
- Kinematical region: all kinematical invariants large compared to the W mass: $M_W^2 \ll s, t, u$
- We expand with respect to $m_s = M_W^2/s$
- Exact analytic result in the high energy limit (up to terms suppressed by powers of m_s)
- Reconstruct (numerically) **full mass dependence** for the whole phase space

The main difficulty

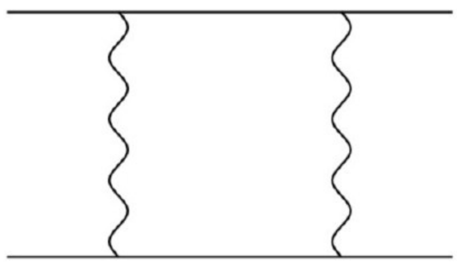
For the 2-loop amplitude
the difficult part here is to compute the
Feynman Integrals

We choose to do that using the
Mellin-Barnes representations
technique

Mellin-Barnes: a simple example

$$\frac{1}{(A+B)^n} = \frac{1}{2\pi i \Gamma(n)} \int_{-i\infty}^{i\infty} dz \frac{A^z}{B^{n+z}} \Gamma(-z)\Gamma(n+z)$$

- Example



$$e^{\epsilon\gamma}\Gamma(2+\epsilon)\int dx_1 ... dx_4 \delta(1 - x_1 - ... - x_4) \frac{1}{(-s x_2 x_3 - t x_1 x_4)^{2+\epsilon}}$$

$$\frac{e^{\epsilon\gamma}}{2\pi i} \frac{1}{(-s)^{2+\epsilon}} \int_{-i\infty}^{i\infty} dz \, \left(\frac{t}{s}\right)^z \frac{\Gamma^2(-1-\epsilon-z)\Gamma(-z)\Gamma^2(1+z)\Gamma(2+\epsilon+z)}{\Gamma(-2\epsilon)}$$

$$\mathrm{Re}\,\epsilon = -\frac{1}{2}, \quad \mathrm{Re}\,z = -\frac{3}{4}$$

Mellin-Barnes

Do a reduction a la Laporta into Masters, then starting from the Feynman parameters representation of a master, “walk” the following Steps:

- produce representations (**MBrepresentations.m**)
- analytically continue in ε to the vicinity of 0 and expand in mass (**MB.m**)
- perform as many as possible integrations using Barnes lemmas (**BarnesRoutines.m**, Kosower)
- resum the remaining integrals by transforming into harmonic series (**Xsummer**)
- resum remaining constants by high-precision numerical evaluation (**quadprec.m**) and fit them to a transcendental basis (**PSLQ**)

Software

MBrepresentations.m (GC, Czakon)

Produces representations for any **multi**-loop, **planar** or **non-planar**, **scalar** or **tensor** integral of any **rank**!

MB.m (Czakon)

Determination of contours, analytic continuation, expansion in a chosen parameter, numerical integration

XSummer (Moch, Uwer)

Evaluation of harmonic sums

PSLQ (Bailey)

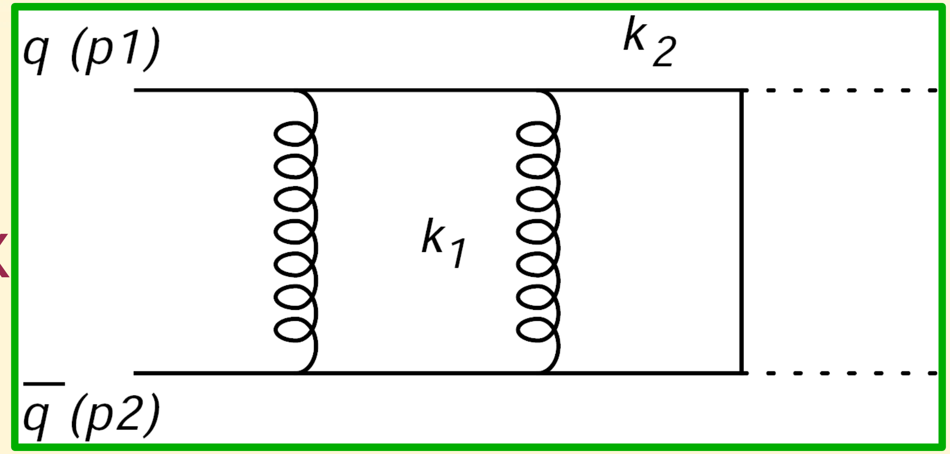
Fitting to a transcendental basis

quadprec.m (Czakon)

High precision numerical evaluation with up to 64 digits

A tensor example:

(k1.p3) x



$$\int\int d^dk_1 d^dk_2 \frac{k_1.p_3}{k_1^2 k_2^2 (k_1+k_2)^2 (k_1+k_2-p_1)^2 (k_2-p_1-p_2)^2 (k_1+k_2-p_1-p_2)^2 (k_2-p_1-p_2+p_3)^2}$$

