Supplemental Assignments for Collider Phenomenology

Tao Han, Aug. 1–3, 2018, The XJU-IMP HEP summer school, Lanzhou, China

Lecture 1: Introduction and Basic Formalism

`Exercise 1.1:` A B-factory (*e. g.* KEKB) is designed for asymmetric head-on collisions between a positron beam of energy 3.5 GeV and an electron beam of energy 8 GeV. Find the center-of-mass energy for the B-factory. Do you understand why to adopt this design for the energy and for the asymmetry?

`Exercise 1.2:` The dominant decay channel of the top quark is $t \to W^+ b$. The partial decay width given in terms of the known mass parameters at the leading order is

$$\Gamma_t = \frac{G_F m_t^3}{8\pi\sqrt{2}}(1 - \frac{m_W^2}{m_t^2})^2(1 + 2\frac{m_W^2}{m_t^2}).$$

Assuming this formula gives its total decay width, estimate the top-quark life-time in units of yocto-second.

If the QCD scale is $\Lambda_{\rm QCD} \approx 200$ MeV, compare the top-quark life-time with the time scale at which the QCD strong interaction sets in.

Also compare with the b-quark life-time, and try to understand the differences between the decays of the two quarks.

(Use the PDG review for the parameters needed.)

`Exercise 1.3:` (**challenging problem**) In the "Standard Model" of elementary particle physics, the amplitude for the scattering of the (longitudinally polarized) weak gauge bosons (the force mediator for the nuclear β decay) $W^+W^+ \to W^+W^+$ is calculated at high energies to be

$$f(k,\ \theta) = \frac{1}{16\pi k}\left(\frac{-M_H^2}{v^2}\right)\left(\frac{t}{t - M_H^2} + \frac{u}{u - M_H^2}\right)$$

where k is the W^+ momentum in the Center-of-Momentum frame, M_H is the mass of the Higgs boson, and $v \approx 250$ GeV is the Higgs vacuum expectation value. The angular-dependent kinematical variables are

$$t = -2k^2(1 - \cos\theta) \quad \text{and} \quad u = -2k^2(1 + \cos\theta).$$

Note that the amplitude is give in the "natural units" where $c = \hbar = 1$, and everything is expressed in terms of the energy units electron-volts: 1 GeV = 10^9 eV.

(a). Ignore spins and take the high-energy limit $2k \gg M_H$, compute the partial wave amplitude a_ℓ. Note that for final state identical particles W^+W^+, the angular integration should be $1/2 \int_{-1}^{1} d\cos\theta$.

(b). Impose the partial wave unitarity condition on a_ℓ for s-wave, determine the bound on the mass of the Higgs boson M_H (in units of GeV).

(c). If the Higgs boson did not exist in Nature, then the amplitude for the weak gauge boson scattering for $W^+W^+ \to W^+W^+$ would be expressed by taking the limit $2k \ll M_H \to \infty$. Using the same procedure above, determine at what energy scale $2k$ the Standard Model theory would break down to violate the partial wave unitarity.

(Remark: The "Large Hadron Collider" (LHC) at CERN, Geneva, provides proton-proton collisions at a c.m. energy of 13,000 GeV, which was designed based on the above physics argument. Consequently, we have witnessed the historical discovery of the Higgs boson!)

`Exercise 1.4:` A 125 GeV Higgs boson will have a production cross section of 20 pb at the LHC. How many events per year do you expect to produce for the Higgs boson with a designed LHC luminosity $10^{33}/\text{cm}^2/\text{s}$? With the expected events, why is the Higgs boson so difficult to observe?

Lecture 2: Relativistic Kinematics and Phase Space, Collider Detectors

`Exercise 2.1:` Show that the phase space element $d\vec{p}/2p^0$ is Lorentz invariant.

`Exercise 2.2:` (**challenging problem**) A particle of mass M decays to two particles isotropically in its rest frame. What does the momentum distribution look like in a frame in which the particle is moving with a speed β_z? Compare the result with your expectation for the shape change for a basket ball.

`Exercise 2.3:` Consider a $2 \to 2$ scattering process $p_a + p_b \to p_1 + p_2$. Assume that $m_a = m_1$ and $m_b = m_2$. Show that

$$\begin{aligned} t &= -2p_{cm}^2(1 - \cos\theta_{a1}^*), \\ u &= -2p_{cm}^2(1 + \cos\theta_{a1}^*) + \frac{(m_1^2 - m_2^2)^2}{s}, \end{aligned}$$

$p_{cm} = \lambda^{1/2}(s, m_1^2, m_2^2)/2\sqrt{s}$ is the momentum magnitude in the c.m. frame.
Note: t is negative definite; $t \to 0$ in the collinear limit, that could be singular for massless-exchange. Comment on the u-channel.

