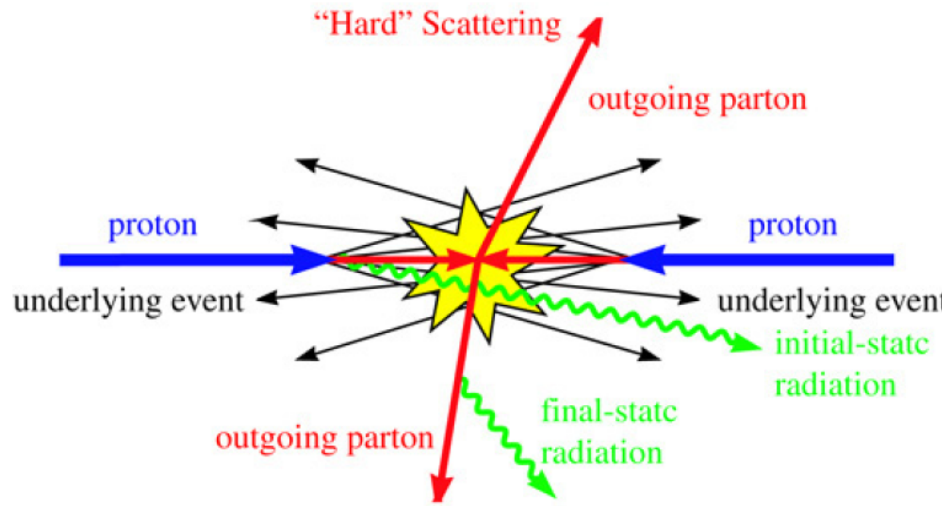
Factorization

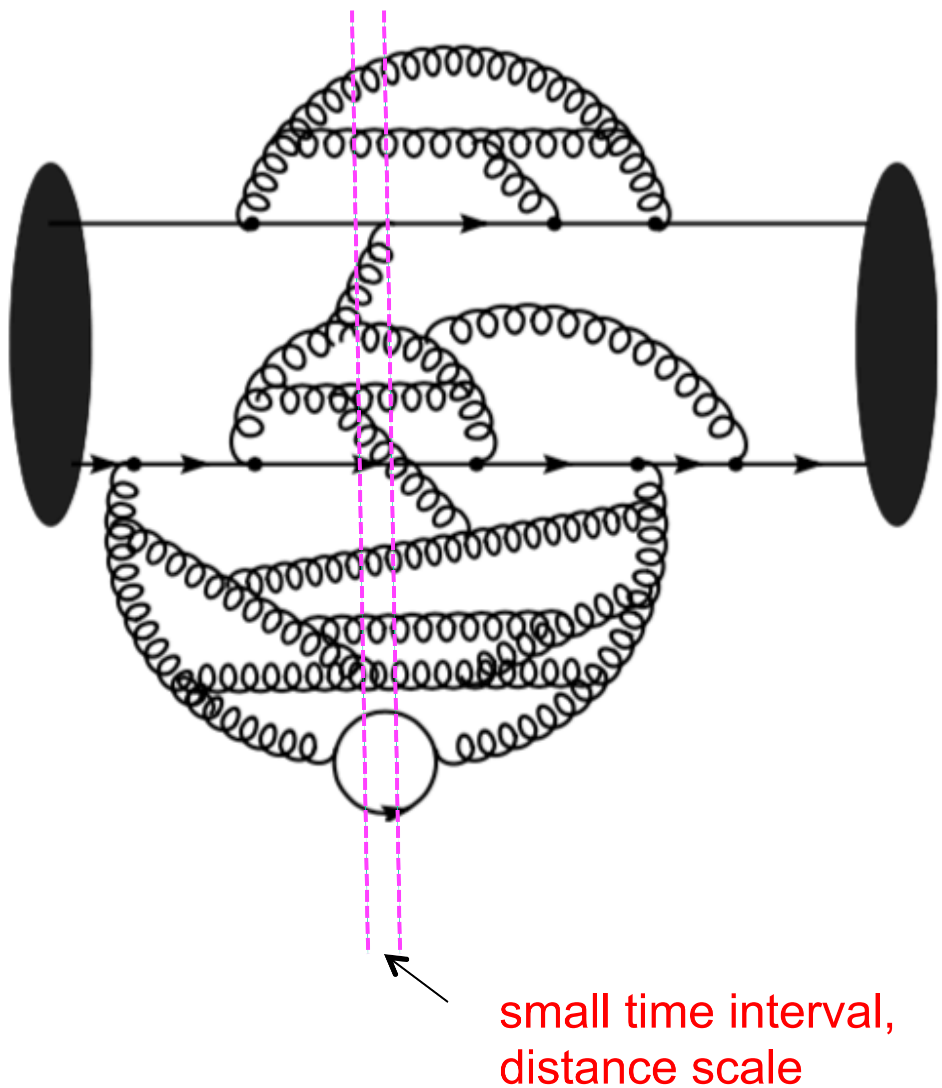
- Factorization is the key to perturbative QCD
  - the ability to separate the short-distance physics and the long-distance physics
- In particular, parton distribution functions are part of the long-distance physics
- Factorization tells us that PDFs determined from one process (or group of processes) can be used for other processes
- So we can determine PDFs from experiments whose data was taken long before you were born (and more recent data as well) and use them for the LHC



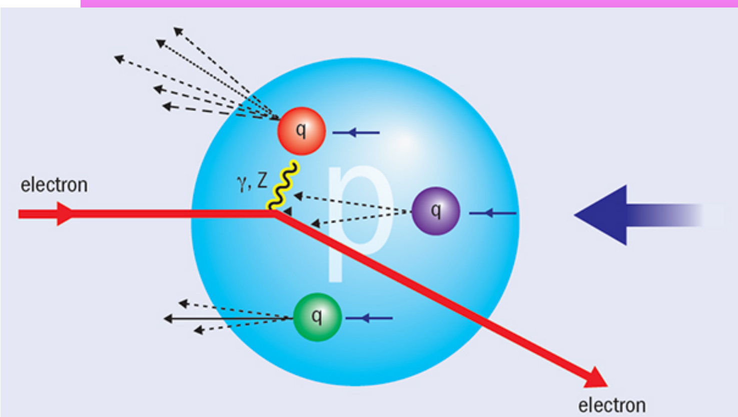
The calculation of hard scattering processes at the LHC requires:
(1)knowledge of the distributions of the quarks and gluons inside the proton, i.e. what fraction of the momentum of the parent proton do they have
->parton distribution functions (pdf' s)
(2) knowledge of the hard scattering cross sections of the quarks and gluons, at LO, NLO, or NNLO in the strong coupling constant α_s

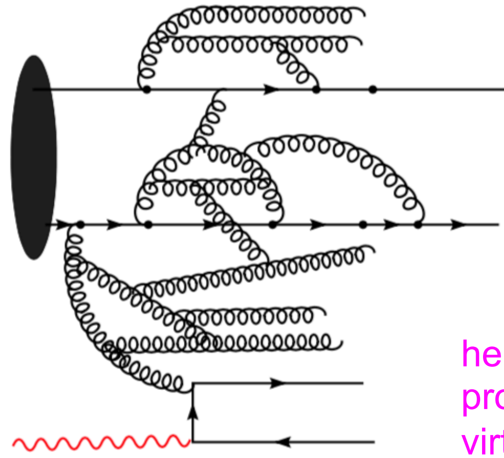
Let’s think about this from a space-time perspective

- Partons in the proton are always emitting virtual gluons/ quark-antiquark pairs which then recombine (the proton remains intact)
- The lifetime of these virtual states depends inversely on the energy of the partons
  - uncertainty principle
- If I can probe smaller and smaller distances (time-scales), then I can resolve more of the radiative structure inside the proton
- I can probe these smaller distances by using higher energies (Q) to probe



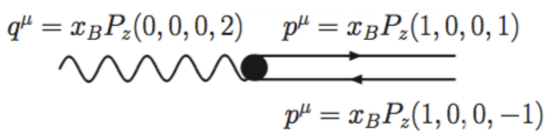
Consider deep-inelastic scattering (DIS)





here I' m actually probing these virtual fluctuations; quark anti-quark pair cannot recombine

- Condition for DIS on protons
  - $Q > m_p \sim 1/R_p$: scale Q (typically the virtual mass of the photon)
- Using Breit frame



brick wall frame of reference

$$P^\mu = (P_0, \vec{0}, P_z) \text{ with } P_z = \sqrt{P_0^2 - m_P^2} \approx P_0$$

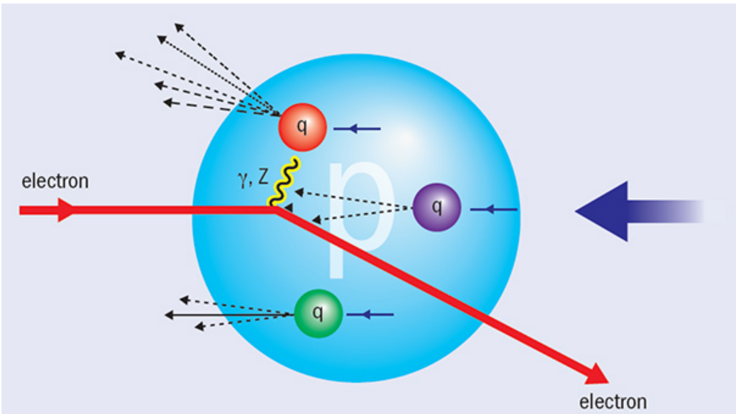
$$q^\mu = (0, \vec{0}, -q_z) \text{ with } Q^2 = -q^2 = q_z^2 .$$

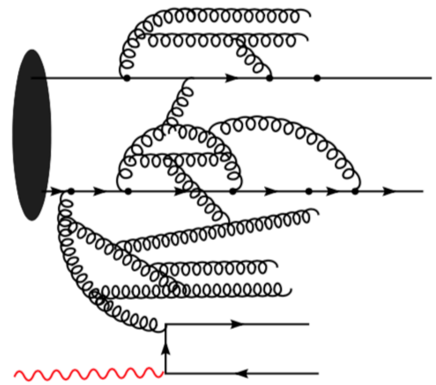
- Introducing Bjorken variable x

$$x_B = -\frac{q^2}{2P \cdot q} = \frac{q_z^2}{2P_z q_z}$$

(we'll call this x from now on, the fraction of the proton's momentum taken by a parton)

Consider the timescales involved





- interaction time between photon and proton from longitudinal wavelength of photon $\tau_{\text{int}} \sim \lambda_z \sim 1/q_z$
- parton wavelength must be as large as photon wavelength for them to "see" each other: $\Longrightarrow \quad p_z = q_z$
- lifetime of parton $\tau_{\text{life}} \sim p_z/p_\perp^2 \ \geq \ \tau_{\text{int}}$
  must be larger than interaction time!

- therefore: $p_z \geq p_\perp$ for interaction to happen
- if this holds: collision of two quasi-free particles
  collinear factorisation
  into hard process and **independent** proton $\rightarrow$ parton transitions

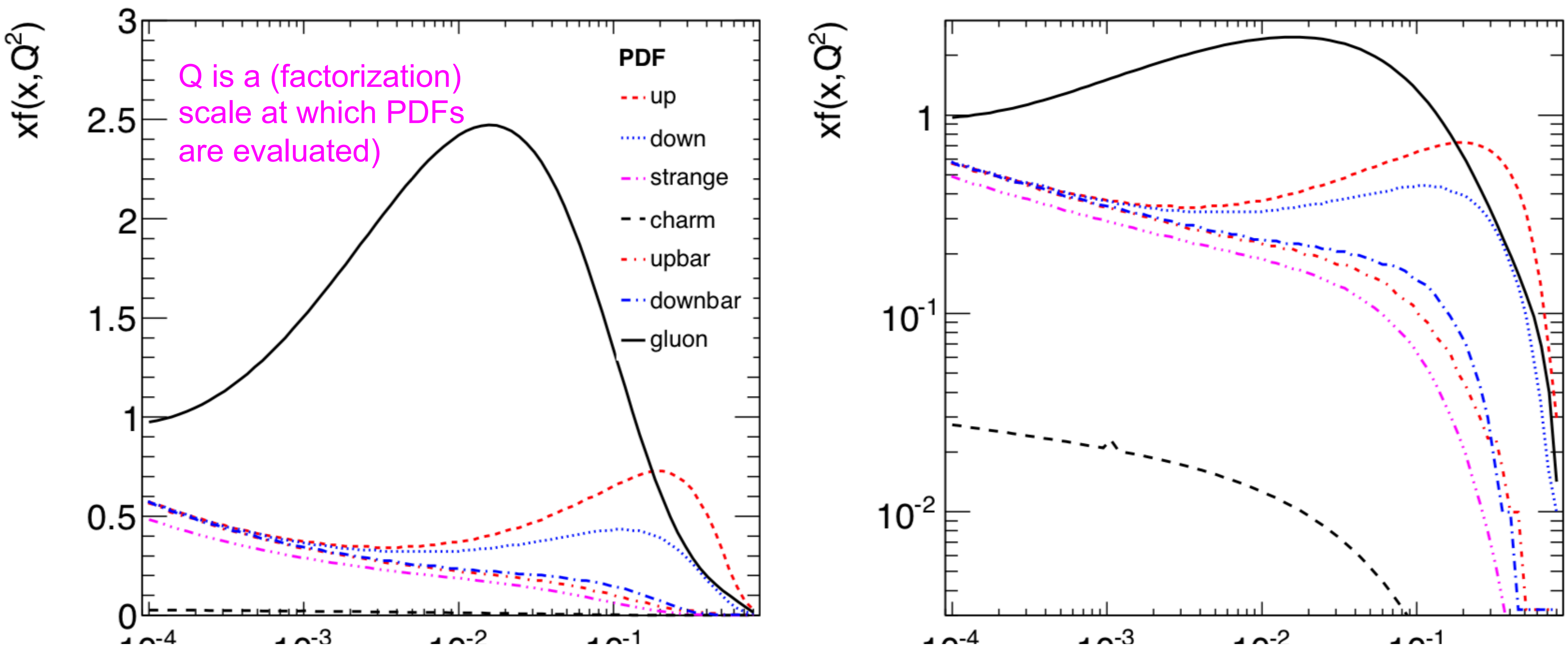
- then cross section proportional to probabilities to find partons in proton, given by the parton distribution functions $f_{q/p}$

$$\sigma_{\text{DIS}} \sim \sum_q e_q^2 f_{q/p}(x,\, Q^2)\,,$$

Parton distributions

- The momentum of the proton is distributed among the quarks and gluons that comprise it
  - about **40%** of the momentum is with gluons, the rest with the quarks
  - note that the quarks at high x tend to be valence quarks (uud), while the quarks at low x tend to be sea quarks produced by gluon splitting into quark-antiquark pairs (u-ubar, d-dbar, s-sbar, etc)

$f(x,Q^2)$ describes the momentum distribution of partons inside a proton



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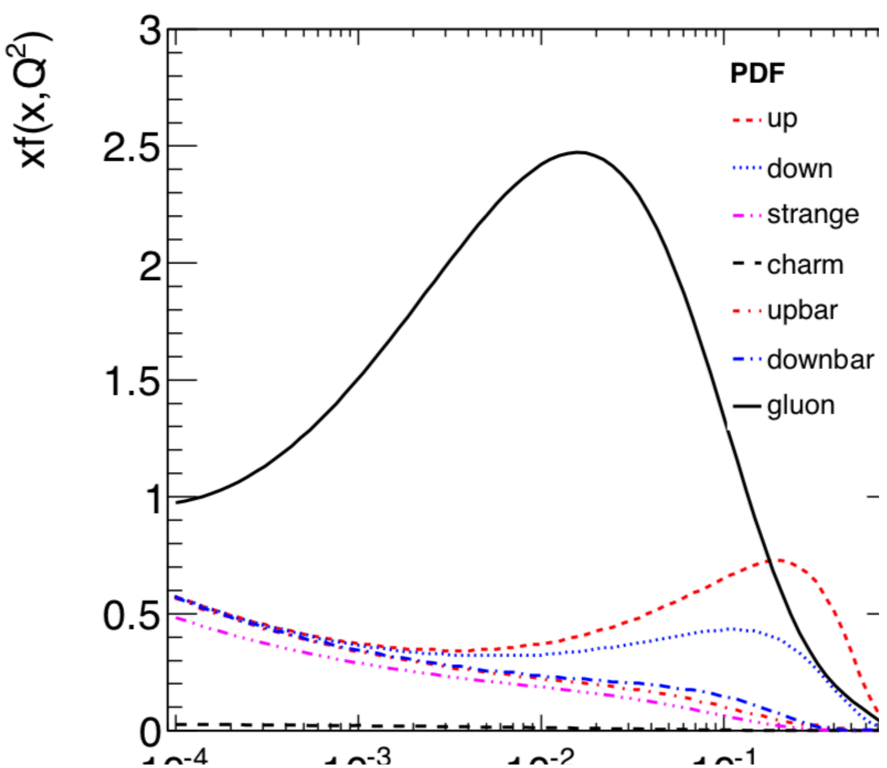
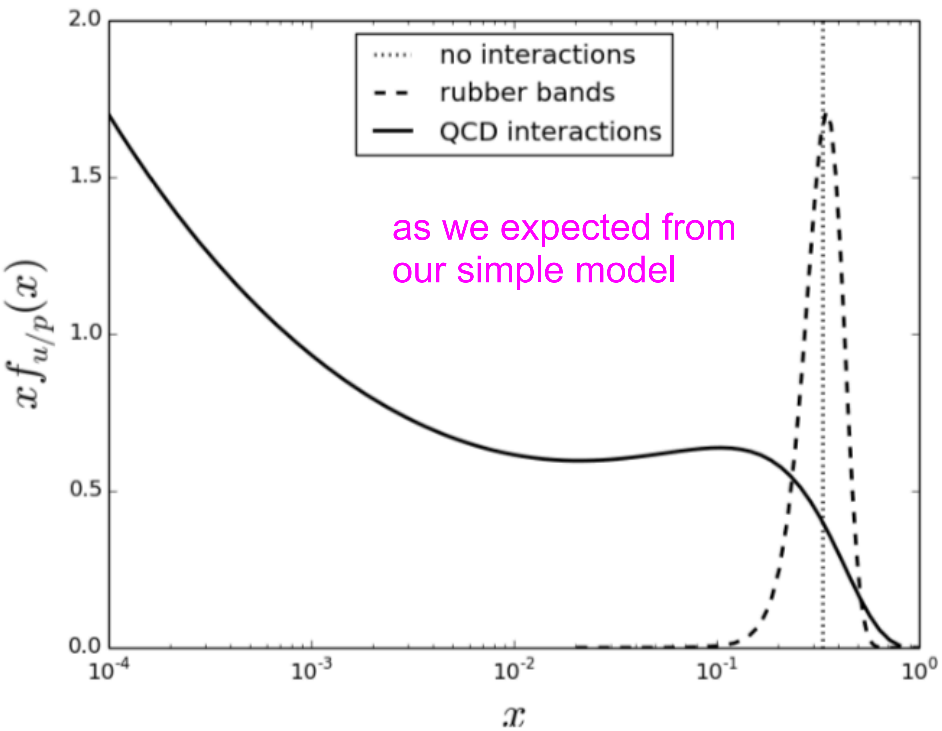
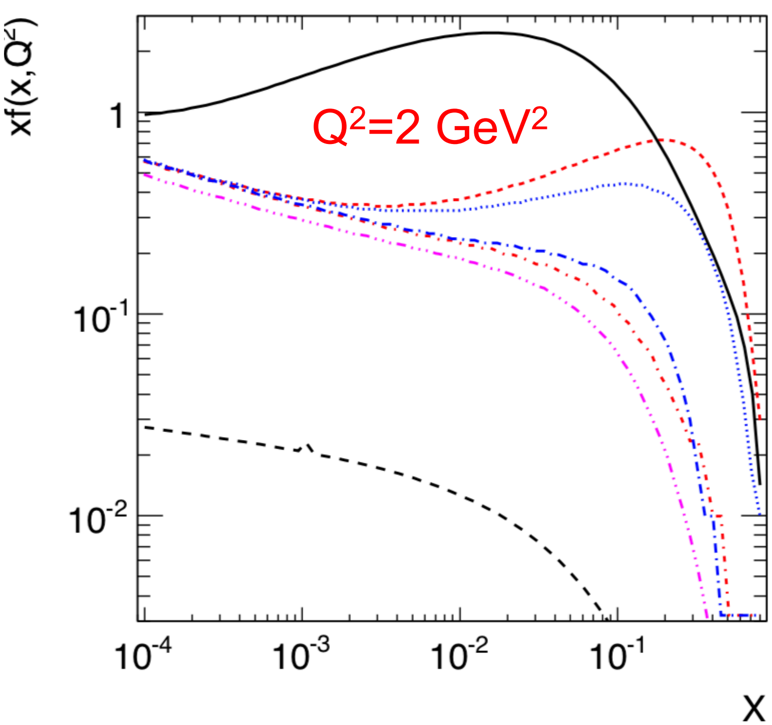