$\int dz_1 \ldots dz_5$

```
{(M2^(-1-ep-z5) S^(-2-ep-z1) U^(z1+z5) (-M2 + S + U) Gamma[-ep + z1] Gamma[2 + z2] Gamma[-1 - ep - z2 - z3]
    Gamma[-z3] Gamma[1 - z1 + z2 + z3] Gamma[-1 - ep - z1 + z3 - z4] Gamma[-z1 + z2 + z3 - z4]
    Gamma[-z4] Gamma[1 + z1 + z4] Gamma[2 + ep + z1 + z4] Gamma[1 - z3 + z4]
    Gamma[-z1 - z5] Gamma[1 - ep - z1 + z2 + z4 - z5] Gamma[1 + ep + z5] Gamma[z1 - z2 + z5]) /
  (2 Gamma[1 - 2 ep] Gamma[1 - z3] Gamma[1 - z1 + z2 + z3 - z4]
    Gamma[1 - 2 ep + z1 + z4] Gamma[2 - z1 + z2 + z4]),

 (M2^(-1-ep-z5) S^(-1-ep-z1) U^(z1+z5) Gamma[-ep + z1] Gamma[1 + z2] Gamma[-1 - ep - z2 - z3] Gamma[-z3]
    Gamma[1 - z1 + z2 + z3] Gamma[-ep - z1 + z3 - z4] Gamma[-z1 + z2 + z3 - z4] Gamma[-z4]
    Gamma[1 + z1 + z4] Gamma[2 + ep + z1 + z4] Gamma[1 - z3 + z4] Gamma[-z1 - z5]
    Gamma[1 - ep - z1 + z2 + z4 - z5] Gamma[1 + ep + z5] Gamma[z1 - z2 + z5]) /
  (2 Gamma[1 - 2 ep] Gamma[1 - z3] Gamma[1 - z1 + z2 + z3 - z4]
    Gamma[1 - 2 ep + z1 + z4] Gamma[2 - z1 + z2 + z4]),

 (M2^(-1-ep-z5) (2 M2 - S) S^(-2-ep-z1) U^(z1+z5) Gamma[1 - ep + z1] Gamma[2 + z2] Gamma[-1 - ep - z2 - z3]
    Gamma[-z3] Gamma[1 - z1 + z2 + z3] Gamma[-1 - ep - z1 + z3 - z4] Gamma[-z1 + z2 + z3 - z4]
    Gamma[-z4] Gamma[1 + z1 + z4] Gamma[2 + ep + z1 + z4] Gamma[1 - z3 + z4]
    Gamma[-z1 - z5] Gamma[1 - ep - z1 + z2 + z4 - z5] Gamma[1 + ep + z5] Gamma[z1 - z2 + z5]) /
  (2 Gamma[1 - 2 ep] Gamma[1 - z3] Gamma[1 - z1 + z2 + z3 - z4]
    Gamma[2 - 2 ep + z1 + z4] Gamma[2 - z1 + z2 + z4]), • • •
```

11 terms in total

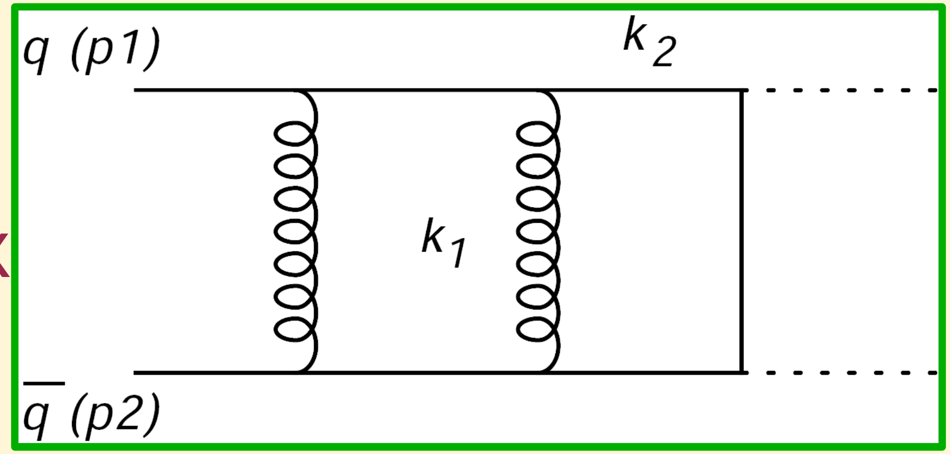
Mellin Barnes representation

-
-

W Pair Production

A tensor example:

(k1.p3) x



$(ms = M_W{}^2/s,\ x = -t/s)$

$$
\begin{aligned}
-\frac{1}{480\,(1-x)}\Big(&40\pi^4 - 132\pi^4 x + 360\pi^2 H[0,1,x] - 80\pi^2 x H[0,1,x] + 160\pi^2 H[1,0,x] - 160\pi^2 x H[1,0,x] + \\
&620\pi^2 H[1,1,x] - 400\pi^2 x H[1,1,x] + 240 H[0,0,1,1,x] + 480 H[0,1,1,0,x] - \\
&480 x H[0,1,1,0,x] + 720 H[0,1,1,1,x] + 480 H[1,0,1,0,x] - 480 x H[1,0,1,0,x] + \\
&960 H[1,0,1,1,x] + 480 H[1,1,0,0,x] - 480 x H[1,1,0,0,x] + 2400 H[1,1,1,0,x] - \\
&2400 x H[1,1,1,0,x] + 1800 H[1,1,1,1,x] + 100\pi^2 H[1,x] \operatorname{Log}[ms] - \\
&240\pi^2 x H[1,x] \operatorname{Log}[ms] - 240 H[0,1,1,x] \operatorname{Log}[ms] + 1440 H[1,1,0,x] \operatorname{Log}[ms] - \\
&1440 x H[1,1,0,x] \operatorname{Log}[ms] - 600 H[1,1,1,x] \operatorname{Log}[ms] + 70\pi^2 \operatorname{Log}[ms]^2 - 300\pi^2 x \operatorname{Log}[ms]^2 - \\
&180 H[1,1,x] \operatorname{Log}[ms]^2 + 260 H[1,x] \operatorname{Log}[ms]^3 + 45 \operatorname{Log}[ms]^4 + 130 x \operatorname{Log}[ms]^4 + \\
&160\pi^2 H[0,x] \operatorname{Log}[1-x] - 160\pi^2 x H[0,x] \operatorname{Log}[1-x] + 80\pi^2 H[1,x] \operatorname{Log}[1-x] - \\
&80\pi^2 x H[1,x] \operatorname{Log}[1-x] + 480 H[0,1,0,x] \operatorname{Log}[1-x] - 480 x H[0,1,0,x] \operatorname{Log}[1-x] + \\
&480 H[1,0,0,x] \operatorname{Log}[1-x] - 480 x H[1,0,0,x] \operatorname{Log}[1-x] + 480 H[1,1,0,x] \operatorname{Log}[1-x] - \\
&480 x H[1,1,0,x] \operatorname{Log}[1-x] - 680\pi^2 \operatorname{Log}[ms] \operatorname{Log}[1-x] + 680\pi^2 x \operatorname{Log}[ms] \operatorname{Log}[1-x] + \\
&1440 H[1,0,x] \operatorname{Log}[ms] \operatorname{Log}[1-x] - 1440 x H[1,0,x] \operatorname{Log}[ms] \operatorname{Log}[1-x] + \\
&40 \operatorname{Log}[ms]^3 \operatorname{Log}[1-x] - 40 x \operatorname{Log}[ms]^3 \operatorname{Log}[1-x] + 300\pi^2 \operatorname{Log}[1-x]^2 - \\
&300\pi^2 x \operatorname{Log}[1-x]^2 + 240 H[0,0,x] \operatorname{Log}[1-x]^2 - 240 x H[0,0,x] \operatorname{Log}[1-x]^2 - \\
&720 H[1,0,x] \operatorname{Log}[1-x]^2 + 720 x H[1,0,x] \operatorname{Log}[1-x]^2 + 720 H[0,x] \operatorname{Log}[ms] \operatorname{Log}[1-x]^2 - \\
&720 x H[0,x] \operatorname{Log}[ms] \operatorname{Log}[1-x]^2 - 60 \operatorname{Log}[ms]^2 \operatorname{Log}[1-x]^2 + 60 x \operatorname{Log}[ms]^2 \operatorname{Log}[1-x]^2 - \\
&560 H[0,x] \operatorname{Log}[1-x]^3 + 560 x H[0,x] \operatorname{Log}[1-x]^3 - 200 \operatorname{Log}[ms] \operatorname{Log}[1-x]^3 + \\
&200 x \operatorname{Log}[ms] \operatorname{Log}[1-x]^3 + 190 \operatorname{Log}[1-x]^4 - 190 x \operatorname{Log}[1-x]^4 - \\
&6320 H[1,x] \operatorname{Zeta}[3] + 1920 x H[1,x] \operatorname{Zeta}[3] + 1280 \operatorname{Log}[ms] \operatorname{Zeta}[3] - \\
&1120 x \operatorname{Log}[ms] \operatorname{Zeta}[3] - 2080 \operatorname{Log}[1-x] \operatorname{Zeta}[3] + 2080 x \operatorname{Log}[1-x] \operatorname{Zeta}[3]\Big)
\end{aligned}
$$

"Analytic result"

Catani's recipe: An important test

One loop: For the IR pole structure of the renormalized amplitude you need the tree level amplitude and $I^{(1)}$: (Catani, Seymour '98)