`Exercise 2.4:` (**challenging problem**) A particle of mass M decays to three particles $M \to abc$. Show that the phase space element can be expressed as

$$dPS_3 = \frac{1}{2^7\pi^3}\ M^2 dx_a dx_b.$$
$$x_i = \frac{2E_i}{M},\ (i = a, b, c,\ \sum_i x_i = 2).$$

where the integration limits for $m_a = m_b = m_c = 0$ are

$$0 \le x_a \le 1, \quad 1 - x_a \le x_b \le 1.$$

Note: For the decay in the M-rest frame, three of the four angular variables can be trivially integrated out (ignoring the spins of the particles).

`Exercise 2.5:` For a π^0, μ^-, or a τ^- respectively, calculate its decay length if the particle has an energy $E = 10$ GeV.

Lecture 3: High Energy Colliders

(Lepton Colliders)

`Exercise 3.1`: For a resonant production $e^+e^- \to V^*$ with a mass M_V and total width Γ_V, derive the Breit-Wigner formula (**If you find it too challenging for the calculation, you may skip this part and move on to the next line.**)

$$\sigma(e^+e^- \to V^* \to X) = \frac{4\pi(2j+1)\Gamma(V \to e^+e^-)\Gamma(V \to X)}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2} \, \frac{s}{M_V^2},$$

Consider a beam energy spread Δ in Gaussian distribution

$$\frac{dL}{d\sqrt{\hat{s}}} = \frac{1}{\sqrt{2\pi}\ \Delta} \exp[\frac{-(\sqrt{\hat{s}} - \sqrt{s})^2}{2\Delta^2}],$$

obtain the appropriate cross section formulas for (a) $\Delta \ll \Gamma_V$ (resonance line-shape) and (b) $\Delta \gg \Gamma_V$ (narrow-width approximation).

`Exercise 3.2`: An event was identified to have a μ^+ and a μ^- along with some missing energy. What can you say about the kinematics of the system of the missing particles? Consider for both an e^+e^- and a hadron collider.

`Exercise 3.3` **(challenging problem)**: Derive the Weizsäcker-Williams spectrum for a photon with an energy xE off an electron with an energy E

$$P_{\gamma/e}(x) \approx \frac{\alpha}{2\pi} \frac{1 + (1-x)^2}{x} \ln \frac{E^2}{m_e^2}.$$

Note that this procedure is the direct analog to deriving the DGLAP $q \to q'g$ splitting in QCD.

(Hadron Colliders)

`Exercise 3.4:` For a four-momentum $p \equiv p^\mu = (E, \vec{p})$, define

$$\begin{aligned} E_T &= \sqrt{p_T^2 + m^2}, \quad p_T^2 = p_x^2 + p_y^2, \quad y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z}, \\ \text{then show } p^\mu &= (E_T \cosh y,\ p_T\cos\phi,\ p_T\sin\phi,\ E_T\sinh y), \\ \text{and, } \frac{d^3\vec{p}}{E} &= p_T dp_T d\phi\ dy = E_T dE_T d\phi\ dy. \end{aligned}$$

Due to the random boost between the Lab-frame (O) and the c.m. frame (O') for every event,

$$y' = \frac{1}{2}\ln\frac{E'+p'_z}{E'-p'_z} = \frac{1}{2}\ln\frac{(1-\beta_{cm})(E+p_z)}{(1+\beta_{cm})(E-p_z)} = y - y_{cm},$$

where β_{cm} and y_{cm} are the speed and rapidity of the c.m. frame w.r.t. the lab frame.

In the massless limit, the rapidity y defines the pseudo-rapidity:

$$y \to \eta = \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} = \ln\cot\frac{\theta}{2}.$$

`Exercise 3.5:` For a two-body massless final state with an invariant mass squared s, show that

$$\frac{d\hat{\sigma}}{dp_T} = \frac{4p_T}{s\sqrt{1-4p_T^2/s}}\ \frac{d\hat{\sigma}}{d\cos\theta^*}.$$

where $p_T = p\sin\theta^*$ is the transverse momentum and θ^* is the polar angle in the c.m. frame. Comment on the apparent singularity at $p_T^2 = s/4$.