Parton distribution functions (PDFs)

- Note the changes in the distributions as Q^2 increases (DGLAP)
- High x distributions decrease, while low x distributions increase



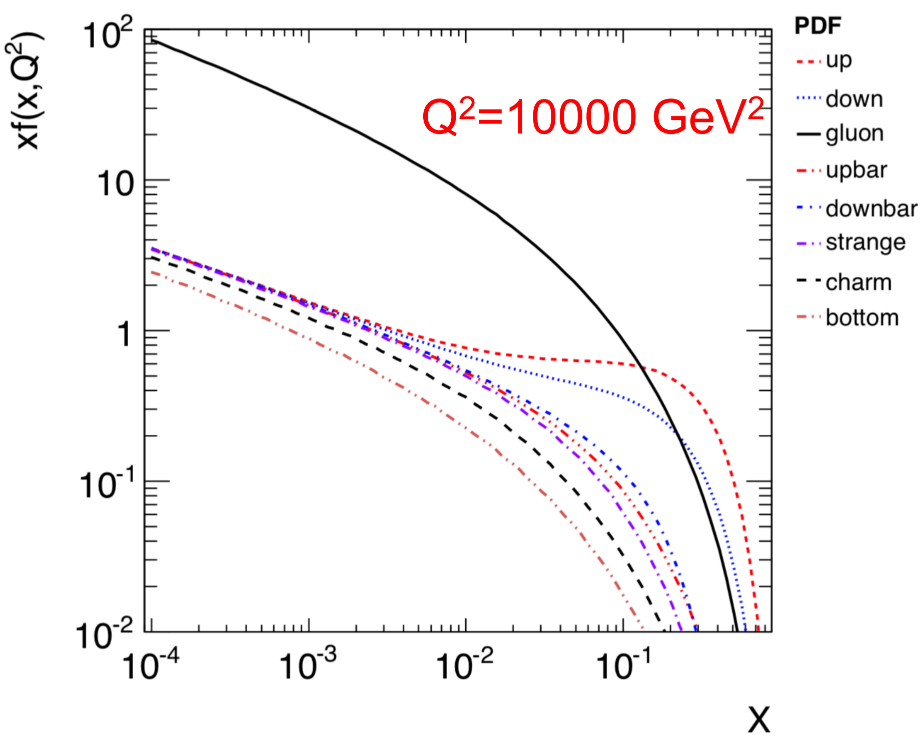
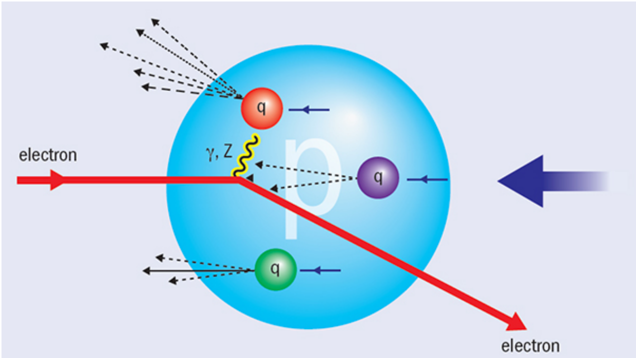
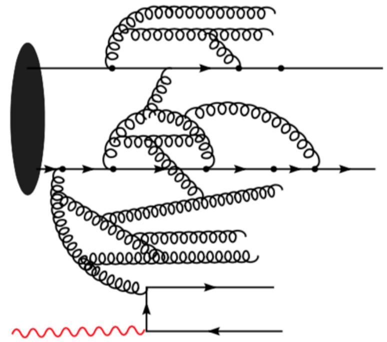
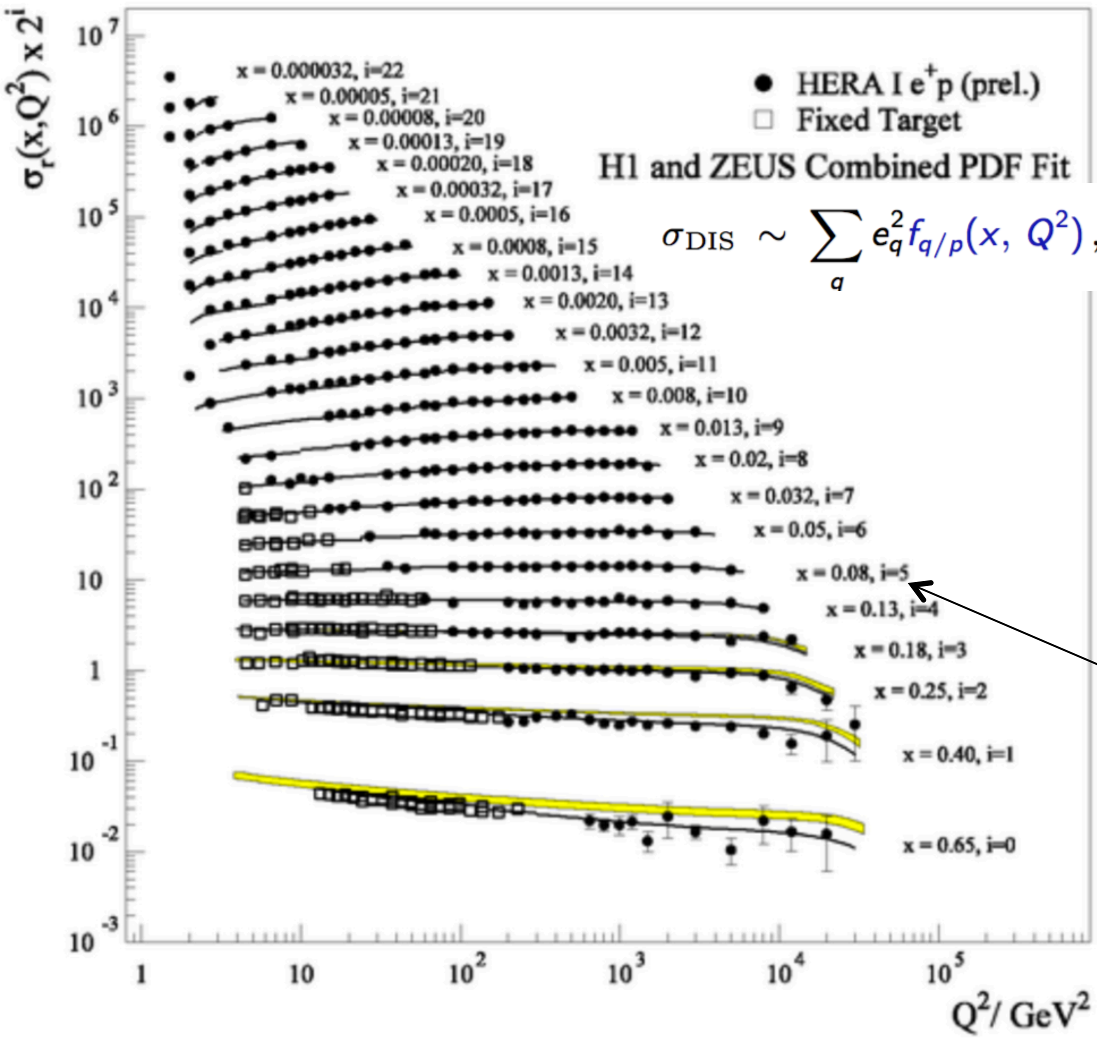


Fig. 6.6 The CT14 NNLO parton distribution functions evaluated at a Q^2 of 10000 GeV^2.

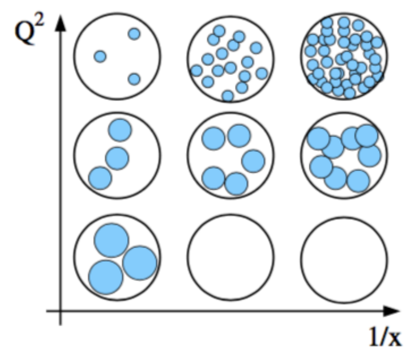




Back to DIS

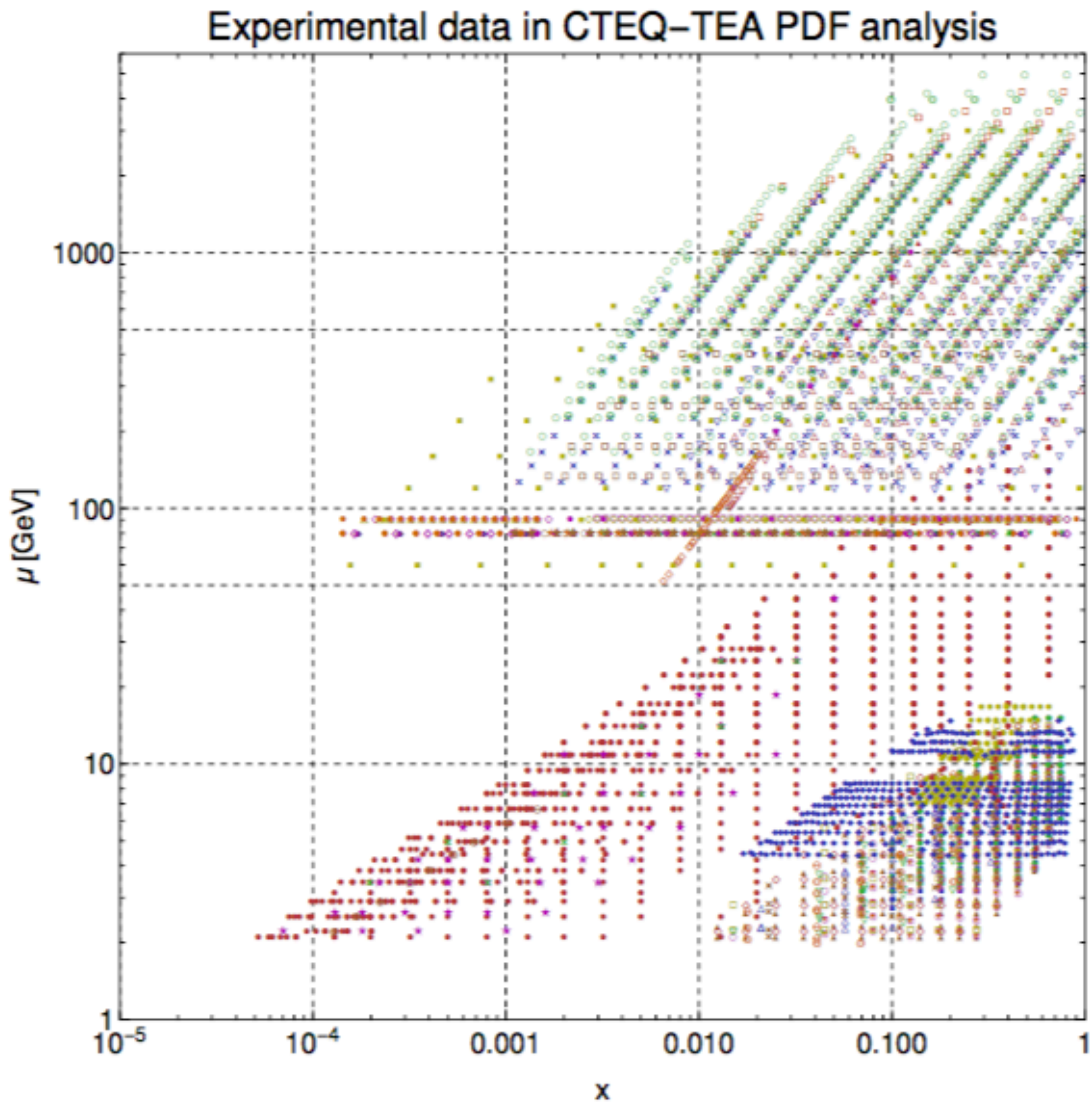


$$\sigma_{\mathrm{DIS}} \sim \sum_q e_q^2 f_{q/p}(x, Q^2),$$

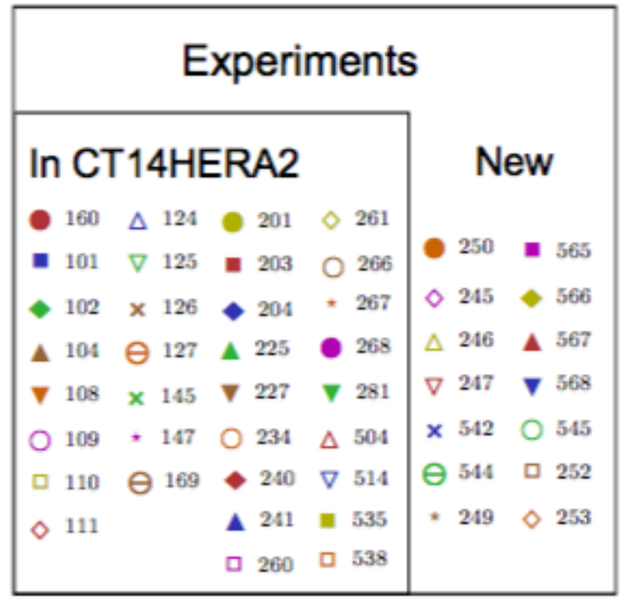


due to DGLAP evolution, number of partons at low x increases with Q^2; the number of partons at high x decreases with Q^2

note that early measurements of DIS at SLAC were for x values around 0.1; at these x values the number of partons is relatively stable with respect to Q^2, i.e. no scaling violations



PDFs are non-perturbative objects. Only evolution is perturbative (DGLAP).



Global PDF fits are carried out using data from a variety of processes, including DIS data taken long before you were born. Increasingly, LHC data are being used in the fits.

FIG. 1: A graphical representation of the space of $\{x, \mu\}$ points probed by the full dataset treated in the present analysis, designated as "CTEQ-TEA". It corresponds to an expansion of the CT14HERA2 data [10] fitted in the most recent CT14 framework [1], including measurements from Run II of HERA [6].

Global fits

- With the DGLAP equations, we know how to evolve PDF's from a starting scale Q_0 to any higher scale
- …but we can't calculate what the PDF's are ab initio
  - one of the goals of lattice QCD
- We have to determine them from a global fit to data
  - factorization theorem tells us that PDF's determined for one process are applicable to another
  - extremely important proof

- So what do we need
  - a value of Q_o (1.3 GeV for CTEQ) lower than the data used in the fit (or any prediction)
  - a parametrization for the PDF's
  - a scheme for the PDF's
  - hard-scattering calculations at the order being considered in the fit
  - PDF evolution at the order being considered in the fit
  - a world average value for α_s
  - a lot of data
    - with appropriate kinematic cuts
  - a treatment of the errors for the experimental data
  - MINUIT

Global fits

- Parametrization: initial form
  - $f(x) \sim x^{\alpha}(1-x)^{\beta}$
  - estimate β from quark counting rules
    - $\beta=2n_s-1$ with n_s being the minimum number of spectator quarks
    - so for valence quarks in a proton (qqq), $n_s=2$, $\beta=3$
    - for gluon in a proton (qqqg), $n_s=3$, $\beta=5$
    - for anti-quarks in a proton (qqqqqbar), $n_s=4$, $\beta=7$
  - estimate α from Regge arguments
    - gluons and anti-quarks have $\alpha \sim -1$ while valence quarks have $\alpha \sim =1/2$
  - but at what Q value are these arguments valid?

- What do we know?

1. we know that the sum of the momentum of all partons in the proton is 1
2. we know the sum of valence quarks is 3
   - and 2 of them are up quarks and 1 of them is a down quark
   - we know that the net number of anti-quarks is 0
   - we know that the net number of strange quarks (charm quarks/bottom quarks) in the proton is 0

This already puts a lot of restrictions on the PDF's

PDF luminosities

- Useful to define PDF luminosities

$$\mathcal{L}_{u\bar{d}}(x_A, x_B, \mu_F) = \frac{1}{2\hat{s}}\left[f_{u/A}(x_A, \mu_F) f_{\bar{d}/B}(x_B, \mu_F) + \{u \longleftrightarrow \bar{d}\}\right]$$
$$= \frac{1}{2x_A x_B s}\left[f_{u/A}(x_A, \mu_F) f_{\bar{d}/B}(x_B, \mu_F) + \{u \longleftrightarrow \bar{d}\}\right]$$

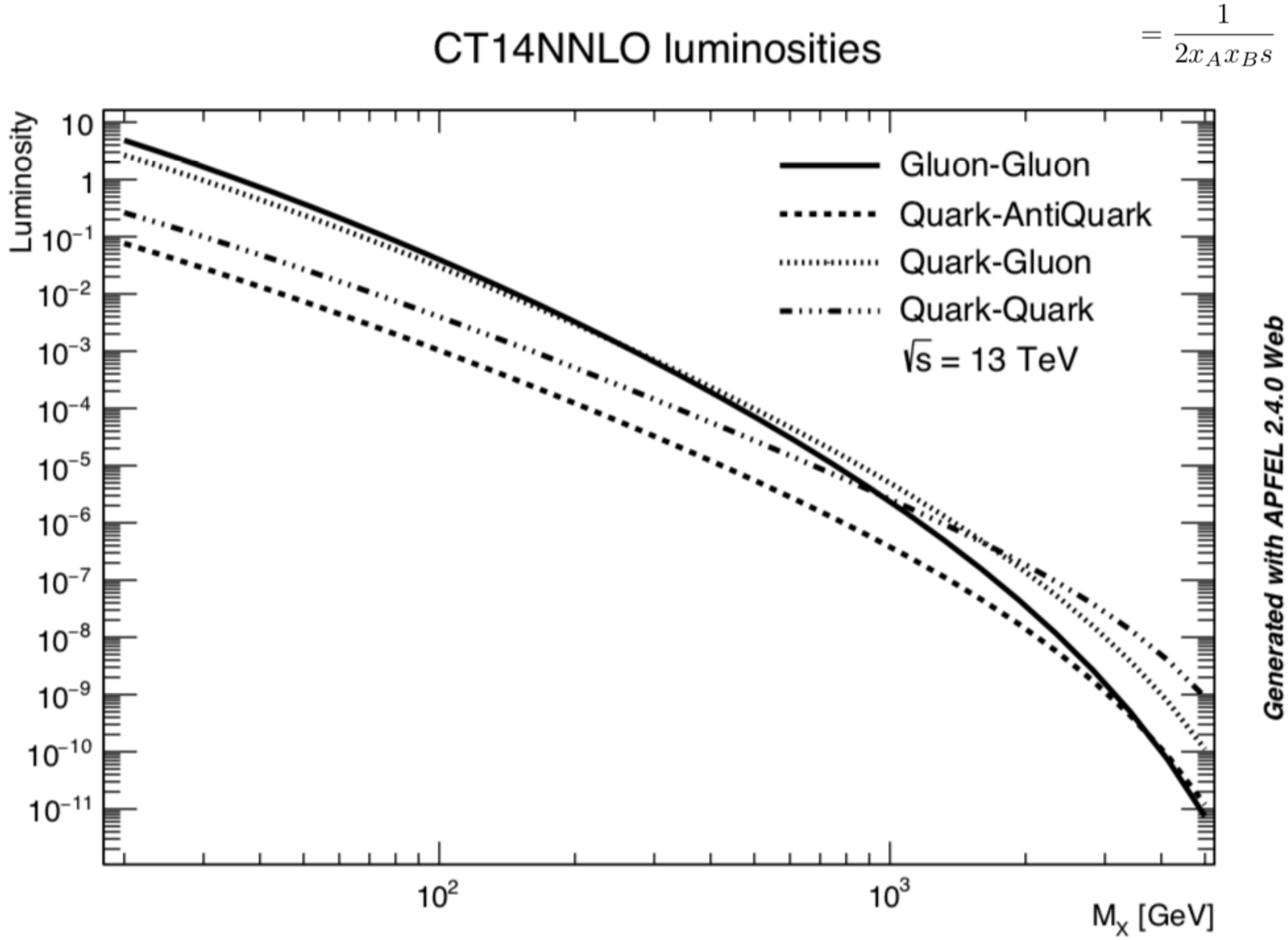


Fig. 6.30 The parton-parton luminosities for CT14 for *pp* collisions at 13 TeV plotted as a function of mass ($M_X \equiv \sqrt{\hat{s}}$).