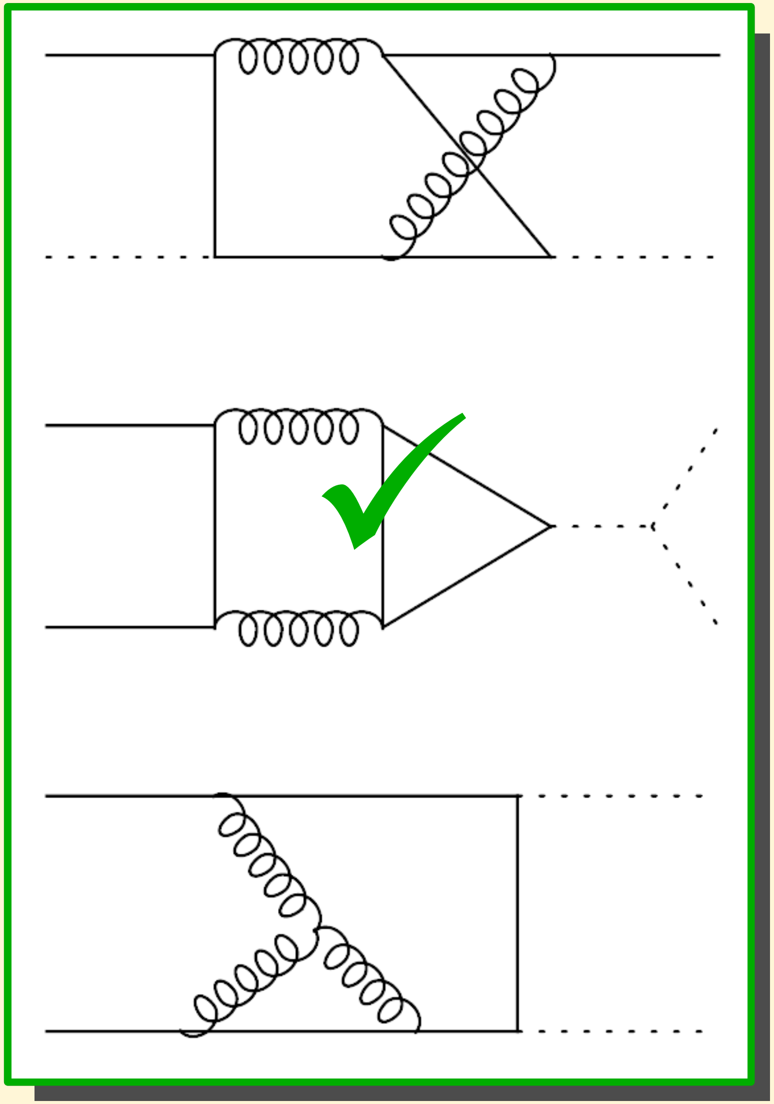
$$|\mathcal{M}_m^{(1)}(\mu^2;\{p\})\rangle_{\rm R.S.} = \boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\;|\mathcal{M}_m^{(0)}(\mu^2;\{p\})\rangle_{\rm R.S.} + |\mathcal{M}_m^{(1)\,\rm fin}(\mu^2;\{p\})\rangle_{\rm R.S.}$$

Two loop: Now you need tree and one loop level amplitude: (Catani '98)

$$\begin{aligned} |\mathcal{M}_m^{(2)}(\mu^2;\{p\})\rangle_{\rm R.S.} &= \boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\;|\mathcal{M}_m^{(1)}(\mu^2;\{p\})\rangle_{\rm R.S.} \\ &+ \boldsymbol{I}_{\rm R.S.}^{(2)}(\epsilon,\mu^2;\{p\})\;|\mathcal{M}_m^{(0)}(\mu^2;\{p\})\rangle_{\rm R.S.} + |\mathcal{M}_m^{(2)\rm fin}(\mu^2;\{p\})\rangle_{\rm R.S.} \end{aligned}$$

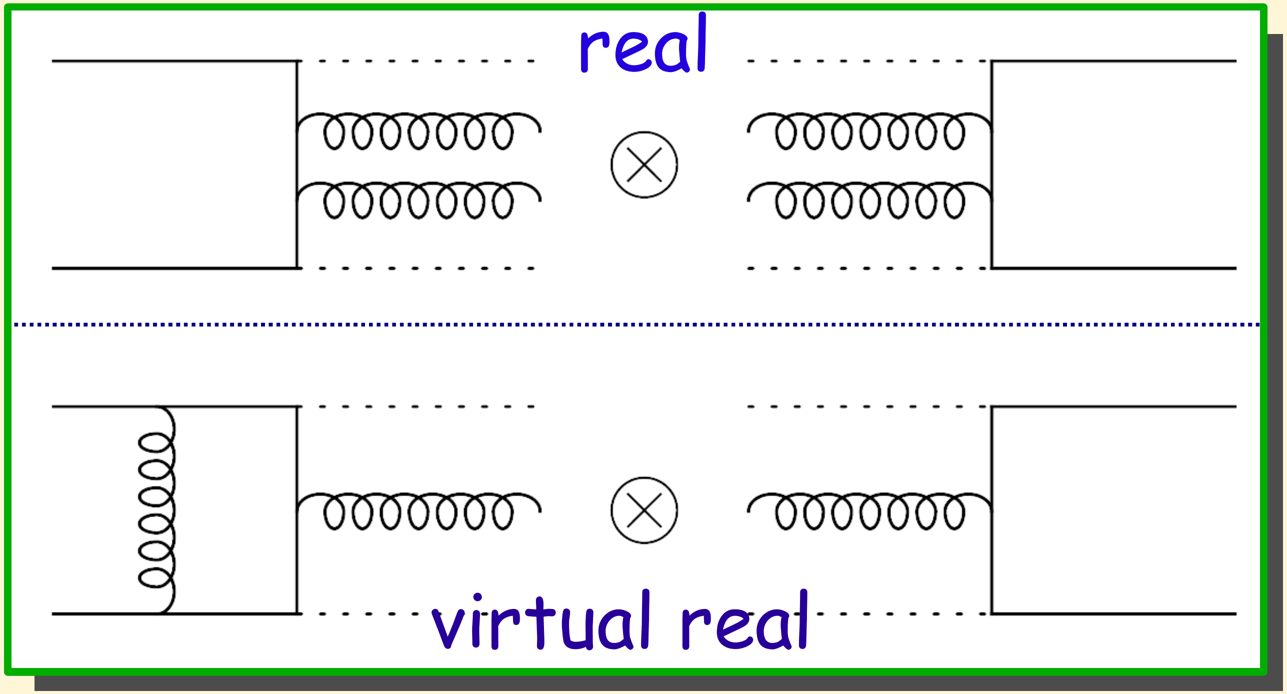
Singular dependence embodied in the operators $I^{(1)}$ and $I^{(2)}$