Solution and keys to the exercises:

`Sol. to Exercise 1.1:`

At high energies, the mass of the beam particles $e^\pm$ is totally negligible, which implies $E_\pm = |\vec{p}_\pm|$. Thus the c.m. energy for a head-on collision is

$$\sqrt{s} = \sqrt{(E_- + E_+)^2 - (\vec{p}_- + \vec{p}_+)^2} = \sqrt{(E_- + E_+)^2 - (E_- - E_+)^2} = \sqrt{4E_-E_+} = 10.57 \text{ GeV}.$$

This energy value is right on the resonance mass of $b\bar{b}$ bound state $\Upsilon(4S)$, and the asymmetry provide the boost factor $\gamma = 11.5/10.57 \approx 1.09$ for the system.

`Sol. to Exercise 1.2:`

Assuming this formula gives its total decay width (accurate to a QCD-factor of 0.9), the partial decay width is

$$\Gamma_t = \frac{G_F m_t^3}{8\pi\sqrt{2}}(1 - \frac{m_W^2}{m_t^2})^2(1 + 2\frac{m_W^2}{m_t^2}) \approx 1.76 \cdot 0.79^2 \cdot 1.42 \approx 1.6 \text{ GeV}.$$

Thus, the life-time is $\tau_t = 1/\Gamma_t \approx (6.6/1.6) \times 10^{-25}$s=0.41 yocto-second!

For the QCD scale $\Lambda_{\rm QCD} \approx 200$ MeV, the time-scale would be 8 times longer comparing with the top-quark life-time. This implies that top quark will undergo the EW decay into Wb before forming any color singlet top hadron.

The b-quark life-time is of the order 10^{-12}s, about 10 orders of magnitude longer than the top decay. This is due to three factors: a much lighter b-mass $(m_b/m_t)^3 \approx (1/35)^3 \approx 2.5 \times 10^{-5}$; an off-shell W-propagator $(2m_b/v)^2 \approx (1/25)^2 \approx 1.6 \times 10^{-3}$; and the $b \to c$ transition $V_{cb} \approx 4 \times 10^{-3}$. All of these effects leads to a factor of $\sim 1.6 \times 10^{-10}$.

`Sol. to Exercise 1.3:` Unitarity bound on Higgs boson mass (see the inserted page.)

`Sol. to Exercise 1.4:`

(To estimate the event rates and understand the background issue.)

Event rate from the cross section and an integrated luminosity: $N = \mathcal{L}\sigma$. For $m_h = 125$ GeV at the 14 TeV LHC, $\sigma(gg \to h) \approx 20$ pb. With the anticipated (low) luminosity at $10^{33}/cm^2/s \Rightarrow 10$ fb^{-1}/yr, then $N_{b\bar{b}}(h) = 2 \cdot 10^5$/yr, about one SM Higgs boson produced every two minutes! A lot produced.

The SM h largely decays to $b\bar{b}$ final state, with about 80% branching fraction, leading to about 160K $b\bar{b}$ events/yr. However, the rate for the QCD processes of $b\bar{b}$ production via gg

(and to a smaller extent the $q\bar{q}$) is overwhelming, $\sigma(b\bar{b}) \approx \mathcal{O}(1\ \mu b)$, even after a selection of $p_{Tb} > 30$ GeV. This yields that $N_{b\bar{b}}$(QCD)$= 10^{10}$!

This is why one will have to look for other "cleaner" channels like $h \to \gamma\gamma$, ZZ^*, WW^* and $\tau\tau$.

`Sol. to Exercise 2.1:` Lorentz invariant phase space element (see the inserted page.)

`Sol. to Exercise 2.2:`

(To compare the "Lorentz contraction" for space-like (x) and time-like (p) vectors.)
For a frame O' moving w.r.t. a rest frame O at a speed β_{cm}, the four-momentum vector transforms as

$$\begin{pmatrix} E' \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\ \beta_{cm} \\ -\gamma\ \beta_{cm} & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_z \end{pmatrix}$$

We then obtain their energy/momenum

$$\Delta E' = \gamma \Delta E - \gamma\beta\Delta p_z,$$
$$\Delta p'_z = -\gamma\beta\Delta E + \gamma\Delta p_z.$$

Knowing the inputs, the lengths $\Delta p'_x = \Delta p'_y = M$ in both frames, and $\Delta p'_z = E' = M$, $\Delta E' = 0$, then $\Delta E = \beta\Delta p_z$, $M = \Delta p'_z = \gamma(\Delta p_z - \beta\Delta E)$, where $\beta > 0$ in this frame setting. Thus $\Delta p'_z = \gamma(1-\beta^2)\Delta p_z = \Delta p_z/\gamma$. Re-written as

$$\Delta p_z = \gamma\Delta p'_z = \gamma M,$$
$$\Delta E = \beta\gamma M.$$

Opposite to the "space contraction", the momentum extends to a long (cigar) shape.