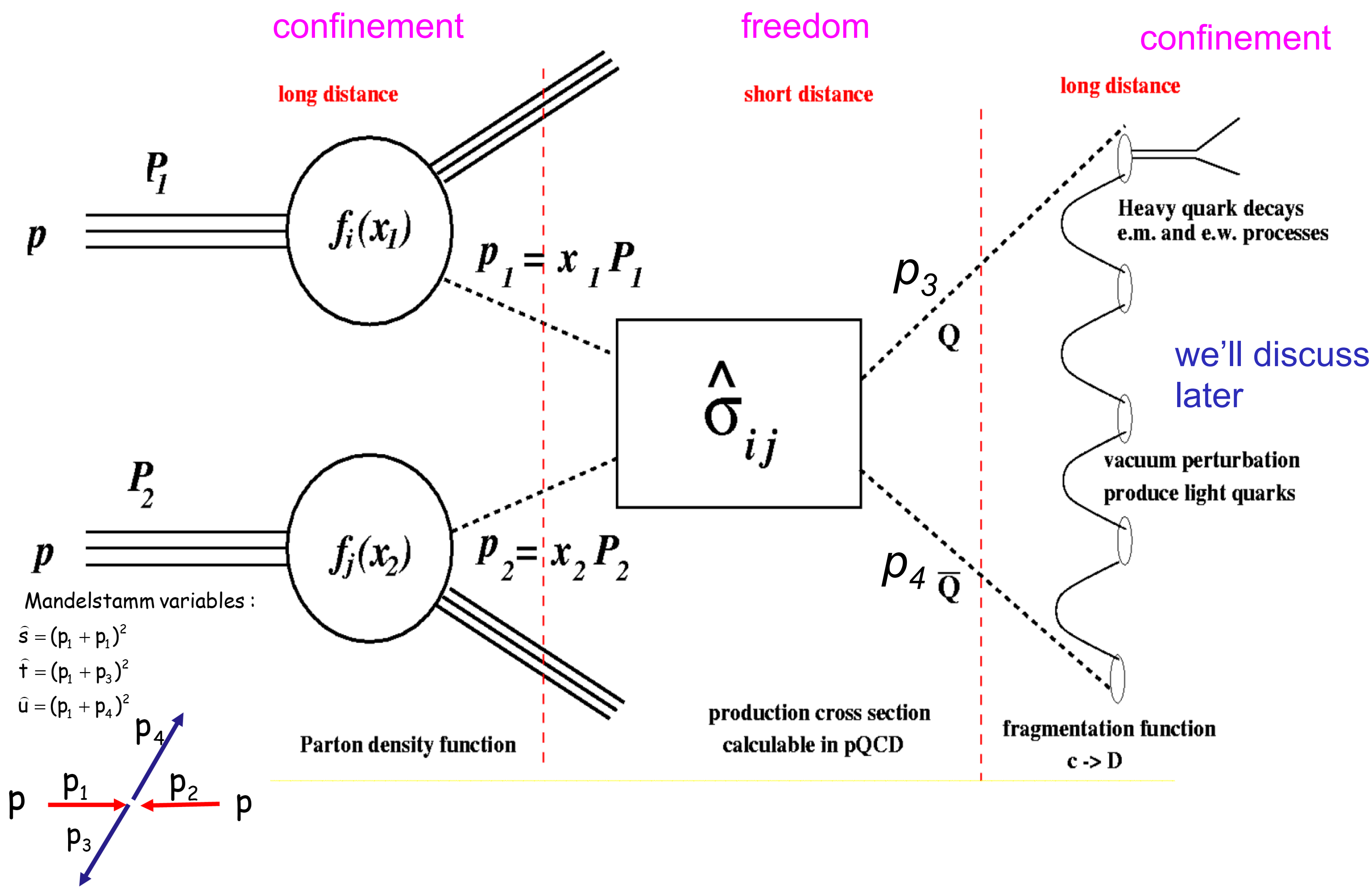
for any 2 partons, A and B

PDF luminosity times partonic cross section equals hadronic cross section

note that gluon-gluon scattering dominates at low mass and quark-quark scattering dominates at high mass

of course, specific PDF luminosity needed depends on which process is being considered

Factorization theorem



Master formula for cross section calculation

a factorization scale μ_F

a renormalization scale μ_R

$$\sigma_{2\to n} = \sum_{a,b} \int_0^1 \mathrm{d}x_a \mathrm{d}x_b \, f_{a/h_1}(x_a, \mu_F) \, f_{b/h_2}(x_b, \mu_F) \, \hat{\sigma}_{ab\to n}(\mu_F, \mu_R)$$

both PDFs and matrix elements can be at LO, NLO and NNLO

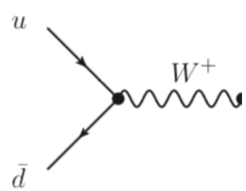
$$= \sum_{a,b} \int_0^1 \underbrace{\mathrm{d}x_a \mathrm{d}x_b}_{\text{IS phase space}} \underbrace{f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F)}_{\text{parton distribution functions}}$$

$$\times \underbrace{\frac{1}{2\hat{s}}}_{\text{incoming flux}} \times \underbrace{\int \mathrm{d}\Phi_n}_{\text{FS phase space}} \underbrace{|\mathcal{M}_{ab\to n}|^2(\Phi_n; \mu_F, \mu_R)}_{\text{amplitude squared}} .$$

We'll start by calculating the matrix element for W boson production at leading order.

W boson production at leading order

- Consider production of an on-shell W^+ boson; only 1 diagram



- The corresponding matrix element reads

CKM matrix element — weak coupling

$$\mathcal{M}_{u\bar{d}\to W^+} = -\frac{iV_{ud}\, g_W\, \delta_{ij}}{\sqrt{2}} \bar{d}_i(p_2)\gamma^\mu \frac{1-\gamma_5}{2} u_j(p_1)\epsilon_\mu^{(W)}$$

quarks have to be a color-anticolor singlet since the W boson has no color

- Summed and squared expression reads

$$\bar{\sum}|\mathcal{M}_{u\bar{d}\to W^+}|^2 = \frac{3}{9\cdot 4}\frac{|V_{ud}|^2 g_W^2}{2}\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\frac{1-\gamma_5}{2}\right]\left[-g_{\mu\nu}+\frac{Q_\mu Q_\nu}{m_W^2}\right]$$

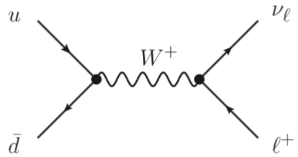
$$= \frac{|V_{ud}|^2 g_W^2}{12} Q^2 = \frac{|V_{ud}|^2 g_W^2}{12} m_W^2\,,$$

$Q=p_1+p_2=m_W{}^2$

factor of 3 comes from sum over 3 possible quark-line colors, 1/9 takes care of averaging over all possible color configurations of the quark and antiquark, and 1/4 takes care of averaging over the incoming quark spins.

...add the decay

- Now let the W^+ decay



- Matrix element can be written as

$$\gamma_{\mu L} = \gamma_\mu \frac{1-\gamma_5}{2}$$

$$\mathcal{M}_{u\bar{d}\to\nu_\ell\bar{\ell}} = \left[\bar{v}_{\bar{d}}\left(\frac{-ig_W V_{ud}}{\sqrt{2}}\gamma_{\mu L}\right)u_u\right]\left[\bar{u}_\nu\left(\frac{-ig_W}{\sqrt{2}}\gamma_{\nu L}\right)v_{\bar{\ell}}\right] \times \frac{-i}{(p_u+p_{\bar{d}})^2 - m_W^2 + i m_W \Gamma_W}\left[g^{\mu\nu} - \frac{(p_u+p_{\bar{d}})^\mu (p_u+p_{\bar{d}})^\nu}{m_W^2}\right]$$

W is no longer a final state particle, but instead a propagator

- Squaring yields

$$\overline{\sum}|\mathcal{M}_{u\bar{d}\to\ell^+\nu_\ell}|^2 = \frac{3}{9\cdot 4}\frac{|V_{ud}|^2 g_W^4}{4}\mathrm{Tr}\left[\not{p}_{\bar{d}}\gamma^\mu \not{p}_u \gamma^\rho \frac{1-\gamma_5}{2}\right]\mathrm{Tr}\left[\not{p}_{\nu_\ell}\gamma^\nu \not{p}_{\bar{\ell}}\gamma^\sigma \frac{1-\gamma_5}{2}\right] \times \frac{\left(g_{\mu\nu} - \frac{Q_\mu Q_\nu}{m_W^2}\right)\left(g_{\rho\sigma} - \frac{Q_\rho Q_\sigma}{m_W^2}\right)}{(Q^2-m_W^2)^2 + m_W^2\Gamma_W^2} = \frac{|V_{ud}|^2 g_W^4}{12}\frac{\hat{t}^2}{(Q^2-m_W^2)^2 + m_W^2\Gamma_W^2},$$

average over initial quarks' spins and colors, and sum over lepton spins implicit

- Define Mandelstam variables

$$\hat{s} = Q^2 = (p_u + p_{\bar{d}})^2 \quad \text{and} \quad \hat{t} = (p_u - p_{\bar{\ell}})^2$$

Phase space

- Write phase-space integral over final state particles as

$$\int d\Phi_n = \int \frac{d^4 p_\ell}{(2\pi)^4}(2\pi)\delta(p_\ell^2)\frac{d^4 p_\nu}{(2\pi)^4}(2\pi)\delta(p_\nu^2)(2\pi)^4\delta^4(p_u+p_d-p_\ell-p_\nu)$$

→ conservation of momentum

$$= \frac{1}{32\pi^2}\int d^2\Omega_\ell^*\,,$$

→ solid angle of outgoing lepton in rest frame of collision

- Then can re-write

$$\hat{\sigma}^{(LO)} = \frac{1}{2\hat{s}}\int \frac{d^2\Omega_\ell^*}{32\pi^2}|\mathcal{M}|^2_{u\bar{d}\to\nu_\ell\bar{\ell}} = \frac{g_W^4\,|V_{ud}|^2}{12\cdot 2\hat{s}}\int_{-1}^{1}\frac{2\pi d\cos\theta^*}{4\cdot 32\pi^2}\frac{\hat{s}^2(1-\cos\theta^*)^2}{\left[(\hat{s}-m_W^2)^2+m_W^2\Gamma_W^2\right]}$$

$$= \frac{g_W^4\,|V_{ud}|^2}{576\pi}\frac{\hat{s}}{(\hat{s}-m_W^2)^2+m_W^2\Gamma_W^2}\,,$$

width of the W boson

Breit-Wigner for W propagator

- And for the final cross section

$$\sigma^{(LO)}_{h_1h_2\to\nu_\ell\bar{\ell}} = \frac{g_W^4\,|V_{ud}|^2}{576\pi}\int dy_W\,d\hat{s}\left[\frac{1}{\left[\hat{s}-m_W^2\right]^2+m_W^2\Gamma_W^2}\right.$$

$$\left.\times\sum_{u,\bar{d}} x_u f_{u/h_1}(x_u,\,\mu_F)\,x_{\bar{d}} f_{\bar{d}/h_2}(x_{\bar{d}},\,\mu_F)\right]$$

some kinematic tricks

$$dx_u dx_{\bar{d}} = \frac{d\hat{s}}{s}\,dy_W$$

$$\hat{s} = x_u x_{\bar{d}} s$$

$$y_{\text{c.m.}} = \frac{1}{2}\log\frac{x_u}{x_{\bar{d}}}\,.$$

Narrow width approximation

- It is often useful to use the narrow width approximation to simplify the calculation, where the propagator (Breit-Wigner) of mass M_X and width Γ_x is replaced by

$$\frac{d\hat{s}}{(\hat{s}-M_X^2)^2+M_X^2\Gamma_X^2} \longrightarrow \frac{\pi}{M_X\Gamma_X}\, d\hat{s}\,\delta(\hat{s}-M_X^2)$$

- The cross section then can be written as

$$\sigma^{(\mathrm{LO})}_{h_1 h_2\to\nu_\ell\bar{\ell}} = \frac{g_W^4\,|V_{ud}|^2}{576 s}\,\frac{m_W}{\Gamma_W}\int_{-y_{\max}}^{y_{\max}} dy_W \sum_{u,\bar{d}} f_{u/h_1}\!\left(\frac{m_W e^{y_W}}{\sqrt{s}},\,\mu_F\right) f_{\bar{d}/h_2}\!\left(\frac{m_W e^{-y_W}}{\sqrt{s}},\,\mu_F\right)$$

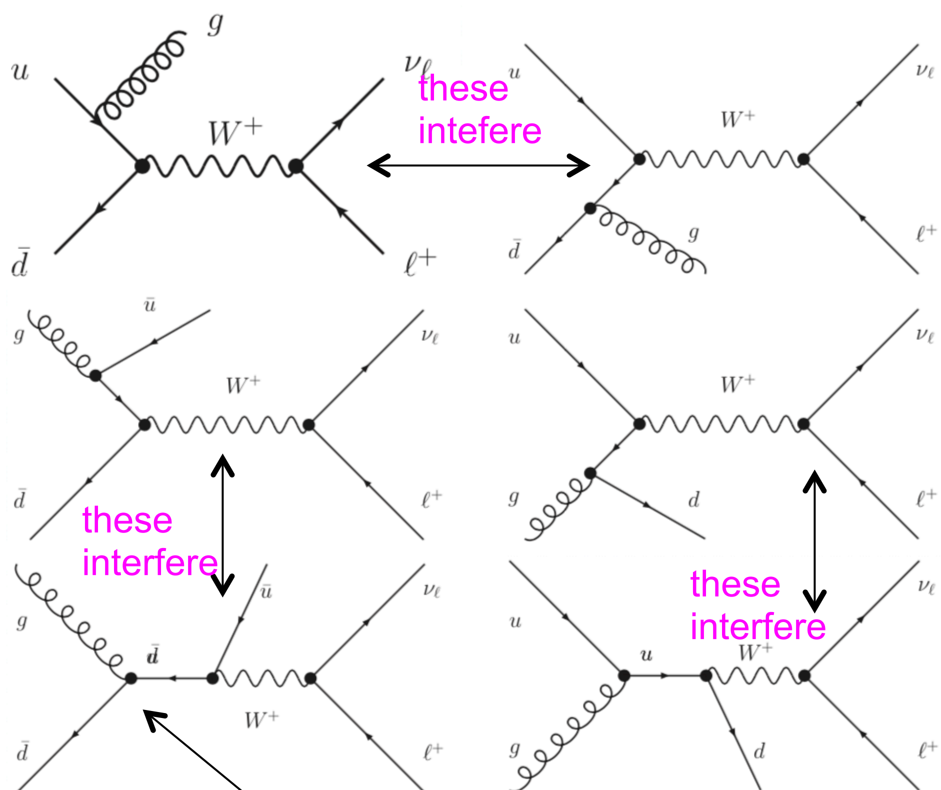
where the W rapidity y_W is constrained by $x_u x_{\bar{d}} s = m_W^2$ and therefore

$$|y_W| \le y_{\max} = \frac{1}{2}\log\frac{s}{m_W^2}\,.$$

- Note this ignores correlations between the initial state particles and the final state particles (spins for example)

Consider the emission of a gluon (or quark from a gq initial state)

- Three sets of diagrams,each with two interfering amplitudes
  - identical initial and final states



$$\mathcal{M}_{u\bar{d}\to gW^+} = \frac{i g_s g_W V_{ud}}{\sqrt{2}}\,\bar{v}_{d,i}\left[\gamma_\nu T^a_{ij}\frac{\not p_{\bar{d}}-\not p_g}{(p_{\bar{d}}-p_g)^2}\gamma_\mu L + \gamma_\mu L\frac{\not p_u-\not p_g}{(p_u-p_g)^2}\gamma_\nu T^a_{ij}\right] u_{u,j}\epsilon^\mu_W \epsilon^{\nu,a}_g$$

$$\mathcal{M}_{ug\to dW^+} = \frac{i g_s g_W V_{ud}}{\sqrt{2}}\,\bar{u}_{d,i}\left[\gamma_\nu T^a_{ij}\frac{\not p_g-\not p_d}{(p_g-p_d)^2}\gamma_\mu L + \gamma_\mu L\frac{\not p_u+\not p_g}{(p_u+p_g)^2}\gamma_\nu T^a_{ij}\right] u_{u,j}\epsilon^\mu_W \epsilon^{*\nu,a}_g$$

$$\mathcal{M}_{\bar{d}g\to \bar{u}W^+} = \frac{i g_s g_W V_{ud}}{\sqrt{2}}\,\bar{v}_{d,i}\left[\gamma_\nu T^a_{ij}\frac{\not p_g+\not p_{\bar{d}}}{(p_g+p_{\bar{d}})^2}\gamma_\mu L + \gamma_\mu L\frac{\not p_g-\not p_{\bar{u}}}{(p_g-p_{\bar{u}})^2}\gamma_\nu T^a_{ij}\right] v_{u,j}\epsilon^\mu_W \epsilon^{*\nu,a}_g$$

color matrix appearing in the quark-quark-gluon vertex

note potentially divergent terms

$$\frac{1}{(p_q-p_g)^2} = -\frac{1}{2E_qE_g(1-\cos\theta)}$$

when parton gets soft, or angle approaches 0

Matrix elements squared

- Square and average/sum over initial/final states polarizations and colors, and performing some color algebra, get

$$|\mathcal{M}|^2_{u\bar{d}\to gW^+} = \frac{4\pi\alpha_s C_F\, g_W^2 |V_{ud}|^2}{12} \, \frac{\hat{t}^2 + \hat{u}^2 + 2m_W^2 \hat{s}}{\hat{t}\hat{u}}$$

and

$$|\mathcal{M}|^2_{ug\to dW^+} = |\mathcal{M}|^2_{\bar{d}g\to \bar{u}W^+} = \frac{4\pi\alpha_s T_R\, g_W^2 |V_{ud}|^2}{12} \, \frac{\hat{s}^2 + \hat{u}^2 + 2m_W^2 \hat{t}}{-\hat{s}\hat{u}} .$$