Check list:



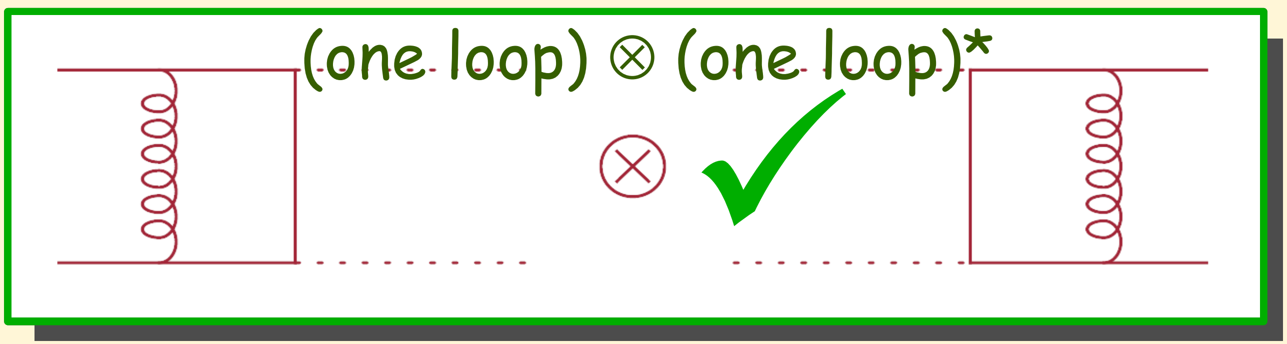
2 loop virtual

G.C., Czakon, Eiras ('08)



Campbell, Ellis, Zanderighi ('07)

Dittmaier, Kallweit, Uwer ('08,'09)



G.C., Czakon, Eiras ('08)

Mid talk summary

- We have the full (virtual) result up to $O(m_s^0)$ in the high energy limit for the virtual corrections.
- Mellin Barnes representations approach is a powerful technique though not an easy one (especially for the non-planar graphs).

Nevertheless, ready for the next step:

Full mass dependence

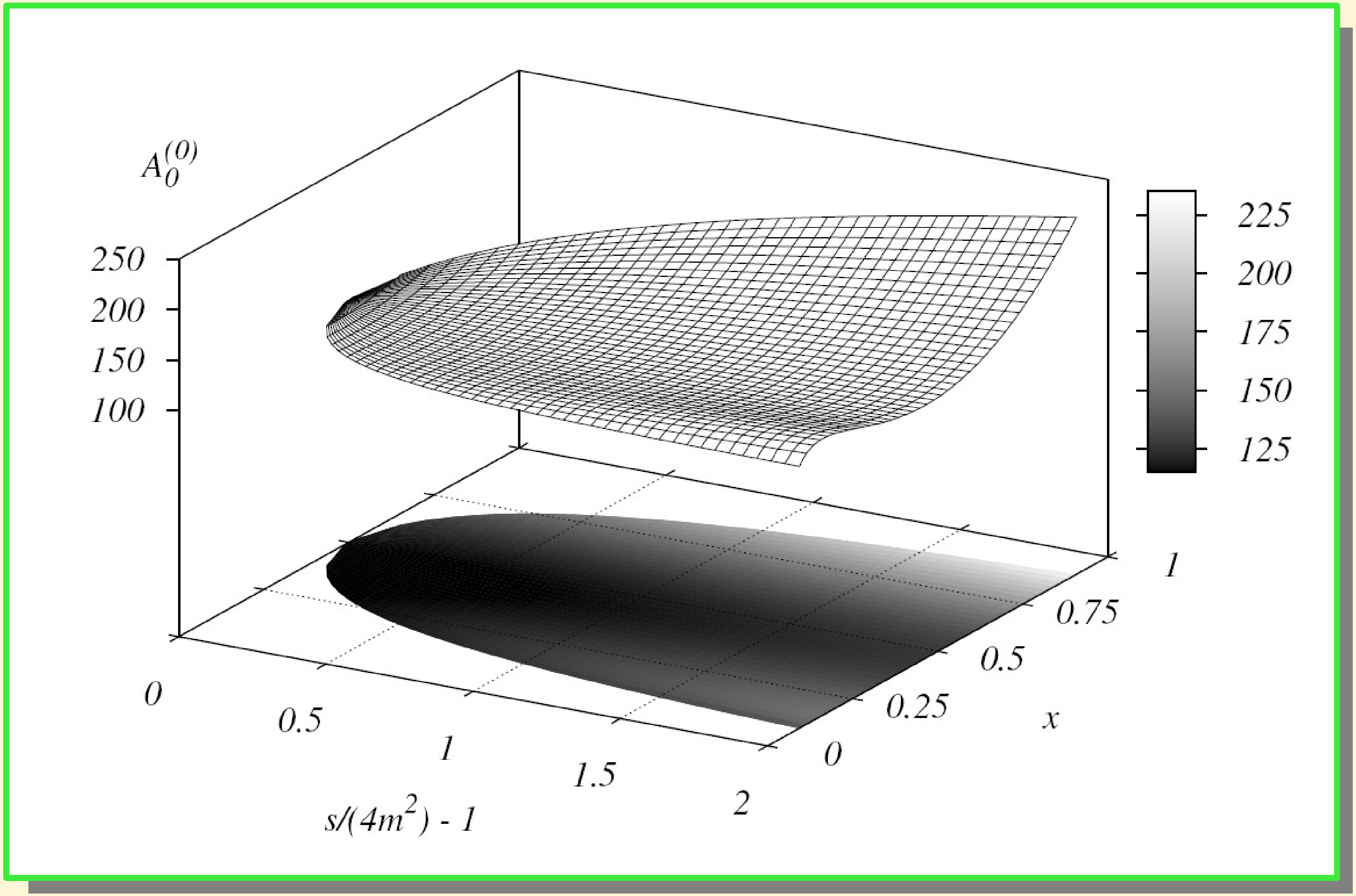
Similar to M. Czakon [arXiv:0803.1400] using