What about the shape beyond the z direction? An isotropic distribution in $\mathcal{O}'$ is given by

$$p'^2_x + p'^2_y + p'^2_z = E'^2 = M^2/4,$$

which results in

$$p^2_x + p^2_y + \gamma(p^2_z - \beta E) = M^2/4.$$

Substituting E by the equation involving $E' = M/2$, one has

$$\frac{p^2_T}{(M/2)^2} + \frac{(p_z - \beta\gamma M/2)^2}{(\gamma M/2)^2} = 1.$$

(i). The cigar shape is transparent: $\tan\theta = (p_T/p_z)_{|axes} = \gamma^{-1}$.

(ii). There exists a dead zone: no events with $p_z < \beta\gamma M/2$.

`Sol. to Exercise 2.3:`

In general, for a process $a + b \to 1 + 2$,

$$
\begin{aligned}
t &= (p_a - p_1)^2 = (p_b - p_2)^2 = m_a^2 + m_1^2 - 2(E_a E_1 - p_a p_1 \cos\theta_{a1}),\\
u &= (p_a - p_2)^2 = (p_b - p_1)^2 = m_a^2 + m_2^2 - 2(E_a E_2 - p_a p_2 \cos\theta_{a2}).
\end{aligned}
$$

In the c.m. frame,

$$
\vec{p}_a = -\vec{p}_b, \quad \vec{p}_1 = -\vec{p}_2, \quad p^2 = \frac{\lambda(s, m_1^2, m_2^2)}{4s}, \quad E_i = \frac{s + m_i^2 - m_j^2}{2\sqrt{s}}.
$$

With $m_a = m_1,\ m_b = m_2$, then

$$
\begin{aligned}
t &= -2(E_1 - m_1^2 - p_1^2 \cos\theta_{a1}) = -2p^2(1 - \cos\theta_{a1}),\\
u &= -2p^2(1 + \cos\theta_{a1}) + (m_1^2 - m_2^2)^2/s.
\end{aligned}
$$

It is only negative-definite if $m_1 = m_2$.

`Sol. to Exercise 2.4:`

(To derive a very useful three-body phase space formula.)

In general,

$$
\begin{aligned}
dPS_3 &\equiv \frac{1}{(2\pi)^5}\, \delta^4\left(P - p_1 - p_2 - p_3\right) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2} \frac{d^3\vec{p}_3}{2E_3}\\
&\doteq \frac{1}{(2\pi)^5}\, \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2}\, \frac{\delta\left(E - E_1 - E_2 - E_3\right)}{2E_3}\\
&\doteq \frac{1}{(2\pi)^5}\, \frac{|\vec{p}_1|\ dE_1\ d\Omega_1}{2}\, \frac{|\vec{p}_2|\ dE_2\ d\Omega_2}{2}\, \frac{\delta\left(E - E_1 - E_2 - E_3\right)}{2E_3}.
\end{aligned}
$$

For an unpolarized process, the squared matrix element can only be a function of the invariant products of the momenta $p_i \cdot p_j$. Furthermore, for a decay process in its c.m. frame, only the energies of final state particles are non-trivial variables. This follows from, in our process under consideration, that

$$
\begin{aligned}
M^2 &= (p_a + p_b + p_c)^2,\\
2p_a \cdot p_b &= M^2\left(1 + \frac{m_c^2}{M^2} - \frac{m_a^2}{M^2} - \frac{m_b^2}{M^2} - \frac{2E_c}{M}\right).
\end{aligned}
$$

Thus the independent angles are trivial, $d\Omega_1 = 4\pi,\ d\Omega_1 = 2\pi d\cos\theta_{ab}$, where

$$|\vec{p}_c|^2 = |\vec{p}_a|^2 + |\vec{p}_b|^2 + 2|\vec{p}_a||\vec{p}_b|\cos\theta_{ab}, \quad d\cos\theta_{ab} = \frac{E_c dE_c}{|\vec{p}_a||\vec{p}_b|}.$$

Thus, we reach

$$\begin{aligned} dPS_3 &\doteq \frac{1}{2^5\pi^3}\,\delta\left(M - E_a - E_b - E_c\right)\ dE_a\ dE_b\ dE_c, \\ &\doteq \frac{1}{2^7\pi^3}\ M^2\ dx_a\ dx_b, \quad x_i = \frac{2E_i}{M},\ (i = a, b, c,\ \sum_i x_i = 2). \end{aligned}$$