In all cases **Mandelstam variables** have been used, namely

$$\begin{aligned} \hat{s} &= (p_a + p_b)^2 = (p_1 + p_2)^2 , \\ \hat{t} &= (p_a - p_1)^2 = (p_b - p_2)^2 , \\ \hat{u} &= (p_a - p_2)^2 = (p_b - p_1)^2 , \end{aligned}$$

$$\hat{s} + \hat{t} + \hat{u} = m_a^2 + m_b^2 + m_1^2 + m_2^2 .$$

- Closer inspection reveals that the squared matrix elements can be written as the leading order matrix element squared (for W production) times a QCD emission term, consisting of the strong coupling and a color factor times an expression representing the kinematics of the extra emission

$$\begin{aligned} |\mathcal{M}|^2_{u\bar{d}\to gW^+} &= \frac{|\mathcal{M}^{(\mathrm{LO})}|^2_{u\bar{d}\to W^+}}{m_W^2} \cdot (4\pi\alpha_s C_F) \, \frac{\hat{t}^2 + \hat{u}^2 + 2m_W^2 \hat{s}}{\hat{t}\hat{u}} \\ |\mathcal{M}|^2_{ug\to dW^+} = |\mathcal{M}|^2_{\bar{d}g\to \bar{u}W^+} &= \frac{|\mathcal{M}^{(\mathrm{LO})}|^2_{u\bar{d}\to W^+}}{m_W^2} \cdot (4\pi\alpha_s T_R) \, \frac{\hat{s}^2 + \hat{u}^2 + 2m_W^2 \hat{t}}{-\hat{s}\hat{u}} \end{aligned}$$

note the divergence when t-hat or u-hat goes to zero

Modern life

- Note that this procedure works for simple processes, 2->n, where n is small (n=1 for W production, n=2 for W+j), but the number of Feynman diagrams increases (more than) factorially with n
- Squaring the amplitudes, taking traces, is just too complex a process for large n
- In modern techniques, alas beyond the scope of these lectures, the focus is on evaluating individual amplitudes as a function of their internal and external degrees of freedom
  - helicity amplitude method: any Feynman amplitude (represented by propagators and vertices for the internal lines and spinors and polarization vectors for the external particles) is translated into a complex number dependent on external helicities and momenta

- Every amplitude becomes just a complex number
- Summation and squaring is then a (more) straightforward exercise

Let's start over, in a somewhat more pedagogical way

- Consider Drell-Yan production
  - write cross section as

$$\sigma_{AB} = \int \mathrm{d}x_a \mathrm{d}x_b \; f_{a/A}(x_a) f_{b/B}(x_b) \; \hat{\sigma}_{ab\to X}$$

  - where $X=l^+l^-$
  - note we're back to the parton model, i.e. no QCD corrections
- Potential problems appeared to arise from when perturbative corrections from real and virtual gluon emissions were calculated
  - but these logarithms were the same as those in structure function calculations and thus can be absorbed, via DGLAP equations in definition of parton distributions, giving rise to logarithmic violations of scaling
  - can now write the cross section as

$$\sigma_{AB} = \int \mathrm{d}x_a \mathrm{d}x_b \; f_{a/A}(x_a, Q^2) f_{b/B}(x_b, Q^2) \; \hat{\sigma}_{ab\to X}$$

where x_a is the momentum fraction of parton a in hadron A, and x_b the momentum fraction of parton b in hadron B, and Q is a scale that measures the hardness of the interaction

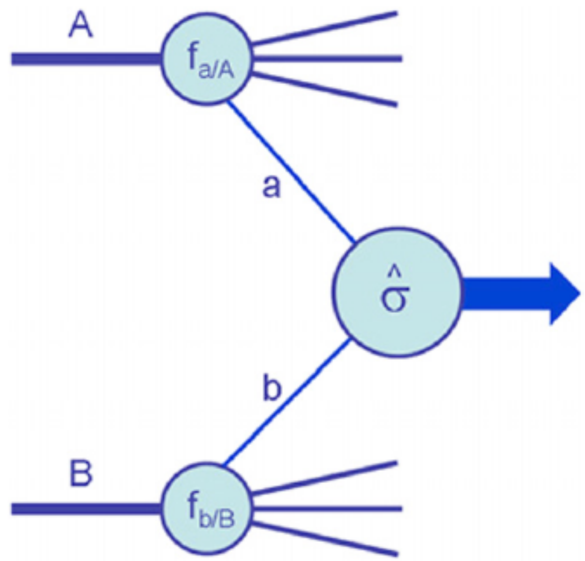


Figure 1. Diagrammatic structure of a generic hard-scattering process.

…but

- Key point is that all logarithms appearing in Drell-Yan corrections can be factored into renormalized (universal) parton distributions
  - factorization
- But finite corrections left behind after the logarithms are not universal and have to be calculated separately for each process, giving rise to order α_s^n perturbative corrections
- So now we can write the cross section as

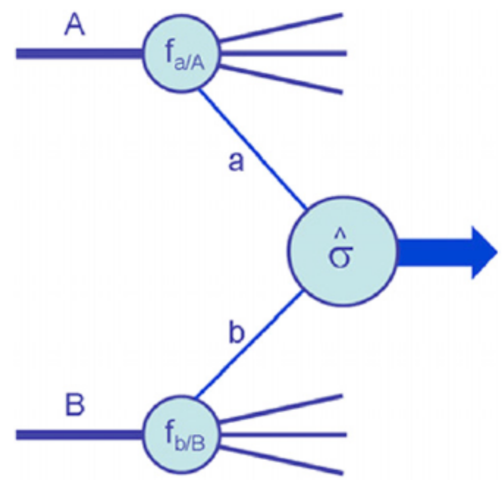


Figure 1. Diagrammatic structure of a generic hard-scattering process.

also depends on μ_R and μ_F, so as to cancel scale dependence in PDF's and α_s, to this order

$$\sigma_{AB} = \int dx_a dx_b \; f_{a/A}(x_a, \mu_F^2) \; f_{b/B}(x_b, \mu_F^2) \; \times \; [\, \hat{\sigma}_0 \, + \, \alpha_S(\mu_R^2) \, \hat{\sigma}_1 \, + \, \cdots \,]_{ab \to X} .$$

- where μ_F is the factorization scale (separates long and short-distance physics) and μ_R is the renormalization scale for α_s
- choose μ_R=μ_F~Q (say,$m_{W/Z}$)

An all-orders cross section has no dependence on μ_F and μ_R; a residual dependence remains (to order α_s^{n+1}) for a finite order (α_s^n) calculation (see later discussion as well)

Kinematics

- Double differential cross section for production of a Drell-Yan pair of mass M and rapidity y is given by

$$\frac{d\sigma}{dM^2 dy} = \frac{\hat{\sigma}_0}{Ns}\left[\sum_k Q_k^2(q_k(x_1, M^2)\bar{q}_k(x_2, M^2) + [1 \leftrightarrow 2])\right]$$

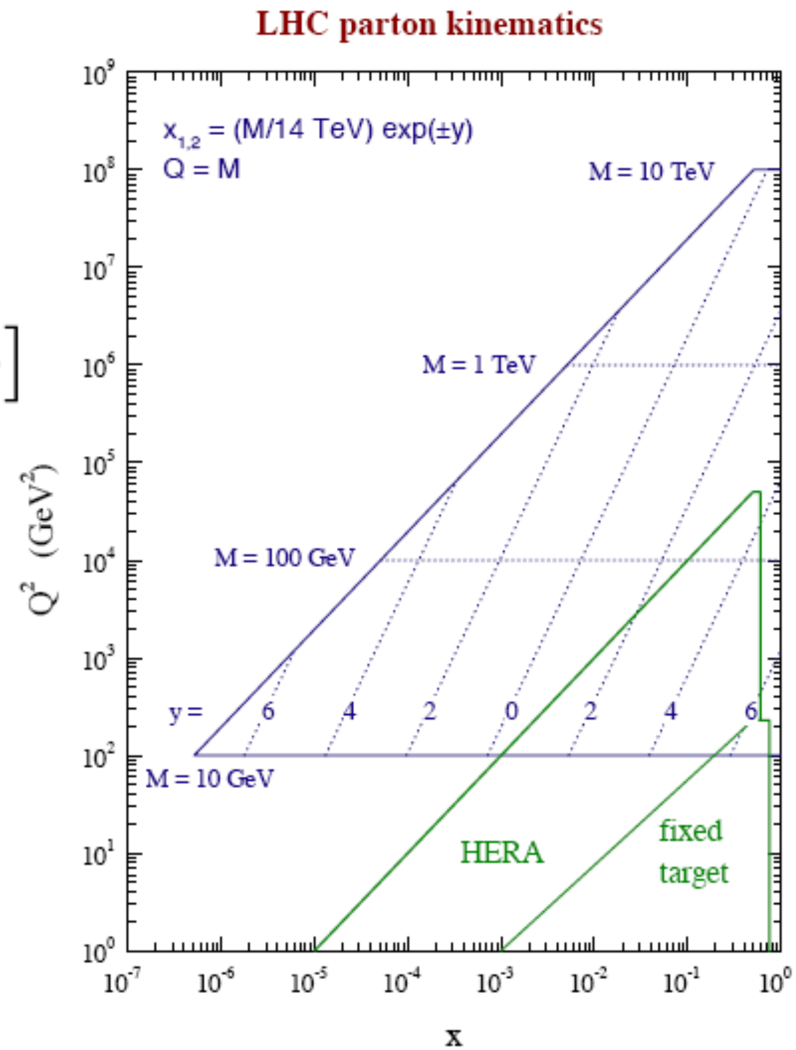
  - where

$$\hat{\sigma}_o = \frac{4\pi\alpha^2}{3M^2}$$

  - and

$$x_1 = \frac{M}{\sqrt{s}} e^y , \qquad x_2 = \frac{M}{\sqrt{s}} e^{-y}.$$

- Thus, different values of M and y probe different values of x and Q^2



W/Z production

- Cross sections for on-shell W/Z production (in narrow width limit) given by

$$\hat{\sigma}^{q\bar{q}'\to W} = \frac{\pi}{3}\sqrt{2}G_F M_W^2 |V_{qq'}|^2 \delta(\hat{s} - M_W^2),$$
$$\hat{\sigma}^{q\bar{q}\to Z} = \frac{\pi}{3}\sqrt{2}G_F M_Z^2 (v_q^2 + a_q^2)\delta(\hat{s} - M_Z^2),$$

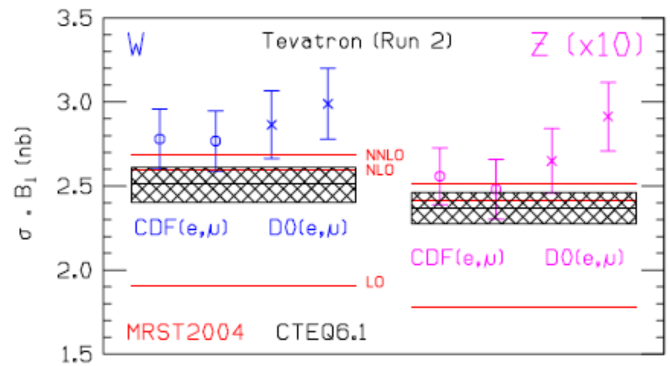


Figure 4. Predictions for the *W* and *Z* total cross sections at the Tevatron and LHC, using MRST2004 [10] and CTEQ6.1 pdfs [11], compared with recent data from CDF and D0. The MRST predictions are shown at LO, NLO and NNLO. The CTEQ6.1 NLO predictions and the accompanying pdf error bands are also shown.

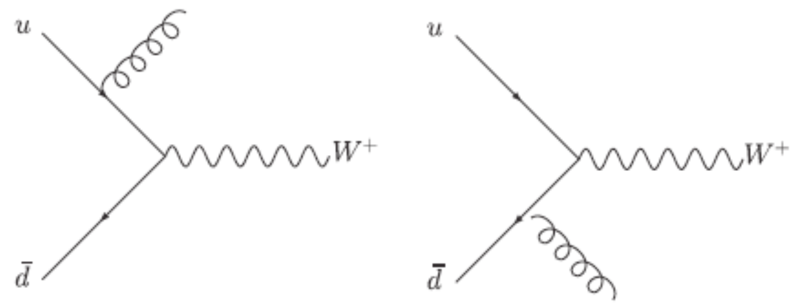
- Where $V_{qq'}$ is appropriate CKM matrix element and v_q and a_q are the vector and axial coupling of the Z to quarks
- Note that at LO, there is no α_s dependence; EW vertex only
- Quark and anti-quark have to be color-anticolor pair
  - factor of 3 suppression
- NLO contribution to the cross section is proportional to α_s; NNLO to α_s^2…

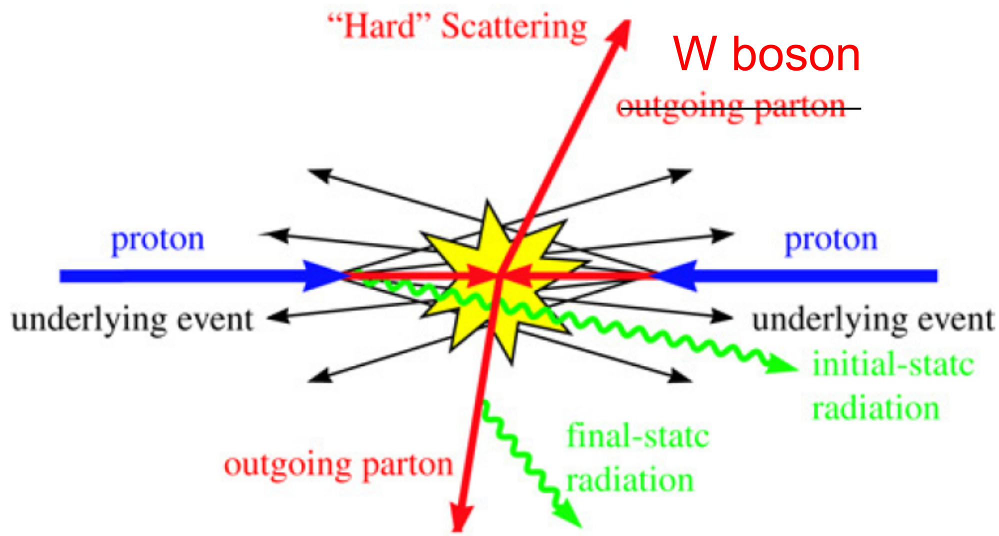
LO->NLO is a fairly large (+) correction

NLO->NNLO is a fairly small (+) correction

W/Z p_T distributions

- Most W/Z produced at low p_T, but can be produced at non-zero p_T due to diagrams such as shown on the right; note the presence of the QCD vertex, where the gluon couples (so one order higher)
- So an example of a 2->2 process





W/Z p_T distributions

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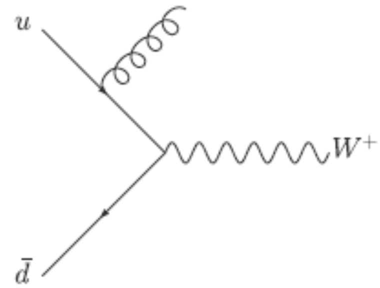
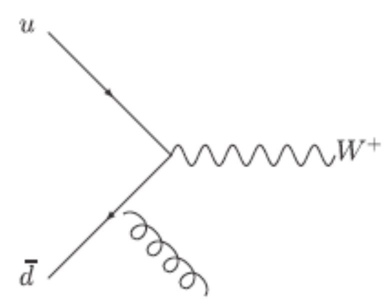




Mandelstam variables

$$\sum|\mathcal{M}^{q\bar{q}'\to Wg}|^2 = \pi\alpha_S\sqrt{2}G_F M_W^2|V_{qq'}|^2\,\frac{8}{9}\,\frac{\hat{t}^2+\hat{u}^2+2M_W^2\hat{s}}{\hat{t}\hat{u}},$$

$$\sum|\mathcal{M}^{gq\to Wq'}|^2 = \pi\alpha_S\sqrt{2}G_F M_W^2|V_{qq'}|^2\,\frac{1}{3}\,\frac{\hat{s}^2+\hat{u}^2+2\hat{t}M_W^2}{-\hat{s}\hat{u}},$$

If this were photon production, and not W, then this last term would not be present

- Sum is over colors and spins in initial state, averaged over same in final state
- Transverse momentum distribution is obtained by convoluting these matrix elements with PDF's in usual way

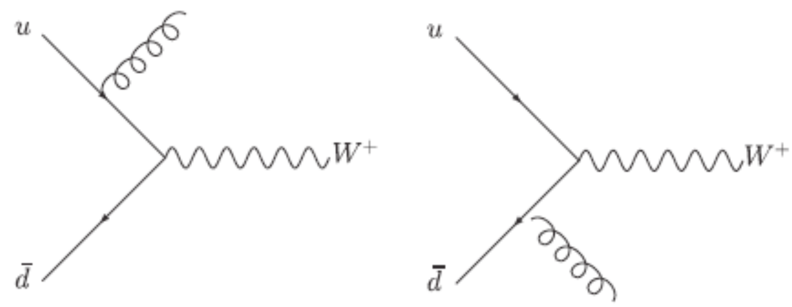
Note that 2->2 matrix elements are singular when final state partons are soft or collinear with initial state partons (soft and collinear->double logarithms)

Related to poles at $\hat{t}=0$ and $\hat{u}=0$

But singularities from real and virtual emissions cancel when all contributions are included, so NLO is finite

Aside

- Can we say which quark the gluon is emitted from?
- No, that's a classical picture (most often adopted in Monte Carlos), but doesn't fit into our quantum mechanical picture
- In a similar way, if we have a diagram with a gluon that can be emitted from either the initial or final state, we can't say from which it was emitted
  - the two diagrams interfere with each other



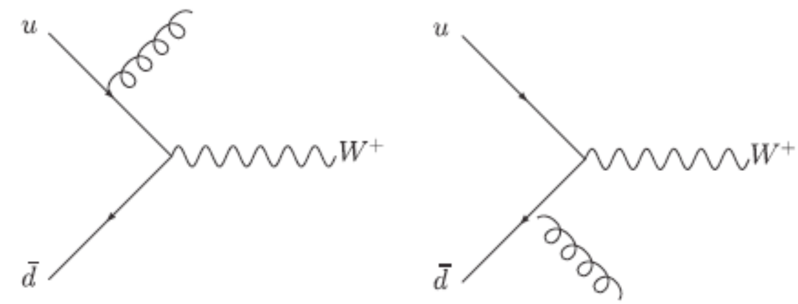
$$\sum |\mathcal{M}^{q\bar{q}'\to Wg}|^2 = \pi \alpha_S \sqrt{2} G_F M_W^2 |V_{qq'}|^2 \, \frac{8}{9} \, \frac{\hat{t}^2 + \hat{u}^2 + 2M_W^2 \hat{s}}{\hat{t}\hat{u}} \, ,$$

$$\sum |\mathcal{M}^{gq\to Wq'}|^2 = \pi \alpha_S \sqrt{2} G_F M_W^2 |V_{qq'}|^2 \, \frac{1}{3} \, \frac{\hat{s}^2 + \hat{u}^2 + 2\hat{t} M_W^2}{-\hat{s}\hat{u}} \, ,$$

W/Z p_T distributions

- Back to the 2->2 subprocess

$$|\mathcal{M}^{u\bar{d}\to W+g}|^2 \sim \left(\frac{\hat{t}^2 + \hat{u}^2 + 2Q^2\hat{s}}{\hat{t}\hat{u}}\right)$$



  - where Q^2 is the virtuality of the W boson

it's pretty clear that $Q \sim m_W$ is a good choice as long as the gluon is reasonably soft

- Convolute with PDFs

$$\sigma = \int dx_1 dx_2 f_u(x_1, Q^2) f_{\bar{d}}(x_2, Q^2) \frac{|\mathcal{M}|^2}{32\pi^2\hat{s}} \frac{d^3 p_W}{E_W} \frac{d^3 p_g}{E_g} \delta(p_u + p_{\bar{d}} - p_g - p_W)$$

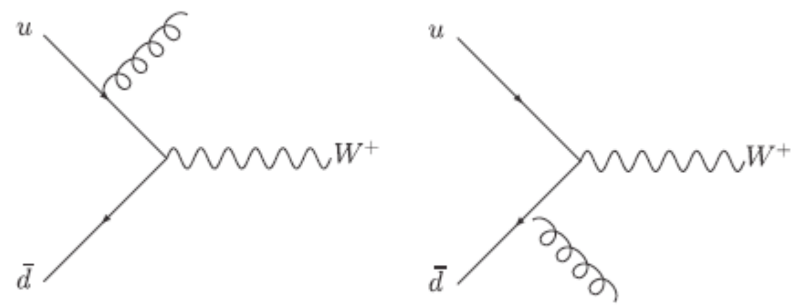
phase space for W and gluon

momentum conservation

W/Z p_T distributions

- Transform into differential cross section

$$\frac{d\sigma}{dQ^2 dy\, dp_T^2} \sim \frac{1}{s}\int dy_g\, f_u(x_1,Q^2) f_{\bar{d}}(x_2,Q^2)\frac{|\mathcal{M}|^2}{\hat{s}}$$