•Numerical Differential Equation method

Caffo, Czyz, Laporta, Remiddi ('98)

Full mass dependence-how to

- Numerical solution of differential equations: an example from Top pair production



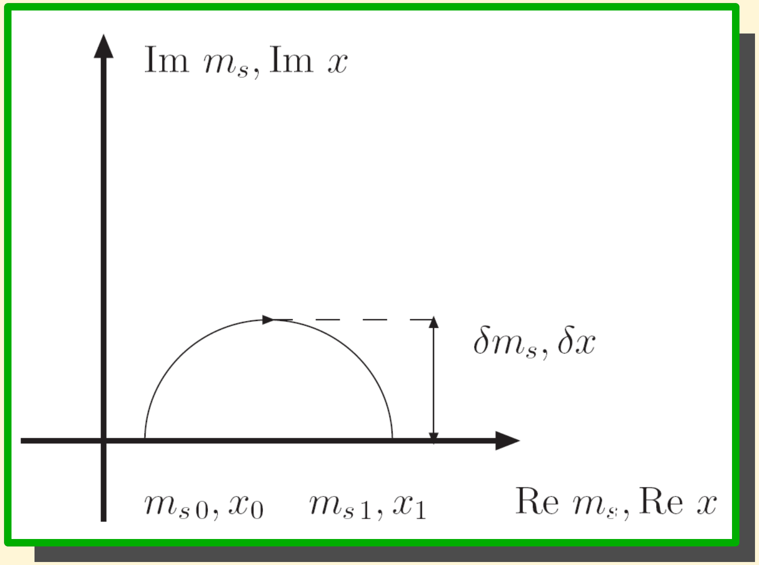
Plot stolen from M. Czakon's paper “Mass effects and...” '08

Towards a numerical solution

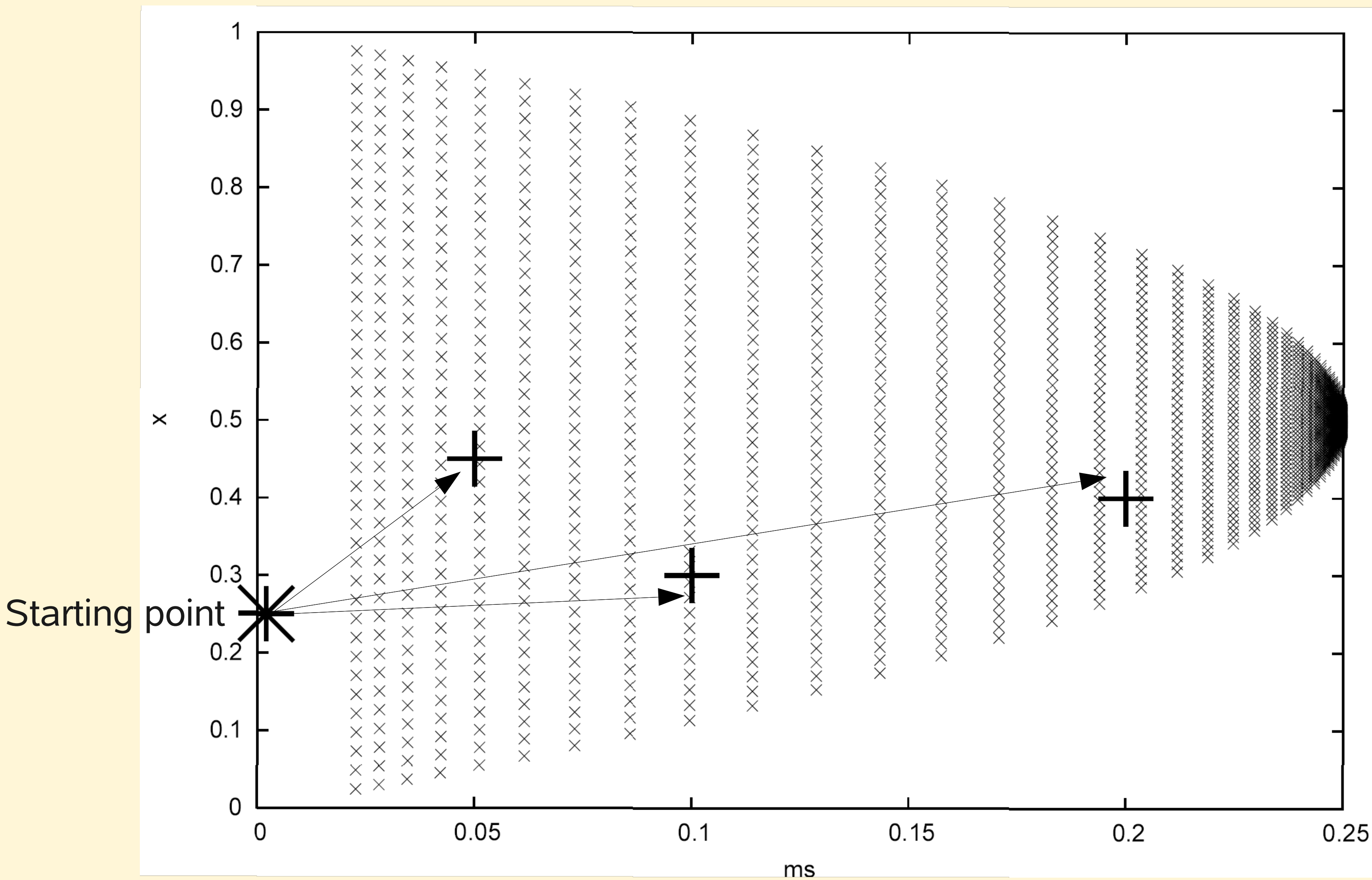
- Compute the high energy asymptotics of the master integrals obtaining the leading behaviour of the amplitude
- Determine the coefficients of the mass expansions using differential equations in m_s obtaining the power corrections