The kinematical region for $x_{a,b,c}$ can be complicated in a general form of a Dalitz plot. Let's consider the simplest case where $m_a = m_b = m_c = 0$. One of the three massless particles may have minimum energy of zero, or maximum energy $M/2$ (in balancing the other two in parallel). Thus the integration limits for $m_a = m_b = m_c = 0$ are

$$0 \le x_a \le 1, \quad 1 - x_a \le x_b \le 1.$$

`Sol. to Exercise 2.5:`

(To learn the "stable/unstable" particles in detectors in terms of their lifetimes.)
Decay length in the lab frame $l = (c\beta)\ \gamma\ \tau_0$, where $\beta = p/E \approx 1,\ \gamma = E/m$.

	τ_0	$c\tau_0$	γ	ℓ	remarks:
π^0	8.4×10^{-17} s	25 nm	74	2μm	prompt decay
$\mu^\pm$	2.2×10^{-6} s	659 m	95	63 km	(quasi) stable
$\tau^\pm$	2.9×10^{-13} s	87μm	5.6 nm	500μm	prompt decay, secondary vertex

`Sol. to Exercise 3.1:`

(First derive a common formula for resonant production, and then understand the signal after convoluting with a realistic beam energy distribution.)
(1). For the process $e^+e^- \to V^* \to X$, the transition matrix element may be written as

$$-i\mathcal{M}(e^+e^- \to V^* \to X) = \frac{J^\mu_{ee}\ g_{\mu\nu}\ J^\nu_X}{(s - M_V^2) + i\Gamma_V M_V} = \frac{-\Sigma^\alpha J_{ee}\cdot\epsilon^\alpha_V\ J_X\cdot\epsilon^{\alpha*}_V}{(s - M_V^2) + i\Gamma_V M_V}.$$

Thus, the cross section is

$$\sigma(e^+e^- \to V^* \to X)\ =\ \frac{\overline{\Sigma}_{spin}|\Sigma^\alpha J_{ee}\cdot\epsilon^\alpha_V\ J_X\cdot\epsilon^{\alpha*}_V|^2}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2}\ \frac{dPS_X}{2s}$$

$$
\begin{aligned}
&= \frac{\overline{\Sigma}_{spin}\Sigma^{\alpha,\alpha'} J_{ee}\cdot\epsilon_V^{\alpha}\ J_X\cdot\epsilon_V^{\alpha *} J_{ee}^{\dagger}\cdot\epsilon_V^{\alpha' *}\ J_X^{\dagger}\cdot\epsilon_V^{\alpha'}}{(s-M_V^2)^2+\Gamma_V^2 M_V^2}\ \frac{dPS_X}{2s} \\
&= \frac{\overline{\Sigma}_{spin}\Sigma^{\alpha,\alpha'} (J_{ee}\cdot\epsilon_V^{\alpha}\ J_{ee}^{\dagger}\cdot\epsilon_V^{\alpha' *})\ (J_X\cdot\epsilon_V^{\alpha *} J_X^{\dagger}\cdot\epsilon_V^{\alpha'})}{(s-M_V^2)^2+\Gamma_V^2 M_V^2}\ \frac{dPS_X}{2s}
\end{aligned}
$$

Like in the case of top decay, ignore the spin correlation of V^*,

$$
\begin{aligned}
\Sigma^{\alpha,\alpha'}(J_{ee}\cdot\epsilon_V^{\alpha}\ J_{ee}^{\dagger}\cdot\epsilon_V^{\alpha' *})\ (J_X\cdot\epsilon_V^{\alpha *} J_X^{\dagger}\cdot\epsilon_V^{\alpha'}) &\approx \Sigma^{\alpha}(J_{ee}\cdot\epsilon_V^{\alpha}\ J_{ee}^{\dagger}\cdot\epsilon_V^{\alpha *})\ \overline{\Sigma}^{\alpha'}(J_X\cdot\epsilon_V^{\alpha' *} J_X^{\dagger}\cdot\epsilon_V^{\alpha'}) \\
&= (2j+1)\overline{\Sigma}^{\alpha}(J_{ee}\cdot\epsilon_V^{\alpha}\ J_{ee}^{\dagger}\cdot\epsilon_V^{\alpha *})\ \overline{\Sigma}^{\alpha'}(J_X\cdot\epsilon_V^{\alpha' *} J_X^{\dagger}\cdot\epsilon_V^{\alpha'}).
\end{aligned}
$$