  - where we have one integral left over, the gluon rapidity
- Note that $p_T^2 = \frac{\hat{t}\hat{u}}{\hat{s}}$
  - thus, leading divergence can be written as $1/p_T^2$
- In this limit, behavior of cross section becomes

$$\frac{d\sigma}{dQ^2 dy\, dp_T^2} \sim \frac{2}{s}\frac{1}{p_T^2}\int dy_g\, f_u(x_1,Q^2) f_{\bar{d}}(x_2,Q^2) + (\text{sub-leading in } p_T^2)$$

- As p_T of W becomes small, limits of y_g integration are given by +/- $\log(s^{1/2}/p_T)$
- The result then is

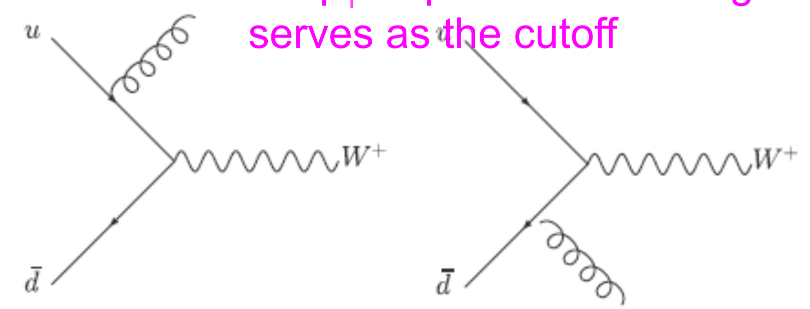
$$\frac{d\sigma}{dQ^2 dy\, dp_T^2} \sim \frac{\log(s/p_T^2)}{p_T^2}$$

…diverges unless we apply a p_T^{min} cut; so we end up with a distribution that depends not only on α_s but on α_s times a logarithm: universal theme

Rapidity distributions

- Now look at rapidity distributions for jet for two different choices of p_T^{min}
- Top diagrams imply that gluon is radiated off initial state parton at an early time (ISR)
- With collinear pole, this would imply that these gluons would be emitted primarily at forward rapidities
- But the distributions look central
- The reason is that we are binning in p_T and not in energy, and the most effective place to convert from E to p_T is at central rapidities
- Suppose I re-draw the Feynman diagrams as shown to the right
  - is there a difference from what is shown at the top of the page?
  - hint: no

the p_T requirement of the gluon serves as the cutoff



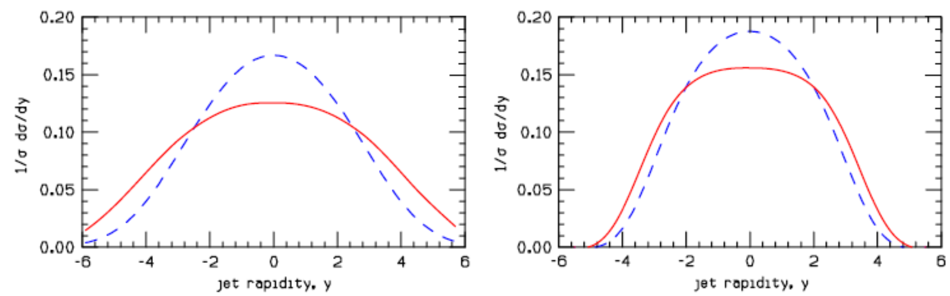
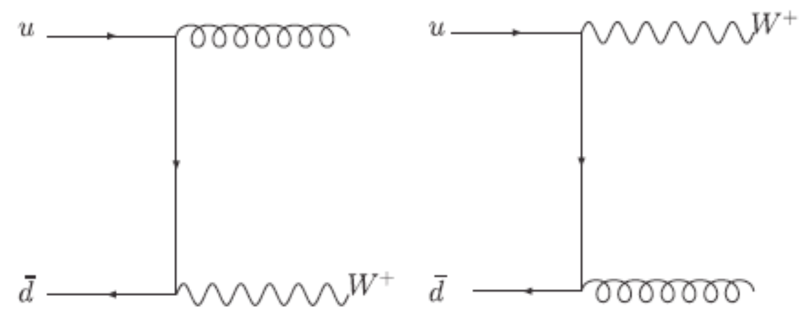
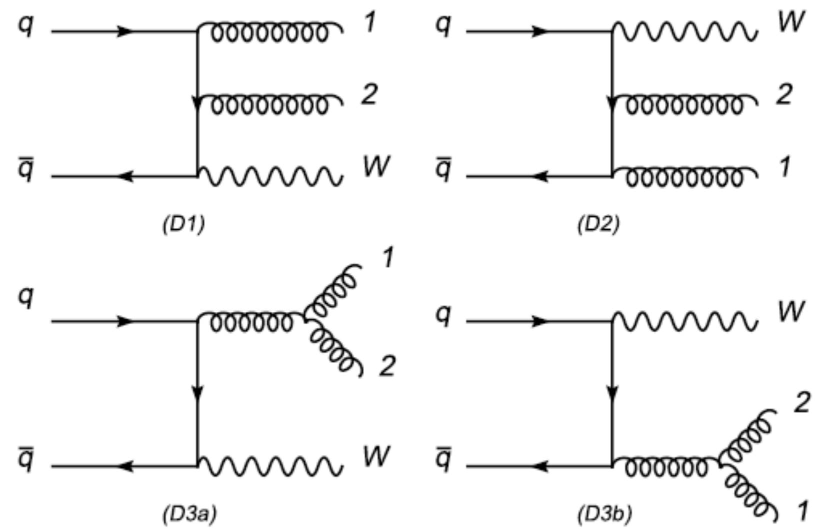


Figure 9. The rapidity distribution of the final-state parton found in a lowest-order calculation of the $W+1$ jet cross section at the LHC. The parton is required to have a p_T larger than 2 GeV (left) or 50 GeV(right). Contributions from $q\bar{q}$ annihilation (solid red line) and the qg process (dashed blue line) are shown separately.



Now on to W + 2 jets

- For sake of simplicity, consider Wgg
- **Let p_1 be soft**
- Then can write



$$\mathcal{M}^{q\bar{q}\to Wgg} = t^A t^B (D_2 + D_3) + t^B t^A (D_1 - D_3).$$

- where t^A and t^B are color labels of p_1 and p_2

so the kinematic structures obtained from the Feynman diagrams are collected in the function D_1,D_2 and D_3, which are called color-ordered amplitudes

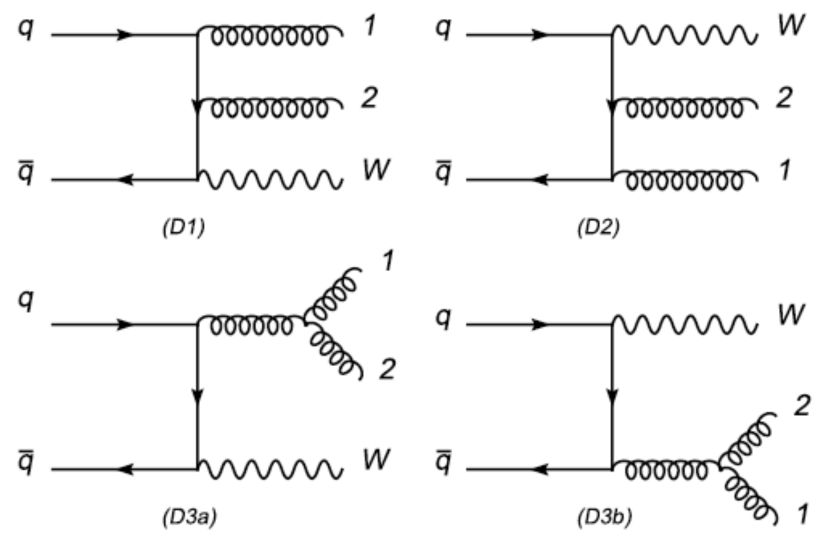
- Square the matrix amplitude to get

using $tr(t^At^Bt^Bt^A)=NC_F^2$ and $tr(t^At^Bt^At^B)=-C_F/2$

$$|\mathcal{M}^{q\bar{q}\to Wgg}|^2 = NC_F^2\left[|D_2 + D_3|^2 + |D_1 - D_3|^2\right] - C_F \,\mathrm{Re}\left[(D_2 + D_3)(D_1 - D_3)^\star\right]$$
$$= \frac{C_F N^2}{2}\left[|D_2 + D_3|^2 + |D_1 - D_3|^2 - \frac{1}{N^2}|D_1 + D_2|^2\right].$$

W + 2 jets

- Since p_1 is soft, can write D's (color-ordered amplitudes) as product of an eikonal term and the matrix elements containing only 1 gluon



$$D_2 + D_3 \longrightarrow \epsilon_\mu \left(\frac{q^\mu}{p_1 \cdot q} - \frac{p_2^\mu}{p_1 \cdot p_2} \right) \mathcal{M}_{q\bar{q}\to Wg},$$

$$D_1 - D_3 \longrightarrow \epsilon_\mu \left(\frac{p_2^\mu}{p_1 \cdot p_2} - \frac{\bar{q}^\mu}{p_1 \cdot \bar{q}} \right) \mathcal{M}_{q\bar{q}\to Wg},$$

  - where ε_μ is the polarization vector for gluon p_1

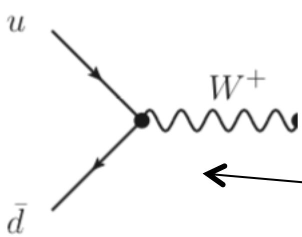
- Summing over gluon polarizations, we get

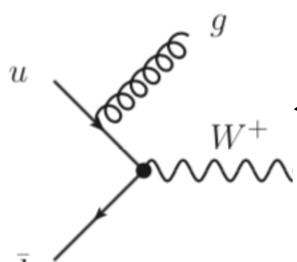
$$|\mathcal{M}^{q\bar{q}\to Wgg}|^2 \xrightarrow{\text{soft}} \frac{C_F N^2}{2} \left[[q\ p_2] + [p_2\ \bar{q}] - \frac{1}{N^2} [q\ \bar{q}] \right] \mathcal{M}^{q\bar{q}\to Wg}$$

  - where

$$\frac{a.b}{p_1.a\ p_1.b} \equiv [a\ b],$$

Observables and orders





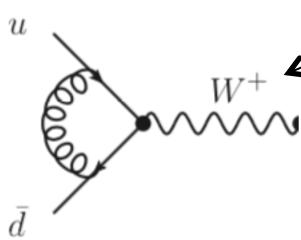
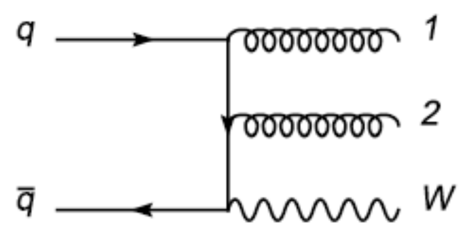


Table 3.1 The order at which various observables related to W production are computed, as a function of the overall power of the strong coupling in the theoretical calculation.

strong coupling order	σ_{tot}, $d\sigma/dy$	$d\sigma/dp_T(W)$	$d\sigma/d\phi_{jj}$
α_s^0	LO	-	-
α_s^1	NLO	LO	-
α_s^2	NNLO	NLO	LO
α_s^3	N^{3}LO	NNLO	NLO



suppose I want to know $d\sigma/d\phi_{jj}$ to NLO; then I would need the 1 loop correction to the diagram on the left, and the W+3 jets real correction; both of order α_s^3

Power series

W production

strong coupling order	$\sigma_{\rm tot}$, $d\sigma/dy$
α_s^0	LO
α_s^1	NLO
α_s^2	NNLO
α_s^3	N^{3}LO

Cross section is a power series in α_s

$$d\sigma = \sum_{n=n_0}^{\infty} \alpha_s^n f_n(\ldots)$$

For perturbation theory to work, need $\alpha_s \equiv g_s^2/4\pi << 1$

Each vertex has g_s in amplitude->g_s^2 is proportional to α_s in cross section

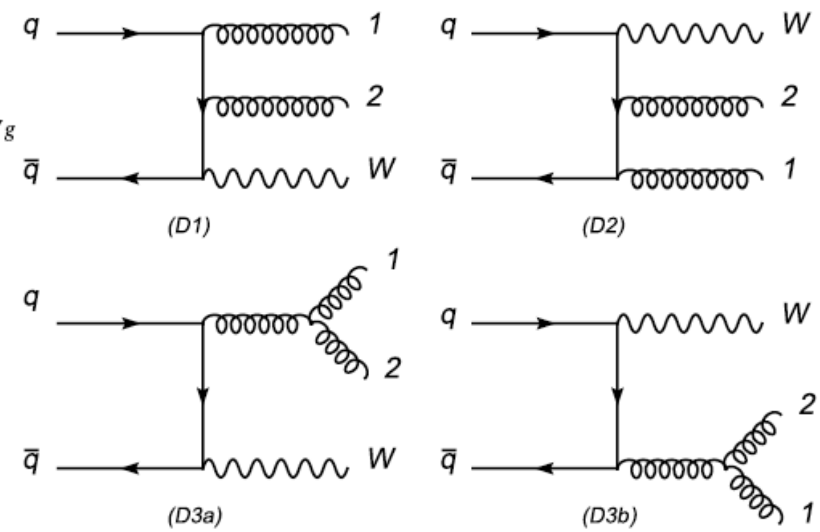
Higher orders->more vertices->more diagrams (n!)->f_n becomes more difficult to calculate

But if α_s<<1, can truncate series (LO, NLO, NNLO,…)

For W production, NNLO corrections are reasonably small; not true for Higgs production, for example

Color flow

$$|\mathcal{M}^{q\bar{q}\to Wgg}|^2 \xrightarrow{\text{soft}} \frac{C_F N^2}{2}\left[[q\ p_2]+[p_2\ \bar{q}]-\frac{1}{N^2}[q\ \bar{q}]\right]\mathcal{M}^{q\bar{q}\to Wg}$$



- The leading term (in number of colors) contains singularities along two lines of color flow-one connecting gluon p_2 to the quark and the other connecting it to the anti-quark
  - sub-leading term has singularities along the line connecting the quark and anti-quark
- It is these lines of color that indicate preferred direction for emission of additional gluons
  - needed by programs like Pythia/Herwig for example
  - sub-leading terms don't correspond to any unique color flow

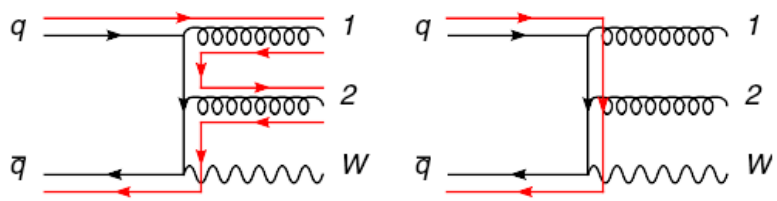


Figure 12. Two examples of colour flow in a W + 2 jet event, shown in red. In the left-hand diagram, a leading colour flow is shown. The right-hand diagram depicts the sub-leading colour flow resulting from interference.

…and thus can't be fed directly into the parton shower Monte Carlo programs

Eikonal factors

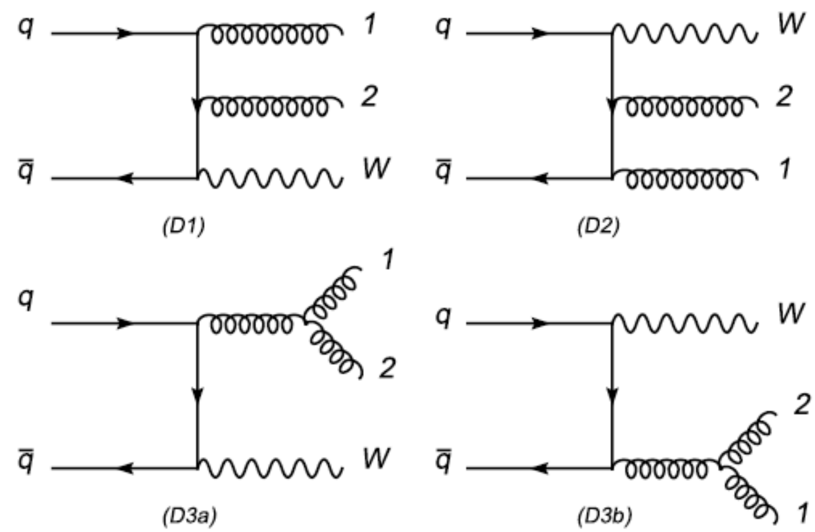
- Re-write

$$\frac{a.b}{p_1.a\; p_1.b} \equiv [a\; b],$$

- As

$$[a\; b]\, \mathrm{d}PS_{\text{gluon}} = \frac{1}{E^2}\frac{1}{1-\cos\theta_a} E\mathrm{d}E\; \mathrm{d}\cos\theta_a$$

- It is clear that the cross section diverges either as $\cos\theta_a$->1 (gluon is collinear to parton a) or as E->0
  - similar for parton b
- Each divergence is logarithmic and regulating the divergence by providing a fixed cutoff (in angle or energy) will produce a single logarithm from collinear configurations and another from soft ones
  - double logs



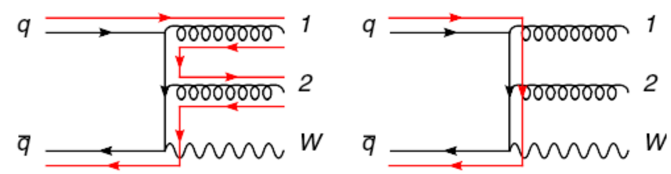
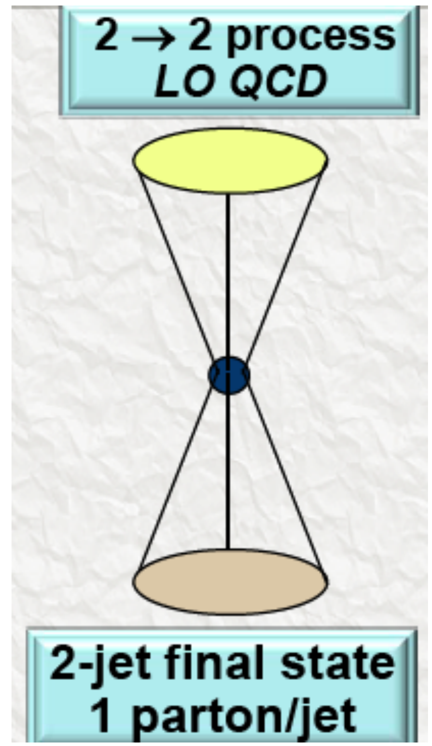
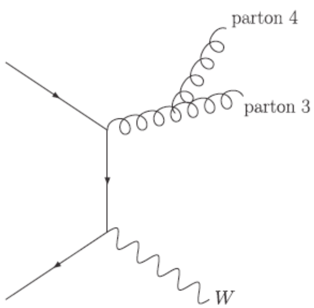


Figure 12. Two examples of colour flow in a $W+2$ jet event, shown in red. In the left-hand diagram, a leading colour flow is shown. The right-hand diagram depicts the sub-leading colour flow resulting from interference.