$$m_s \frac{d}{d m_s} M_i(m_s, x, \epsilon) = \sum_j C_{ij}(m_s, x, \epsilon) M_j(m_s, x, \epsilon)$$

- Evaluate the expansions for $m_s \ll 1$ to obtain the desired numerical precision of the boundaries
- Evolve the functions from the boundary point with differential equations first in m_s and then in x (**ZVODE**)



Phase space and some sample points-Numerics works!



DE numerics

- Find the optimal set of masters
- Solve the differential equations for a grid of points
- Run numerical checks (e.g. use different contours) and control errors
- Renormalize
- Pass once more through Catani's purgatory
- Interpolate

Conclusions - Outlook

- Working numerics for full mass dependence- further checks need to be done though!
- The first steps have been done, many more are to follow for the full **NNLO corrections** in the q q $\longrightarrow$ W W channel

  Next: **g g → W W channel**

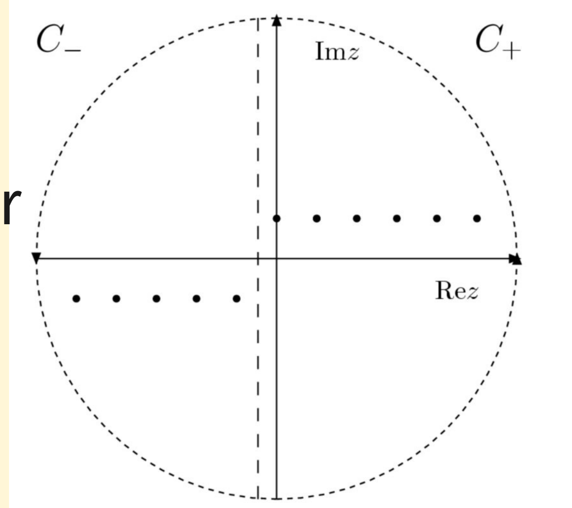
  and Z pair production
- The ultimate goal is to have a

  NNLO Monte Carlo generator

  for gauge boson pair production

Back up slides

Mellin-Barnes representations



Under the assumption that n>0 and that the contour separates the poles of the Gamma functions

$$\frac{1}{(A+B)^n} = \frac{1}{2\pi i\Gamma(n)}\int_{-i\infty}^{i\infty} dz \frac{A^z}{B^{n+z}}\Gamma(-z)\Gamma(n+z)$$

Behaviour of the Gamma functions around non-positive arguments

$$z\Gamma(z) = \Gamma(1+z) \qquad \Rightarrow \qquad \Gamma(-n+z) = \frac{\Gamma(1+z)}{(-n+z)...(z)} \sim \frac{(-)^n}{n!}\frac{1}{z}$$

Take residues depending on the values of A and B

For $A > B \quad \Rightarrow \quad z < 0 \quad \Rightarrow \quad z = -N-n,\ N = 0, 1, ...$

$$\sum_{N=0}^{\infty} \frac{(-)^N}{N!}\frac{\Gamma(N+n)}{\Gamma(n)}\frac{B^N}{A^{N+n}} = \frac{1}{A^n}\sum_{N=0}^{\infty}\frac{(N+n-1)!}{(n-1)!N!}\left(-\frac{B}{A}\right)^N = \frac{1}{A^n}\frac{1}{\left(1+\frac{B}{A}\right)^n} = \frac{1}{(A+B)^n}$$

For $A < B \quad \Rightarrow \quad z > 0 \quad \Rightarrow \quad z = N,\ N = 0, 1, ...$

$$\sum_{N=0}^{\infty} \frac{(-)^N}{N!}\frac{\Gamma(N+n)}{\Gamma(n)}\frac{A^N}{B^{N+n}} = \frac{1}{B^n}\sum_{N=0}^{\infty}\frac{(N+n-1)!}{(n-1)!N!}\left(-\frac{A}{B}\right)^N = \frac{1}{B^n}\frac{1}{\left(1+\frac{A}{B}\right)^n} = \frac{1}{(A+B)^n}$$