Using 2-body phase space volume $dPS_2 \doteq 1/8\pi$,

$$
\begin{aligned}
&\Gamma(V^*\to e^+e^-) = \frac{1}{2\sqrt{s}}\ \overline{\Sigma}^{\alpha}_{spin}|J_{ee}\cdot\epsilon_V^{\alpha}|^2\ dPS_2 \\
\Rightarrow\quad &\overline{\Sigma}^{\alpha}_{spin}|J_{ee}\cdot\epsilon_V^{\alpha}|^2 = 16\pi\sqrt{s}\ \Gamma(V^*\to e^+e^-).
\end{aligned}
$$

and

$$
\Gamma(V^*\to X) = \frac{1}{2\sqrt{s}}\ \overline{\Sigma}^{\alpha'}_{spin}|J_X\cdot\epsilon_V^{\alpha'}|^2\ dPS_X,
$$

one obtains

$$
\begin{aligned}
\sigma(e^+e^-\to V^*\to X) &= \frac{1}{2s}\ \frac{2j+1}{2\lambda_e+1}\ \frac{16\pi\sqrt{s}\Gamma(V^*\to e^+e^-)\ 2\sqrt{s}\Gamma(V^*\to X)}{(s-M_V^2)^2+\Gamma_V^2 M_V^2} \\
&= \frac{4\pi(2j+1)\Gamma(V\to e^+e^-)\Gamma(V\to X)}{(s-M_V^2)^2+\Gamma_V^2 M_V^2}\ \frac{s}{M_V^2},
\end{aligned}
$$

where the factor s/M_V^2 is from the $V^*\to V$ conversion.

(2). In reality, the beam energy always has a spread Δ, approximately in Gaussian distribution around the designed energy $\sqrt{s}$:

$$
\frac{dL}{d\sqrt{\hat{s}}} = \frac{1}{\sqrt{2\pi}\ \Delta}\exp[\frac{-(\sqrt{\hat{s}}-\sqrt{s})^2}{2\Delta^2}].
$$

(a) If $\Delta \ll \Gamma_V$,

$$
\frac{dL}{d\sqrt{\hat{s}}} \approx \delta(\sqrt{\hat{s}}-\sqrt{s}),
$$

thus

$$
\sigma = \int \sigma(\hat{s})\ \frac{dL}{d\sqrt{\hat{s}}}\ d\sqrt{\hat{s}} = \sigma(s).
$$

With such a good resolution, a detailed resonance line-shape mapped out.

(b) $\Delta \gg \Gamma_V$, the Breit-Wigner shape dominant and the narrow-width approximation valid:

$$\frac{1}{(s-M_V^2)^2+\Gamma_V^2 M_V^2} \to \frac{\pi}{\Gamma_W M_V}\ \delta(s-M_V^2).$$

Then,

$$\sigma = \int \sigma(\hat{s})\ \frac{dL}{d\sqrt{\hat{s}}}\ d\sqrt{\hat{s}} = 2\pi^2(2j+1)\ \Gamma(V\to e^+e^-)\ BR(V\to X)\ \frac{1}{M_V^2}\ \frac{dL}{d\sqrt{\hat{s}}}|_{\hat{s}=M_V^2}.$$

`Sol. to Exercise 3.2:`

(To learn a missing particle system in a well-constrained e^+e^- collider and in a less-constrained hadron collider.)

For a process

$$e^+(p_1)e^-(p_2) \to f^+(q_1)f^-(q_2) + E_T^{miss},$$

The four-energy momentum conservation reads

$$p_{1\mu}+p_{2\mu} = q_{1\mu}+q_{2\mu}+E_{T\mu}^{miss}. \tag{1}$$

(a). Thus, in e^+e^- collisions, the our-momentum of the missing particle system is fully determined. In particular, the system mass is also known as

$$m^2_{miss}. = (p_1+p_2-q_1+q_2)^2 = E_{cm}^2 + m_{ff}^2 - 2E_{cm}E_{ff}. \tag{2}$$

(b). In hadronic collisions however, the longitudinal components of the initial momenta p_{1z}, p_{2z} and E_{cm} are unknown. Thus only the two transverse components are fixed by measurements:

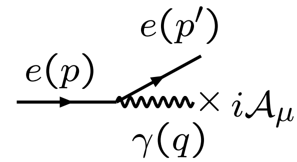
$$(p_1+p_2)_{x,y} = 0; \quad (q_1+q_2)_{x,y} = E_{x,y}^{miss}. \tag{3}$$

`Sol. to Exercise 3.3:`

(Derive the very useful Weizsäcker-Williams spectrum, and understand the behavior.)