Brief interlude: jet definitions and algorithms

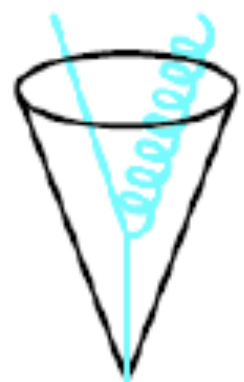
- At (fixed) LO, 1 parton = 1 jet
  - why not more than 1? I have to put a ΔR cut on the separation between two partons; otherwise, there's a collinear divergence. LO parton shower programs effectively put in such a cutoff





$$\log\left(\frac{1}{\Delta R_{34}}\right)$$

- But at NLO, I have to deal with more than 1 parton in a jet, and so now I have to talk about how to cluster those partons
  - i.e. jet algorithms



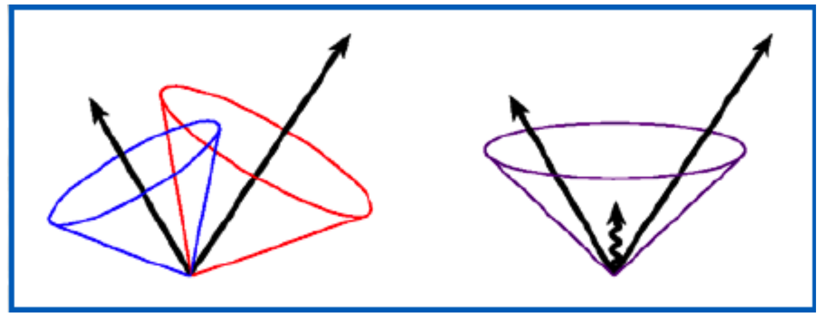
Jet algorithms at NLO

- At NLO (NNLO), there can be two(three) partons in a jet, life becomes more interesting and we have to start talking about jet algorithms to define jets
  - we will see that the addition of the extra parton(s) and virtual terms will cancel the divergence mentioned on the previous slide
- A jet algorithm is based on some measure of localization of the expected collinear spray of particles
- Start with an inclusive list of partons (fixed order), particles (PS shower Monte Carlos, and data)
- End with lists of same for each jet
- …and a list of particles… not in any jet; for example, remnants of the initial hadrons
- Two broad classes of jet algorithms
  - cluster according to proximity in space: cone algorithms
  - cluster according to proximity in momenta: k_T algorithms

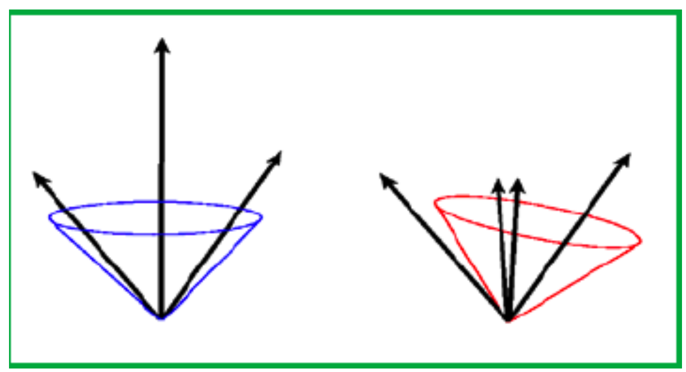
What do I want out of a jet algorithm?

- It should be fully specified, including defining in detail any pre-clustering, merging and splitting issues
- It should be simple to implement in an experimental analysis, and should be independent of the structure of the detector
- It should be boost-invariant
- It should be simple to implement in a theoretical calculation
  - it should be defined at any order in perturbation theory
  - it should yield a finite cross section at any order in perturbation theory
  - it should yield a cross section that is relatively insensitive to hadronization effects

- It should be IR safe, i.e. adding a soft gluon should not change the results of the jet clustering

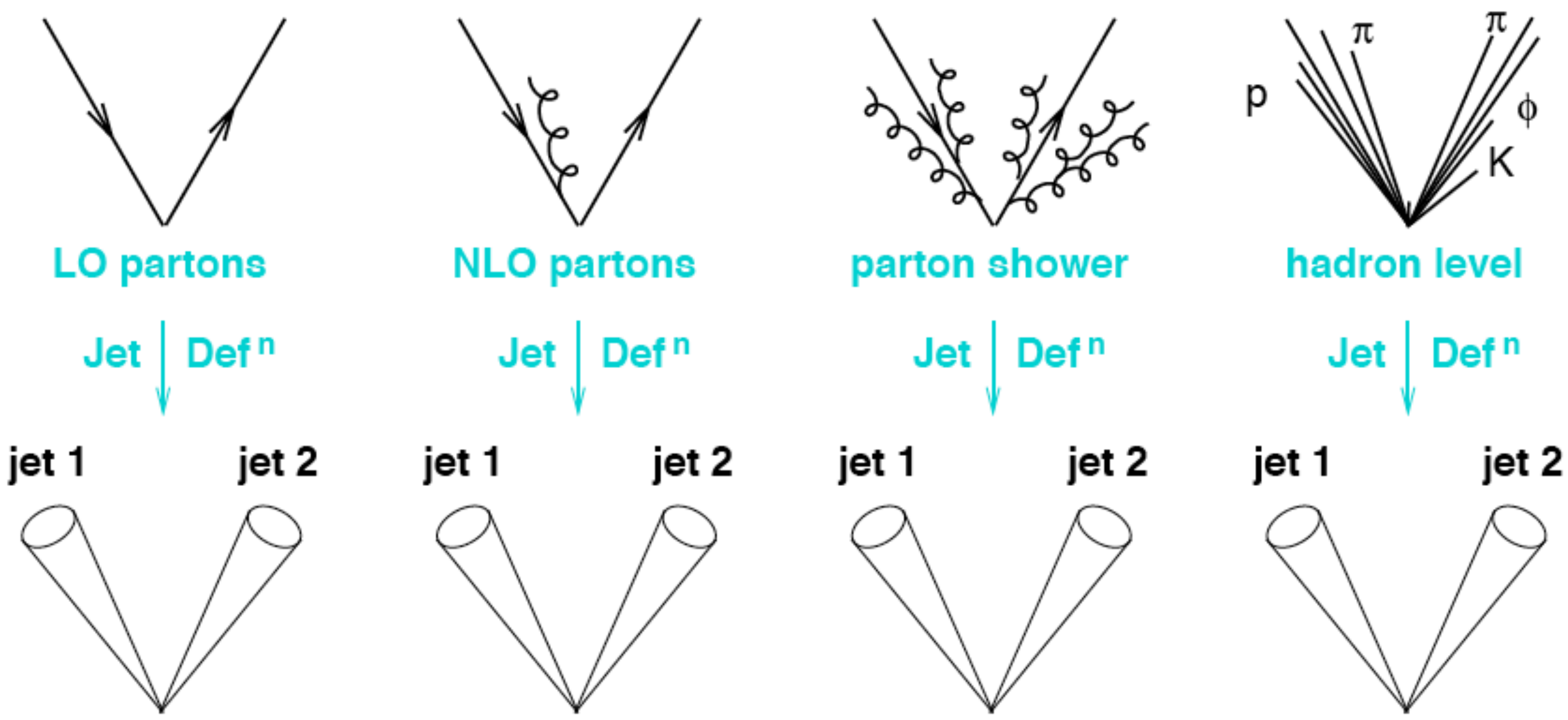


- It should be collinear safe, i.e. splitting one parton into two collinear partons should not change the results of the jet clustering



Jet algorithms

- The algorithm should behave in a similar manner (as much as possible) at the parton, particle and detector levels. Note that differences between levels can unavoidably creep in.



Projection to jets should be resilient to QCD effects

The k_T family of jet algorithms

- p=1
  - the regular k_T jet algorithm
- p=0
  - Cambridge-Aachen algorithm
- p=-1
  - anti-k_T jet algorithm
  - Cacciari, Salam, Soyez '08
  - also P-A Delsart '07 (reverse k_T)
  - soft particles will first cluster with hard particles before clustering among themselves
  - no split/merge
  - leads mostly to constant area hard jets

d=distance measure

$$d_{ij} = \min\left(p_{T,i}^{2p}, p_{T,j}^{2p}\right)\frac{\Delta R_{ij}^2}{D^2}$$

$$d_{ii} = p_{T,i}^{2p}$$

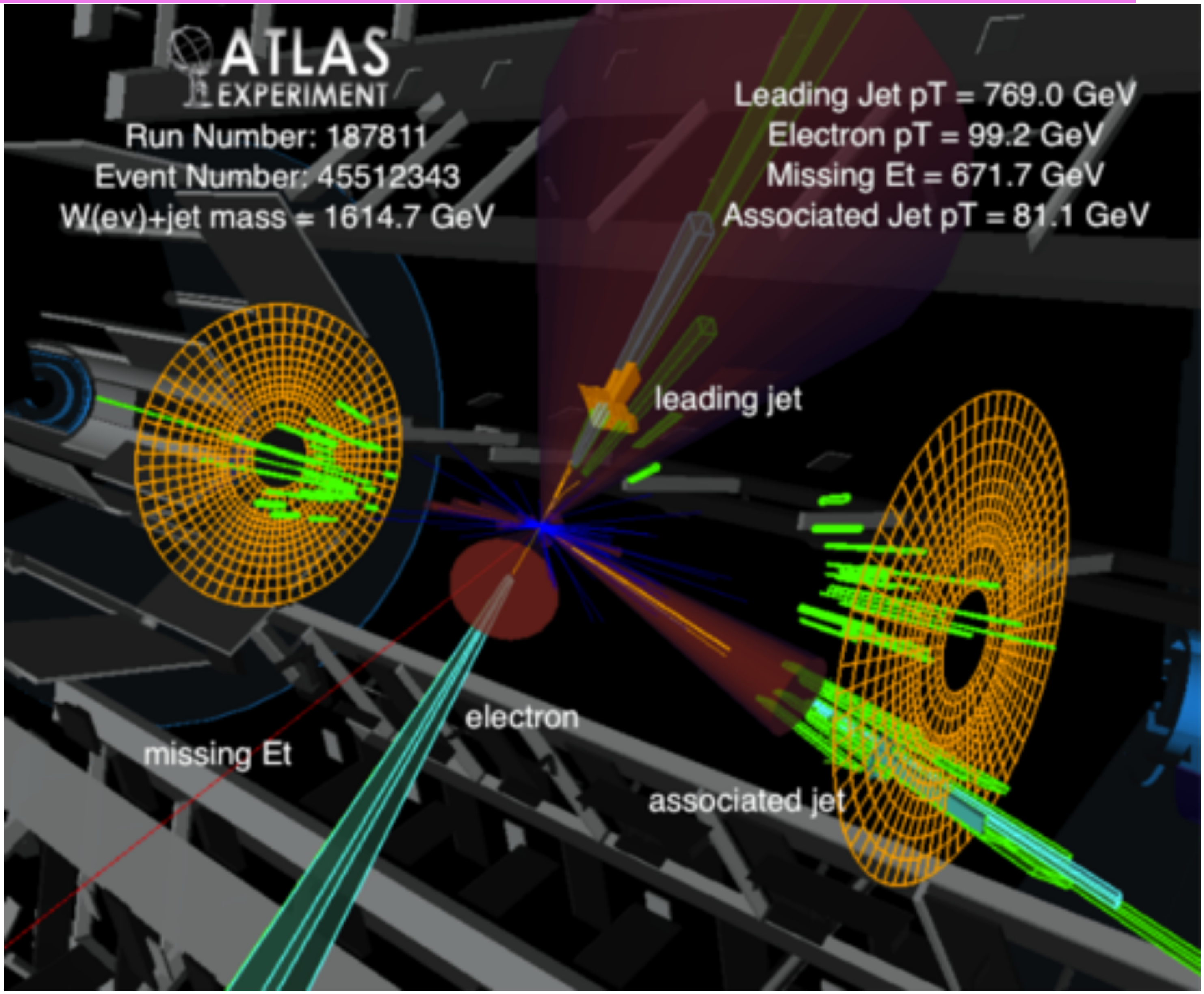
size of jet in Δy-$\Delta\phi$ space

- #1 algorithm for ATLAS, CMS
- Actually, seems to be the only algorithm used

ATLAS W + 2 jet event

.with the W boson decaying into an electron and a neutrino

...and the 2 jets defined with the antikT algorithm with R=0.4



Back to logarithms

- You can keep applying this argument at higher orders of perturbation theory
- Each gluon that is added yields an additional power of α_s, and via the eikonal factorization outlined, can produce an additional two logarithms (soft and collinear)
- So can write the W + jets cross section as

$$d\sigma = \sigma_0(W + 1\text{ jet})\left[1 + \alpha_S(c_{12}L^2 + c_{11}L + c_{10}) + \alpha_S{}^2(c_{24}L^4 + c_{23}L^3 + c_{22}L^2 + c_{21}L + c_{20}) + \cdots\right]$$

  - where L represents the logarithm controlling the divergence, either soft or collinear (Sudakov logs)
  - note that α_s and L appear together as $\alpha_s L$

- Size of L depends on criteria used to define the jets (min p_T, cone size)
- Coefficients c_{ij} depend on color factors
- Thus, addition of each gluon results in additional factor of α_s times logarithms
- In many (typically exclusive) cases, the logs can be large, leading to an enhanced probability for gluon emission to occur
- For most inclusive cases, logs are small and α_s counting may be valid estimator for production of additional jets
- For completely inclusive cross sections, the logs vanish

Specific example

- Remember we encounter logs whenever an emitted gluon becomes soft and/ or collinear

$$[a\ b]\,\mathrm{d}PS_{\mathrm{gluon}} = \frac{1}{E^2}\frac{1}{1-\cos\theta_a}E\mathrm{d}E\ \mathrm{d}\cos\theta_a$$

- We said the c_{ij} were color factors
- So for emission of parton 5 from parton 1, color factor is C_F
- For emission of parton 4 from parton 3, C_A
- If parton 5 is soft, and collinear with parton 1, and parton 4 is soft, and is collinear with parton 3, have 4 powers of logs

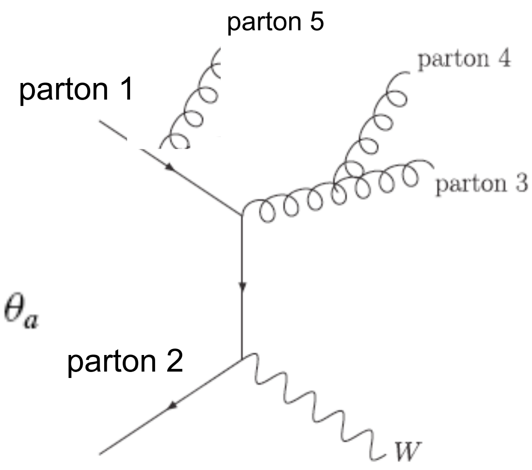


Figure 13. A final-state configuration containing a W and 2 partons. After the jet definition has been applied, either zero, one or two jets may be reconstructed.

not present since have 2 extra gluons, not 1

$$\mathrm{d}\sigma = \sigma_0(W+1\ \mathrm{jet})\left[1+\alpha_S(c_{12}L^2+c_{11}L+c_{10}) \right.$$
$$\left. +\alpha_S{}^2(c_{24}L^4+c_{23}L^3+c_{22}L^2+c_{21}L+c_{20})+\cdots\right]$$

- If one of the partons is not soft or collinear, then only 3 powers of logs
- …and so on
- Factors of 2, π, etc ignored

for W + jets

Re-shuffling

$$d\sigma = \sigma_0(W + 1\ \text{jet})\left[1 + \alpha_S(c_{12}L^2 + c_{11}L + c_{10}) + \alpha_S{}^2(c_{24}L^4 + c_{23}L^3 + c_{22}L^2 + c_{21}L + c_{20}) + \cdots\right]$$

each gluon added has an additional factor of α_s and two additional logs (soft and collinear)

c_{ij} depend on color factors

- re-write the term in brackets as

$$\left[\cdots\right] = 1 + \alpha_S L^2 c_{12} + (\alpha_S L^2)^2 c_{24} + \alpha_S L c_{11}(1 + \alpha_S L^2 c_{23}/c_{11} + \cdots) + \cdots$$
$$= \exp\left[c_{12}\alpha_S L^2 + c_{11}\alpha_S L\right],$$

- Where the infinite series has been resummed into an exponential form
  - first term in expansion is called leading logarithm term, 2nd next-to-leading logarithm, etc
- Now can write out each contribution as a combination of terms in powers of α_s and logarithms

$$\sigma_W = \sigma_{W+0j} + \sigma_{W+1j} + \sigma_{W+2j} + \sigma_{W+3j} + \cdots$$

$$\sigma_{W+0j} = a_0 + \alpha_S(a_{12}L^2 + a_{11}L + a_{10}) + \alpha_S{}^2(a_{24}L^4 + a_{23}L^3 + a_{22}L^2 + a_{21}L + a_{20}) + \cdots$$

$$\sigma_{W+1j} = \alpha_S(b_{12}L^2 + b_{11}L + b_{10}) + \alpha_S{}^2(b_{24}L^4 + b_{23}L^3 + b_{22}L^2 + b_{21}L + b_{20}) + \cdots$$

$$\sigma_{W+2j} = \cdots .$$

as jet definitions change, size of the logs shuffle the contributions from one jet cross section to another, keeping the sum over all contributions the same; for example, as R decreases, L increases, contributions shift towards higher jet multiplicities

Re-shuffling

- Configuration shown to the right can be reconstructed as an event containing up to 2 jets (0,1,2), depending on jet definition and momenta of the partons.
- For a large value of R_{cone}, this is one jet; for a smaller value, it may be two jets
- The matrix elements for this process contain terms proportional to $\alpha_s \log(p_{T3}/p_{T4})$ and as $\log(1/\Delta R_{34})$, so min values for transverse momentum and separation must be imposed
- Suppose that I consider completely inclusive cross sections ($\sigma_{W+>=0\ jets}$)
- Then the logs vanish

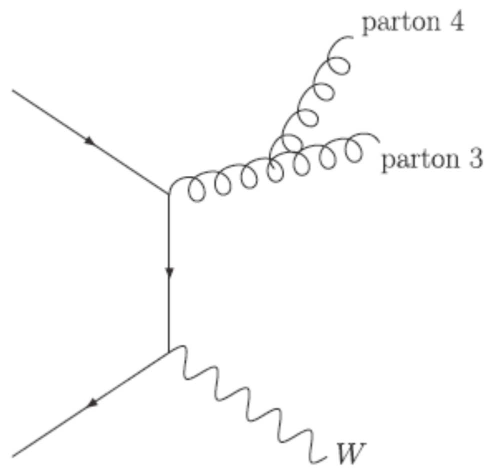


Figure 13. A final-state configuration containing a W and 2 partons. After the jet definition has been applied, either zero, one or two jets may be reconstructed.