1st Barnes' lemma

$$\int_{-i\infty}^{i\infty} dz \, \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

2nd Barnes' lemma

$$\int_{-i\infty}^{i\infty} dz \, \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)} = \frac{\Gamma(a+d)\Gamma(a+e)\Gamma(b+d)\Gamma(b+e)\Gamma(c+d)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}.$$

$$I^{(1)}(\varepsilon) = -C_F \frac{e^{\varepsilon\gamma}}{\Gamma(1-\varepsilon)} \left(\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon}\right) \left(-\frac{\mu^2}{s}\right)^{\varepsilon}$$

$$\begin{aligned} I^{(2)}(\varepsilon) &= -\frac{1}{2} I^{(1)}(\varepsilon) \left(I^{(1)}(\varepsilon) + \frac{2\beta_0}{\varepsilon} \right) + \frac{e^{-\varepsilon\gamma}\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)} \left(\frac{\beta_0}{\varepsilon} + K \right) I^{(1)}(2\varepsilon) \\ &\quad + H^{(2)}(\varepsilon), \end{aligned}$$

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_F n_f.$$

$$\begin{aligned} H^{(2)}(\varepsilon) &= 2 \frac{e^{\varepsilon\gamma}}{4\varepsilon\Gamma(1-\varepsilon)} \left(-\frac{\mu^2}{s}\right)^{2\varepsilon} \left\{ \left(\frac{\pi^2}{2} - 6\,\zeta_3 - \frac{3}{8} \right) C_F^2 \right. \\ &\quad \left. + \left(\frac{13}{2}\zeta_3 + \frac{245}{216} - \frac{23}{48}\pi^2 \right) C_A C_F + \left(-\frac{25}{54} + \frac{\pi^2}{12} \right) C_F T_F n_f \right\} \end{aligned}$$

$$
\begin{aligned}
&\jmath^{3}(m_s,x) = \\
&C_A C_F\left\{\frac{1}{ms^2}\left[\frac{31}{240}(1-x)x\pi^4-\frac{107}{72}(1-x)x\pi^2-\frac{51157(1-x)x}{1296}+\frac{659}{36}(1-x)x\zeta_3+\frac{44}{3}(1-x)x\mathrm{L}_s\right]\right.\\
&+\frac{1}{ms}\left[\frac{31}{240}\left(4x^2-4x+3\right)\pi^4-\frac{107}{72}\left(4x^2-4x+3\right)\pi^2-\frac{51157\left(4x^2-4x+3\right)}{1296}+\frac{659}{36}\left(4x^2-4x+3\right)\zeta_3\right.\\
&\left.+\frac{44}{3}\left(4x^2-4x+3\right)\mathrm{L}_s\right]+\left[-\frac{31}{20}\left(x^2-x+1\right)\pi^4+\frac{107}{6}\left(x^2-x+1\right)\pi^2+\frac{1}{108}\left(51157x^2-51157x\right.\right.\\
&\left.\left.\left.+\left(-23724x^2+23724x-23724\right)\zeta_3+51157\right)-176\left(x^2-x+1\right)\mathrm{L}_s\right]\right\}\\
&+C_F{}^2\left\{\frac{1}{ms^2}\left[-\frac{11}{90}(1-x)x\pi^4+\frac{29}{12}(1-x)x\pi^2+\frac{255}{16}(1-x)x-15(1-x)x\zeta_3\right]\right.\\
&+\frac{1}{ms}\left[-\frac{11}{90}\left(4x^2-4x+3\right)\pi^4+\frac{29}{12}\left(4x^2-4x+3\right)\pi^2+\frac{255}{16}\left(4x^2-4x+3\right)-15\left(4x^2-4x+3\right)\zeta_3\right]\\
&\left.+\left[\frac{22}{15}\left(x^2-x+1\right)\pi^4-29\left(x^2-x+1\right)\pi^2+\frac{45}{4}\left(-17x^2+17x+\left(16x^2-16x+16\right)\zeta_3-17\right)\right]\right\}\\
&+n_f T_F C_F\left\{\frac{1}{ms^2}\left[\frac{7}{18}(1-x)x\pi^2+\frac{4085}{324}(1-x)x-\frac{1}{9}(1-x)x\zeta_3-\frac{16}{3}(1-x)x\mathrm{L}_s\right]\right.\\
&+\frac{1}{ms}\left[\frac{7}{18}\left(4x^2-4x+3\right)\pi^2+\frac{4085}{324}\left(4x^2-4x+3\right)+\frac{1}{9}\left(-4x^2+4x-3\right)\zeta_3-\frac{16}{3}\left(4x^2-4x+3\right)\mathrm{L}_s\right]\\
&\left.+\left[-\frac{14}{3}\left(x^2-x+1\right)\pi^2+\frac{1}{27}\left(-4085x^2+4085x+\left(36x^2-36x+36\right)\zeta_3-4085\right)+64\left(x^2-x+1\right)\mathrm{L}_s\right]\right\}
\end{aligned}
$$

$$
\mathrm{L}_m=\log\left(m_s\right),\quad \mathrm{L}_s=\log\left(\frac{s}{\mu^2}\right),\quad \mathrm{L}_x=\log\left(x\right),\quad \mathrm{L}_y=\log\left(1-x\right)
$$