To obtain the photon distribution function in the collinear radiation $e^-(p) \to e^-(p')\gamma^*(q)$, write the full matrix element as

$$-i\mathcal{M} = \frac{\overline{u_{e'}}(-ie\gamma^\mu)u_e\ (i\mathcal{A}_\mu)}{q^2} = \frac{J_e^\mu\ \mathcal{A}_\mu}{q^2} = \frac{-\sum_\alpha J_e\cdot\epsilon^\alpha\ \mathcal{A}\cdot\epsilon^\alpha}{q^2}.$$



The full e^- a scattering is written as

$$\sigma(e^-(p)a \to e^-(p')X) = \frac{1}{2S}\overline{\Sigma}|\mathcal{M}|^2 \frac{d\vec{p}'}{(2\pi)^3 2E'}\ dPS_X \equiv \int dx\ P_{\gamma/e}(x)\ \sigma(\gamma a),$$

where $P_{\gamma/e}(x)$ is defined to be the probability distribution of finding a photon with an energy xE off an electron with an energy E. To find this function, we need to work on the factors on both sides of the identity.

First, for a polarized on-shell photon, the squared matrix element for the full process reads

$$|\mathcal{M}^\alpha|^2 = e^2 \frac{|\overline{u_{e'}}\ \not\epsilon^\alpha u_e|^2\ |\mathcal{A}\cdot\epsilon^\alpha|^2}{(q^2)^2}.$$

Using the subprocess cross section

$$\sigma(\gamma^\alpha a) = \frac{1}{2xS}\ |\mathcal{A}\cdot\epsilon^\alpha|^2\ dPS_X,$$

we have

$$\sigma(e^- a \to e^- X)^\alpha = 4\pi\alpha_{em} x\ \frac{|\overline{u_{e'}}\ \not\epsilon^\alpha u_e|^2}{(q^2)^2}\ \frac{d\vec{p}'}{(2\pi)^3 2E'}\ \sigma(\gamma^\alpha a).$$

Dynamics:

$$\overline{\Sigma}_{spin}|\overline{u_{e'}}\ \not\epsilon^\alpha u_e|^2 = \frac{1}{2}\ Tr(\not p'\gamma^\mu \not p\gamma^\nu)\ \epsilon^\alpha_\mu\epsilon^\alpha_\nu = 2(2p'\cdot\epsilon^\alpha p\cdot\epsilon^\alpha - p'\cdot p\ \epsilon^\alpha\cdot\epsilon^\alpha). \qquad (4)$$

Kinematics:

$$p = (E,0,0,E), \quad p' = (E', E'\sin\theta, 0, E'\cos\theta), \quad E' = (1-x)E,$$
$$q = p - p' = (E-E', -E'\sin\theta, 0, E - E'\cos\theta) = (xE, -E(1-x)\sin\theta, 0, E(1-(1-x)\cos\theta)),$$
$$q^2 = -2p\cdot p' = -2EE'(1-\cos\theta) \approx -EE'\theta'^2, \quad q_T = q_x, \quad |\vec{q}|^2 = E^2 + E'^2 - 2EE'\cos\theta.$$

Physical polarizations for the photon ($\alpha = x, y$):

$$\epsilon^x = (|\vec{q}|q_T)^{-1}\ (0, q_x q_z, q_y q_z, -q_T^2) = |\vec{q}|^{-1}\ (0, q_z, 0, -q_x),$$
$$\epsilon^y = q_T^{-1}\ (0, -q_y, q_x, 0) = (0,0,1,0), \quad \epsilon^\alpha\cdot\epsilon^{\alpha'} = -\delta_{\alpha\alpha'}.$$

Thus the matrix element factor leads to

$$\begin{aligned}\overline{\Sigma}_{spin}|\overline{u_{e'}}\ \not\epsilon^{\alpha}u_e|^2 &= 4p'\cdot\epsilon^{\alpha}p\cdot\epsilon^{\alpha} - q^2\delta_{\alpha\alpha'} \\ &= 4\frac{E^2E'^2\sin^2\theta}{|\vec{q}|^2} - q^2 \quad (\alpha = x), \quad \text{or} \quad -q^2 \quad (\alpha = y).\end{aligned}$$

Thus the summed matrix element over α is

$$\begin{aligned}\overline{\Sigma}^{\alpha}_{spin}|\overline{u_{e'}}\ \not\epsilon^{x}u_e|^2 &= 4EE'\ \left(\frac{EE'\sin^2\theta}{|\vec{q}|^2} + (1-\cos\theta)\right) \\ &= 4EE'(1-\cos\theta)\ \left(\frac{EE'(1+\cos\theta)}{E^2+E'^2-2EE'\cos\theta} + 1\right).\end{aligned}$$

Note this $1-\cos\theta$ factor keeps the collinear divergence logarithmic, which is due to the vector coupling, not there for a scalar coupling.