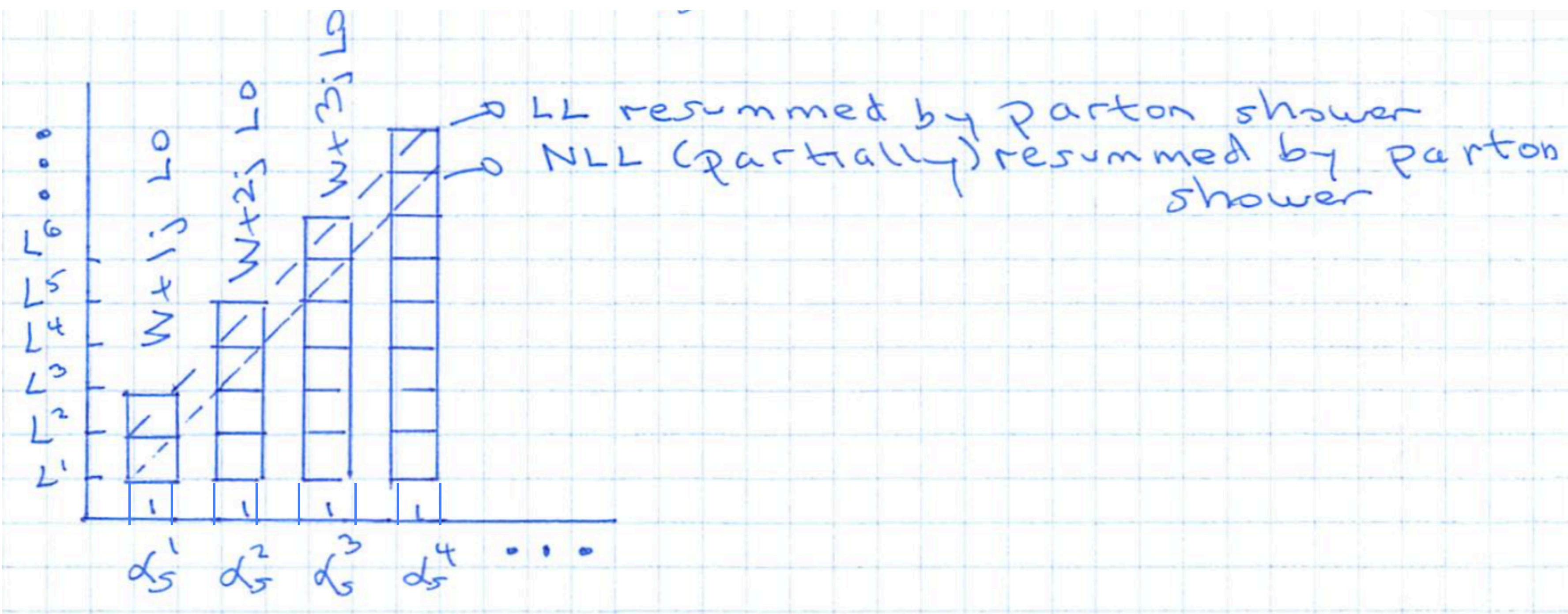
$$\begin{aligned}
\sigma_{W+0j} &= a_0 + \alpha_S(a_{12}L^2 + a_{11}L + a_{10}) \\
&\quad + \alpha_S{}^2(a_{24}L^4 + a_{23}L^3 + a_{22}L^2 + a_{21}L + a_{20}) + \cdots \\
\sigma_{W+1j} &= \alpha_S(b_{12}L^2 + b_{11}L + b_{10}) \\
&\quad + \alpha_S{}^2(b_{24}L^4 + b_{23}L^3 + b_{22}L^2 + b_{21}L + b_{20}) + \cdots \\
\sigma_{W+2j} &= \cdots.
\end{aligned}$$

Reviewing

$$\sigma_{W+0j} = a_0 + \alpha_S(a_{12}L^2 + a_{11}L + a_{10}) + \alpha_S{}^2(a_{24}L^4 + a_{23}L^3 + a_{22}L^2 + a_{21}L + a_{20}) + \cdots$$

$$\sigma_{W+1j} = \alpha_S(b_{12}L^2 + b_{11}L + b_{10}) + \alpha_S{}^2(b_{24}L^4 + b_{23}L^3 + b_{22}L^2 + b_{21}L + b_{20}) + \cdots$$

$$\sigma_{W+2j} = \cdots.$$



NLO calculations

- NLO calculation requires consideration of all diagrams that have an extra factor of α_s
    - real radiation, as we have just discussed
    - virtual diagrams (with loops)
- For virtual diagram, have to integrate over loop momentum
    - but result contains IR singularities (soft and collinear), just as found for tree-level diagrams

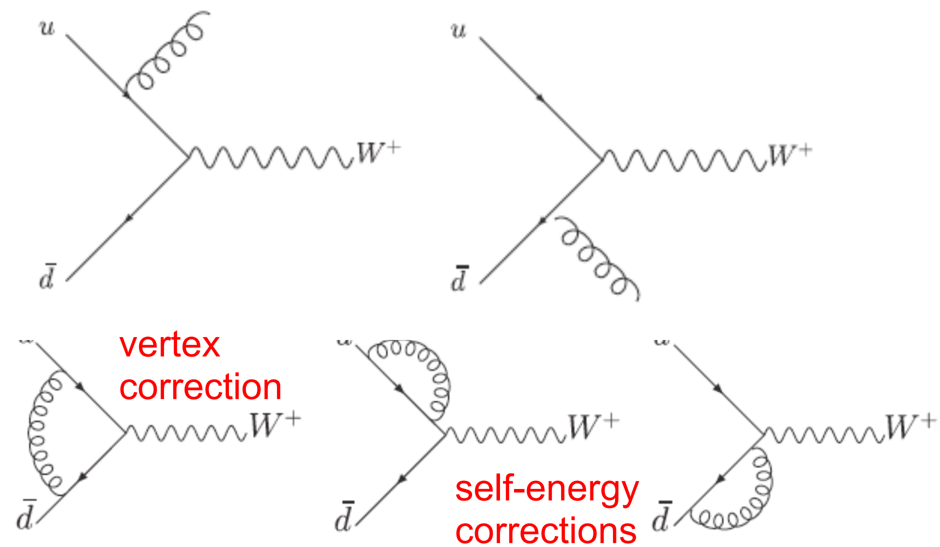


Figure 14. Virtual diagrams included in the next-to-leading order corrections to the Drell–Yan production of a W at hadron colliders.

$O(\alpha_s)$ virtual corrections in NLO cross section arise from interference between tree level and one-loop virtual amplitudes

If we add the real+virtual contributions, we find that the singularities will cancel, for inclusive cross sections. We have to be more clever for differential distributions.

Scale choices

- We know that we have two scales, μ_R and μ_F
- We know that they should be associated with the relevant scale in the hard scattering process
  - sometime this scale is evident, like m_W for W production, p_T^{jet} for inclusive jet production
  - but what if I have a process like W+jet(s)
    - there I have both m_W and p_T^{jet}, and these scales can be very different->very different answers
    - we'll see that for some cases, general scales like H_T may work best
- Often μ_R and μ_F are taken equal to each other, but the physics associated with each is a bit different, so they can be varied separately…as long as the ratio between the two scales is not too large (>2)
- For then, we would introduce a new log into the calculation, the log of the ratio of the two scales
- These logarithms would then have to be re-summed to restore precision to the measurement
- We don't want to have to do that

→ H_T: sum of transverse momenta of all objects in event

Scale uncertainties

- We try to estimate the uncertainty due to uncalculated higher order terms by varying μ_R,μ_F over some range, typically a factor of 2
- This is normally the best we can do, but we have to keep in mind that higher order corrections can arise from a number of other sources such as Sudakov effects, large color factors, large π^2 terms, the opening of new channels
- These contributions are not estimated by the variation of the scale logarithms and can be larger than the variation

What does the scale dependence for a cross section look like?

- Here, we're specifically looking at inclusive jet production, but this holds for other collider processes
- Write cross section indicating explicit scale-dependent terms for NLO
- First term (lowest order) in (3) leads to monotonically decreasing behavior as scale increases (the LO piece)
- Second term is negative for $\mu < p_T$, positive for $\mu > p_T$
- Third term is negative for factorization scale $M < p_T$
- Fourth term has same dependence as lowest order term
- Thus, lines one and four give contributions which decrease monotonically with increasing scale while lines two and three start out negative, reach zero when the scales are equal to p_T, and are positive for larger scales
- At NLO, result is a roughly parabolic behavior

Consider a large transverse momentum process such as the single jet inclusive cross section involving only massless partons. Furthermore, in order to simplify the notation, suppose that the transverse momentum is sufficiently large that only the quark distributions need be considered. In the following, a sum over quark flavors is implied. Schematically, one can write the lowest order cross section as

$$E\frac{d^3\sigma}{dp^3} \equiv \sigma = a^2(\mu)\,\hat{\sigma}_B \otimes q(M) \otimes q(M) \tag{1}$$

where $a(\mu) = \alpha_s(\mu)/2\pi$ and the lowest order parton-parton scattering cross section is denoted by $\hat{\sigma}_B$. The renormalization and factorization scales are denoted by μ *and* M, respectively. In addition, various overall factors have been absorbed into the definition of $\hat{\sigma}_B$. The symbol $\otimes$ denotes a convolution defined as

$$f \otimes g = \int_x^1 \frac{dy}{y} f(\frac{x}{y})\, g(y). \tag{2}$$

When one calculates the $\mathcal{O}(\alpha_s^3)$ contributions to the inclusive cross section, the result can be written as

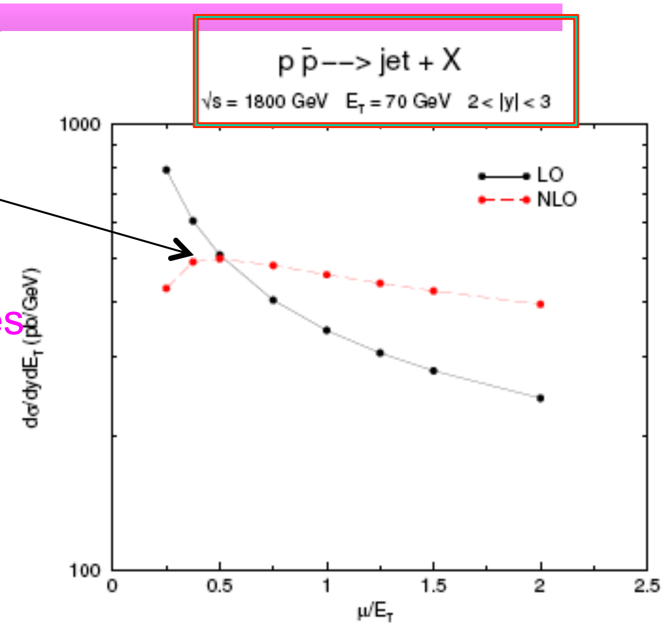
$$\begin{aligned}
(1)\quad \sigma &= a^2(\mu)\,\hat{\sigma}_B \otimes q(M) \otimes q(M) \\
(2)\quad &+ 2a^3(\mu)\, b \ln(\mu/p_T)\hat{\sigma}_B \otimes q(M) \otimes q(M) \\
(3)\quad &+ 2a^3(\mu)\, \ln(p_T/M) P_{qq} \otimes \hat{\sigma}_B \otimes q(M) \otimes q(M) \\
(4)\quad &+ a^3(\mu)\, K \otimes q(M) \otimes q(M).
\end{aligned} \tag{3}$$

In writing Eq. (3), specific logarithms associated with the running coupling and the scale dependence of the parton distributions have been explicitly displayed; the remaining higher order corrections have been collected in the function K in the last line of Eq. (3). The μ

Why does the scale dependence have the shape it does?

- Write cross section indicating explicit scale-dependent terms
- First term (lowest order) in (3) leads to monotonically decreasing behavior as scale increases (the LO piece)
- Second term is negative for $\mu<p_T$, positive for $\mu>p_T$
- Third term is negative for factorization scale $M < p_T$
- Fourth term has same dependence as lowest order term
- Thus, lines one and four give contributions which decrease monotonically with increasing scale while lines two and three start out negative, reach zero when the scales are equal to p_T, and are positive for larger scales
- At NLO, result is a roughly parabolic behavior

Note that NLO=LO for a scale of about $p_T/2$; for other scales NLO>LO, or NLO<LO



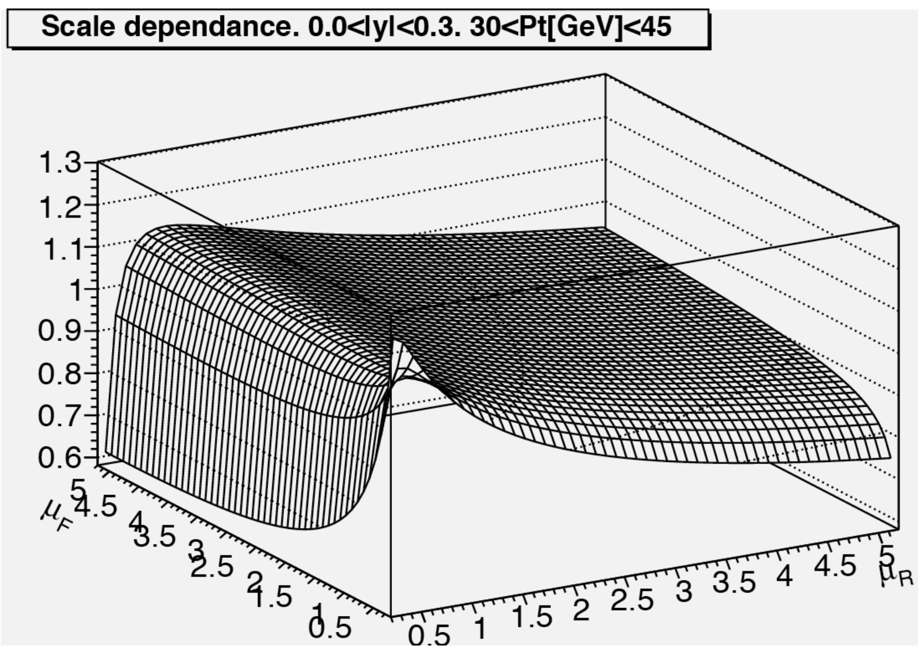
When one calculates the $\mathcal{O}(\alpha_s^3)$ contributions to the inclusive cross section, the result can be written as

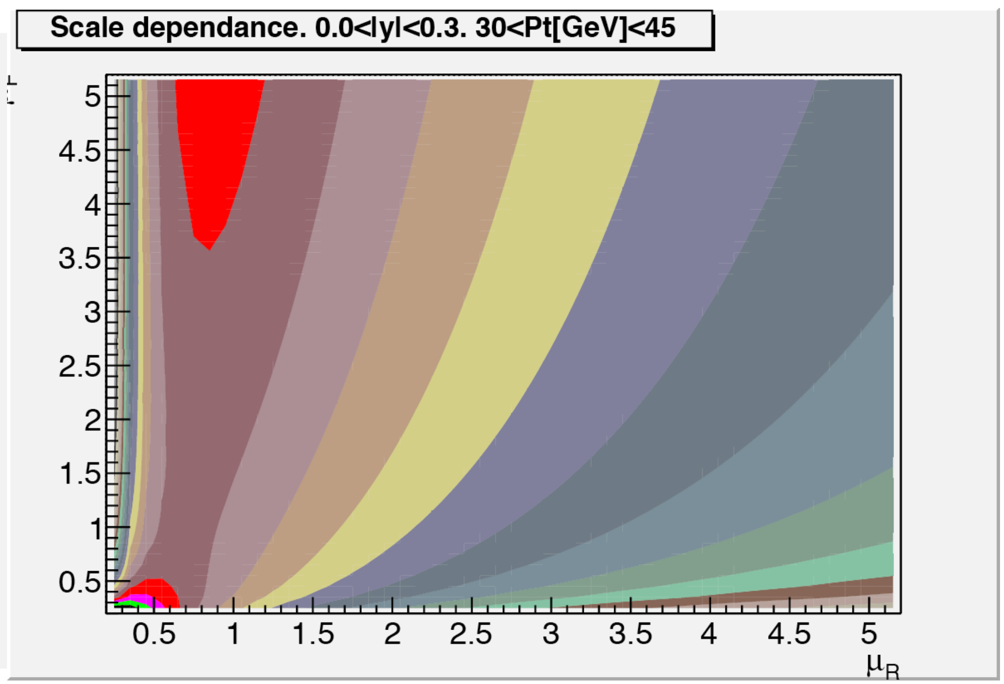
$$
\begin{aligned}
(1)\quad \sigma &= a^2(\mu)\,\hat{\sigma}_B \otimes q(M) \otimes q(M) \\
(2)\quad &+ 2a^3(\mu)\, b \ln(\mu/p_T)\hat{\sigma}_B \otimes q(M) \otimes q(M) \\
(3)\quad &+ 2a^3(\mu)\, \ln(p_T/M) P_{qq} \otimes \hat{\sigma}_B \otimes q(M) \otimes q(M) \\
(4)\quad &+ a^3(\mu)\, K \otimes q(M) \otimes q(M). \qquad (3)
\end{aligned}
$$

In writing Eq. (3), specific logarithms associated with the running coupling and the scale dependence of the parton distributions have been explicitly displayed; the remaining higher order corrections have been collected in the function K in the last line of Eq. (3). The μ

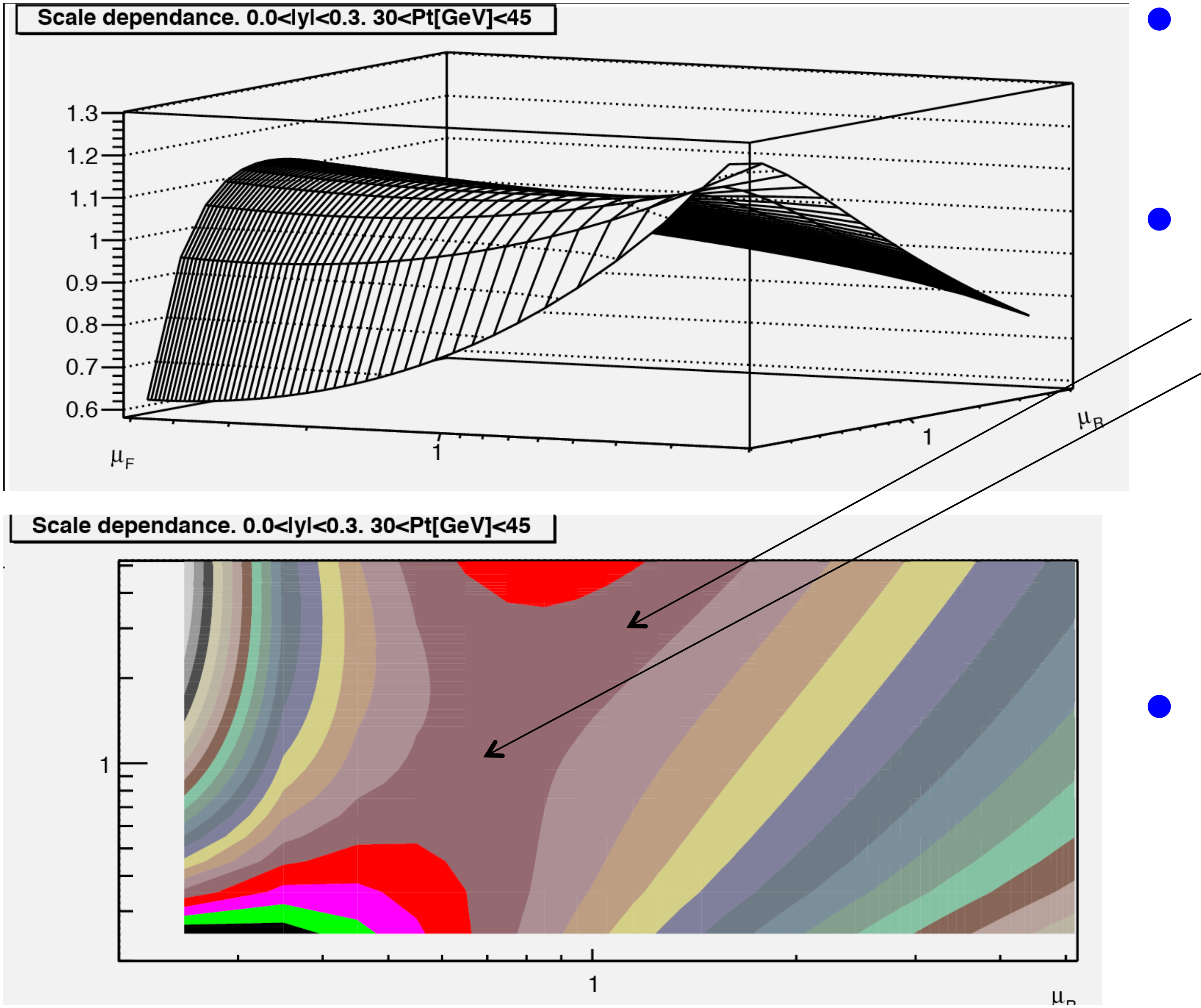
Look at scale dependence in 2-D

Jet production at the LHC



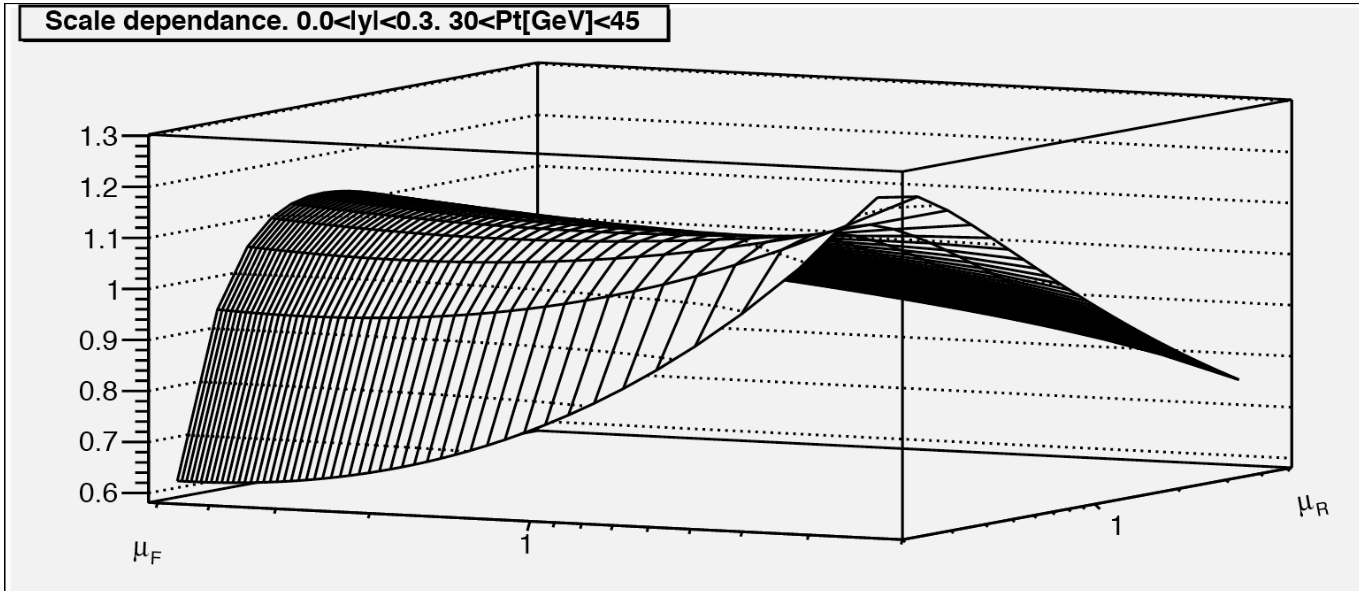


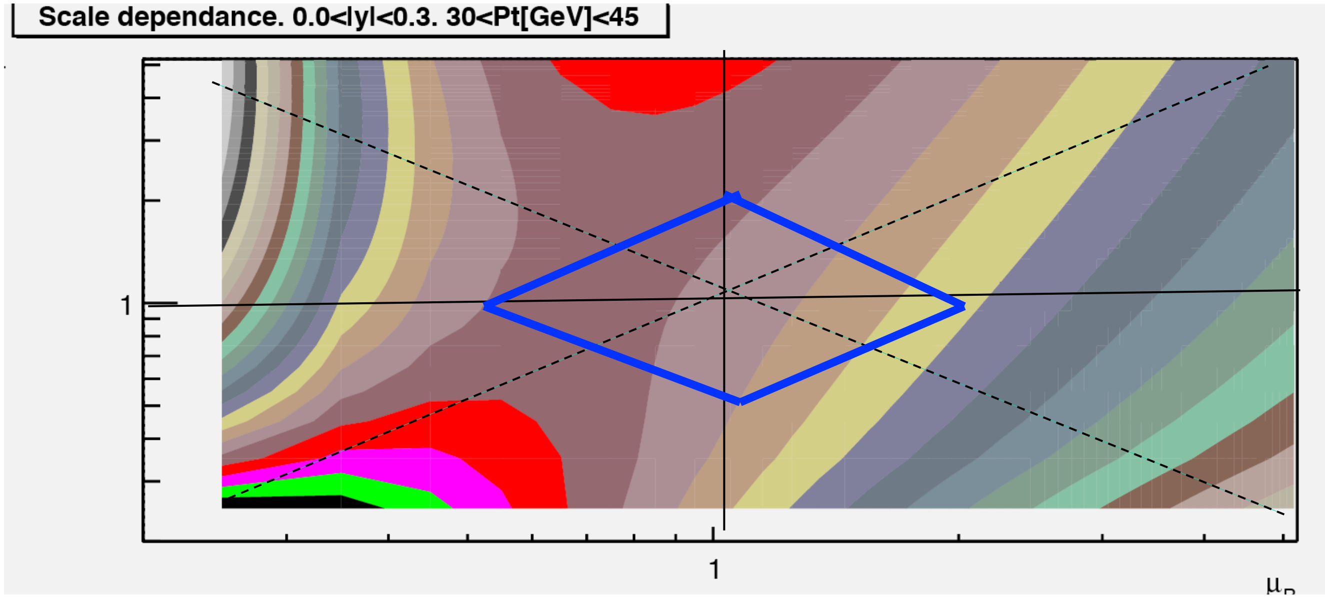
It’s also useful to use a log-log scale



- …since perturbative QCD is logarithmic
- Note that there’s a saddle region, and a saddle point, where locally there is little slope for the cross section with respect to the two scales
- This is kind of the ‘golden point’ and typically around the expected scale (p_T^{jet} in this case)

It’s also useful to use a log-log scale



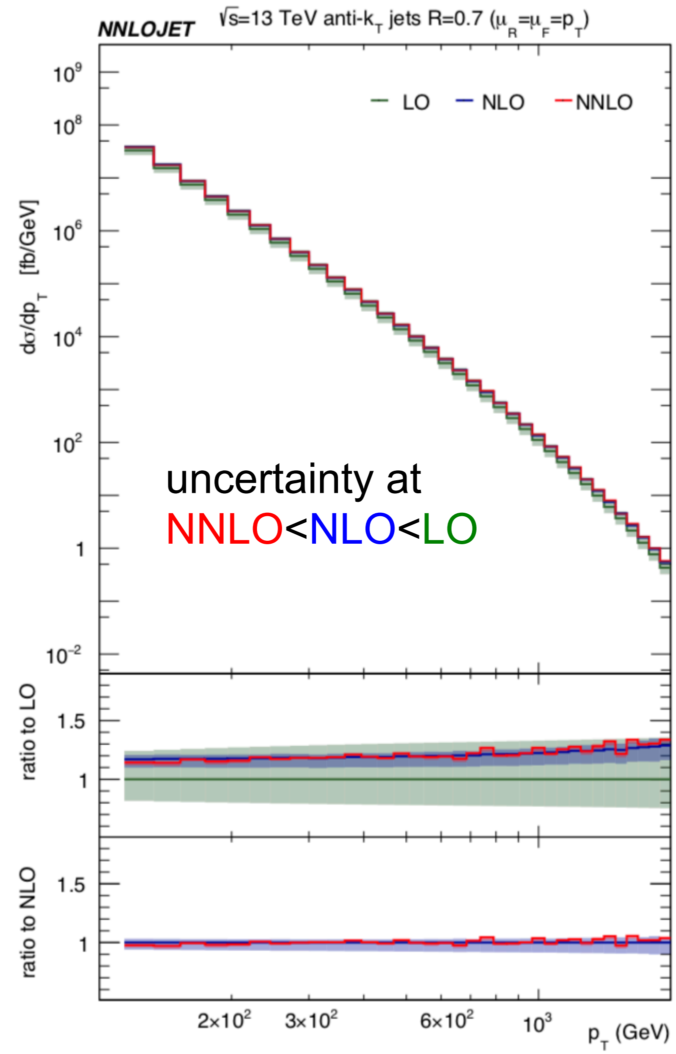


- Choose $p_T j^{et}$ as the central scale
- The scale variation represents an estimate of the uncalculated higher orders
- Typically vary both μ_R and μ_F up and down from their central values to estimate the scale uncertainty
- ...sometimes making sure that the ratio of the two scales is never larger than two, creating the diamond

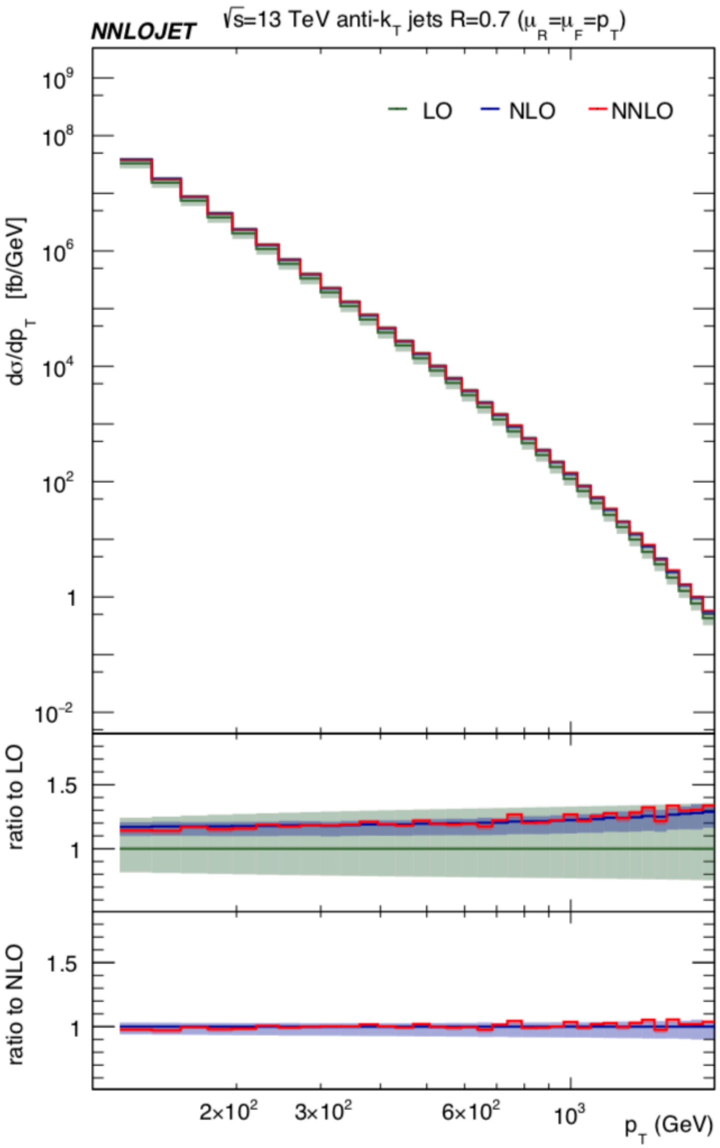
Advantages of higher orders

- Less sensitivity to unphysical input scales, i.e. renormalization and factorization scales
- NLO is first level of prediction where normalization (and sometimes shape) can be taken seriously
- At NNLO can take uncertainties more seriously
- More physics
  - parton merging gives structure in jets
  - more species of incoming partons

consider inclusive jet prod at LO, NLO, NNLO



More scale terms in NNLO expression



$$
\begin{aligned}
&\sigma(\mu_R,\mu_F,\alpha_s(\mu_R),L_R,L_F) = \\
&\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^2 \hat\sigma^{(0)}_{ij} \otimes f_i(\mu_F)\otimes f_j(\mu_F) \\
&+\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^3 \hat\sigma^{(1)}_{ij} \otimes f_i(\mu_F)\otimes f_j(\mu_F) \\
&\quad +L_R\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^3 2\beta_0\,\hat\sigma^{(0)}_{ij} \otimes f_i(\mu_F)\otimes f_j(\mu_F) \\
&\quad +L_F\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^3 \Big[-\hat\sigma^{(0)}_{ij}\otimes f_i(\mu_F)\otimes\left(P^{(0)}_{jk}\otimes f_k(\mu_F)\right) \\
&\qquad\qquad -\hat\sigma^{(0)}_{ij}\otimes\left(P^{(0)}_{ik}\otimes f_k(\mu_F)\right)\otimes f_j(\mu_F)\Big] \\
&+\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^4 \hat\sigma^{(2)}_{ij} \otimes f_i(\mu_F)\otimes f_j(\mu_F) \\
&\quad +L_R\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^4 \left(3\,\beta_0\,\hat\sigma^{(1)}_{ij}+2\,\beta_1\,\hat\sigma^{(0)}_{ij}\right)\otimes f_i(\mu_F)\otimes f_j(\mu_F) \\
&\quad +L_R^2\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^4 3\,\beta_0^2\,\hat\sigma^{(0)}_{ij}\otimes f_i(\mu_F)\otimes f_j(\mu_F) \\
&\quad +L_F\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^4 \Big[-\hat\sigma^{(1)}_{ij}\otimes f_i(\mu_F)\otimes\left(P^{(0)}_{jk}\otimes f_k(\mu_F)\right) \\
&\qquad\qquad -\hat\sigma^{(1)}_{ij}\otimes\left(P^{(0)}_{ik}\otimes f_k(\mu_F)\right)\otimes f_j(\mu_F) \\
&\qquad\qquad -\hat\sigma^{(0)}_{ij}\otimes f_i(\mu_F)\otimes\left(P^{(1)}_{jk}\otimes f_k(\mu_F)\right) \\
&\qquad\qquad -\hat\sigma^{(0)}_{ij}\otimes\left(P^{(1)}_{ik}\otimes f_k(\mu_F)\right)\otimes f_j(\mu_F)\Big] \\
&\quad +L_F^2\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^4 \Big[\hat\sigma^{(0)}_{ij}\otimes\left(P^{(0)}_{ik}\otimes f_k(\mu_F)\right)\otimes\left(P^{(0)}_{jl}\otimes f_l(\mu_F)\right) \\
&\qquad\qquad +\frac{1}{2}\hat\sigma^{(0)}_{ij}\otimes f_i(\mu_F)\otimes\left(P^{(0)}_{jk}\otimes P^{(0)}_{kl}\otimes f_l(\mu_F)\right) \\
&\qquad\qquad +\frac{1}{2}\hat\sigma^{(0)}_{ij}\otimes\left(P^{(0)}_{ik}\otimes P^{(0)}_{kl}\otimes f_l(\mu_F)\right)\otimes f_j(\mu_F) \\
&\qquad\qquad +\frac{1}{2}\beta_0\,\hat\sigma^{(0)}_{ij}\otimes f_i(\mu_F)\otimes\left(P^{(0)}_{jk}\otimes f_k(\mu_F)\right) \\
&\qquad\qquad +\frac{1}{2}\beta_0\,\hat\sigma^{(0)}_{ij}\otimes\left(P^{(0)}_{ik}\otimes f_k(\mu_F)\right)\otimes f_j(\mu_F)\Big] \\
&\quad +L_F L_R\left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^4 \Big[-3\,\beta_0\,\hat\sigma^{(0)}_{ij}\otimes f_i(\mu_F)\otimes\left(P^{(0)}_{jk}\otimes f_k(\mu_F)\right) \\
&\qquad\qquad -3\,\beta_0\,\hat\sigma^{(0)}_{ij}\otimes\left(P^{(0)}_{ik}\otimes f_k(\mu_F)\right)\otimes f_j(\mu_F)\Big]
\end{aligned}
$$

Back to W production to NLO

- In 4-dimensions, the contribution of the real diagrams can be written (ignoring diagrams with incoming gluons for simplicity)

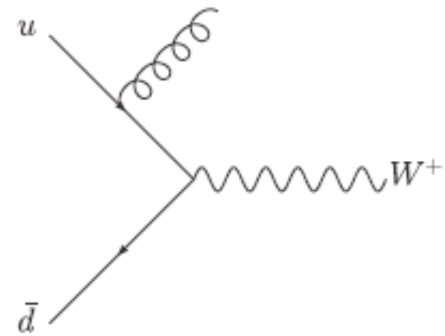
$$\left|M(u\bar{d} \rightarrow W^+ g\right|^2 \sim g^2 C_F \left[\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} + \frac{2Q^2\hat{s}}{\hat{u}\hat{t}}\right]$$

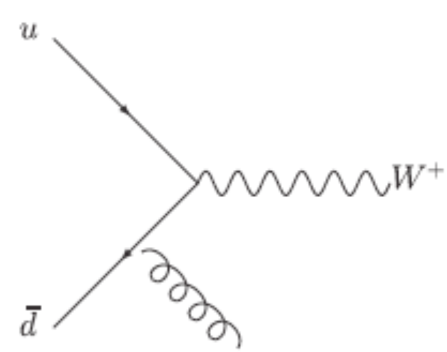
$$\sim g^2 C_F \left[\left(\frac{1+z^2}{1-z}\right)\left(\frac{-\hat{s}}{\hat{t}} + \frac{-\hat{s}}{\hat{u}}\right) - 2\right]$$

  - where

$$z = \frac{Q^2}{s} \text{ and } \hat{s} + \hat{t} + \hat{u} = Q^2$$

- Note that the real diagrams contain collinear singularities, $\hat{u}$->0, $\hat{t}$->0, and soft singularities, z->1





and don't sweat the details; I just want you to see in general terms how a NLO calculation is carried out

Aside: dimensional regularization

- Suppose we have an integral of the form, typical of the integrals in a NLO calculation

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 - m^2\right)^2}$$

- We get infinity if we integrate this in 4 dimensions, so go to 4-2ε dimensions

$$\int \frac{d^4k}{(2\pi)^4} \to (\mu)^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \to (\mu)^{2\varepsilon} \int \frac{d\Omega_{4-2\varepsilon}}{(2\pi)^{4-2\varepsilon}} \int dk_E k_E^{\,3-2\varepsilon}$$

$$\int \frac{d\Omega_{4-2\varepsilon}}{(2\pi)^{4-2\varepsilon}} = \frac{2}{(4\pi)^{2-\varepsilon}} \frac{1}{\Gamma(2-\varepsilon)}$$

$$(\mu)^{2\varepsilon} \int_0^\infty dk_E \frac{k_E^{\,3-2\varepsilon}}{\left(k_E^{\,2} + m^2\right)^2} = \frac{(\mu)^{2\varepsilon}}{2(m)^{2\varepsilon}} \int_0^1 dz z^{1-\varepsilon} (1-z)^{\varepsilon-1} = \frac{1}{2}\left(\frac{\mu}{m}\right)^{2\varepsilon} \frac{\Gamma(\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(2)}$$

- Using

$$\Gamma(1+z) = z\Gamma(z); \Gamma'(1) = -\gamma_E = -0.5772\ldots$$

Dimensional regularization, continued

- Find

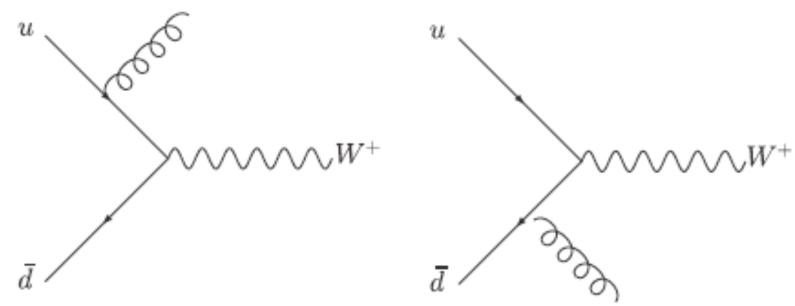
$$I = \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\left(\frac{\mu}{m}\right)^{2\varepsilon} \underset{\varepsilon\text{->}0}{\longrightarrow} \frac{1}{(4\pi)^2}\left[+\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + 2\ln\left(\frac{\mu}{m}\right) + O(\varepsilon)\right]$$

  - singular bits, plus finite bits as ε->0, plus log singularity as m->0

- Define $\overline{\text{MS}}$ scheme: subtract (absorb) 1/ε pole, γ_E, and ln(4π) bits

Now do the dimension trick for the real part

- Problem: if I work in 4 dimensions, I get divergences
- Solution: working in 4-2ε dimensions, to control the divergences (dimensional reduction)



$$\sigma_{real} = \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\varepsilon} c_\Gamma \left[\left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - \frac{\pi^2}{3}\right)\delta(1-z) - \frac{2}{\varepsilon} P_{qq}(z) - 2(1-z) + 4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2\frac{1+z^2}{1-z}\ln z \right]$$

- with

$$c_\Gamma = \frac{(4\pi)^{\varepsilon}}{\Gamma(1-\varepsilon)}$$

"+ distribution"

$$\left(\frac{\log(1-z)}{1-z}\right)_+ \equiv \lim_{\beta \to 0} \left\{ \frac{\log(1-z)}{1-z}\theta(1-z-\beta) + \frac{1}{2}\log^2(\beta)\delta(1-z-\beta) \right\}$$

We get 1/ε terms from individual soft and collinear singularities
We get $1/\varepsilon^2$ terms for overlapping IR singularities.