Phase space integral:

$$\frac{d\vec{p}'}{(2\pi)^3 2E'} = \frac{|\vec{p}'|^2 d\vec{p}'\ d\Omega'}{(2\pi)^3 2E'} = \frac{E'dE'\ d\cos\theta}{2(2\pi)^2} = \frac{EE'dx\ d\cos\theta|^{1-\delta}_{const.}}{8\pi^2}.$$

Under collinear approximation and take the dominant contribution near $\cos\theta = 1-\delta$, expressed with the dimensionless quantity $\delta \approx m_e^2/E^2$, we have

$$\begin{aligned}\frac{\overline{\Sigma}^{\alpha}_{spin}|\overline{u_{e'}}\ \not\epsilon^{x}u_e|^2}{(q^2)^2} &= 4EE'(1-\cos\theta)\ \left(\frac{EE'(1+\cos\theta)}{E^2+E'^2-2EE'\cos\theta} + 1\right)/4E^2E'^2(1-\cos\theta)^2 \\ &= \left(\frac{EE'(1+\cos\theta)}{E^2+E'^2-2EE'\cos\theta} + 1\right)\frac{1}{EE'(1-\cos\theta)} \\ &\approx \left(\frac{2}{(E-E')^2} + \frac{1}{EE'}\right)\frac{1}{(1-\cos\theta)}.\end{aligned}$$

Putting everything together, the cross section for the full process summing over the transverse photos ($\alpha = 1, 2$) reads

$$\begin{aligned}\sigma(e^- a \to e^- X) &= 4\pi\alpha_{em}\ \frac{x\ \overline{\Sigma}^{\alpha}_{spin}|\overline{u_{e'}}\ \not\epsilon^{\alpha}u_e|^2}{(q^2)^2}\ \frac{d\vec{p}'}{(2\pi)^3 2E'}\ \sigma(\gamma^{\alpha}a) \\ &= \frac{\alpha_{em}}{2\pi}\ \frac{d\cos\theta|^{1-\delta}_{const.}}{1-\cos\theta}\ \left(\frac{2EE'}{(E-E')^2} + 1\right)xdx\ \sigma(\gamma^{\alpha}a) \\ &\approx \frac{\alpha_{em}}{2\pi}\ \ln\frac{E^2}{m_e^2}\ \left(\frac{2(1-x)}{x^2} + 1\right)xdx\ \sigma(\gamma^{\alpha}a).\end{aligned}$$

The final answer for the photo spectrum as a distribution function is

$$\begin{aligned}P_{\gamma/e}(x) &\approx \frac{\alpha}{2\pi}\frac{x^2+2(1-x)}{x}\ln\frac{E^2}{m_e^2} \\ &= \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\ln\frac{E^2}{m_e^2}.\end{aligned}$$

`Sol. to Exercise 4.1:`

(To get familiar with the commonly used kinematical variables at hadron colliders.)

For a four-momentum $p \equiv p^\mu = (E, \vec{p})$, define

$$E_T = \sqrt{p_T^2 + m^2}, \quad p_T^2 = p_x^2 + p_y^2, \quad y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z}.$$

Then

$$\sinh y = \frac{e^y - e^{-y}}{2} = \frac{1}{2} \left[\sqrt{\frac{E + p_z}{E - p_z}} - \sqrt{\frac{E - p_z}{E + p_z}} \right] = \frac{p_z}{E_T}, \quad \cosh y = \frac{E}{E_T},$$

and we have

$$p^\mu = (E_T \cosh y,\ p_T \cos\phi,\ p_T \sin\phi,\ E_T \sinh y),$$
$$\frac{d^3\vec{p}}{E} = p_T dp_T d\phi\ dy = E_T dE_T d\phi\ dy.$$

Due to the random boost between the Lab-frame (O) and the c.m. frame (O') for every event,

$$y' = \frac{1}{2} \ln \frac{E' + p'_z}{E' - p'_z} = \frac{1}{2} \ln \frac{E - \beta_{cm} p_z + p_z - \beta_{cm} E}{E - \beta_{cm} p_z - p_z + \beta_{cm} E} = \frac{1}{2} \ln \frac{(1 - \beta_{cm})(E + p_z)}{(1 + \beta_{cm})(E - p_z)} = y - y_{cm},$$

where $\beta_{cm} = P_z/E_{cm}$ and y_{cm} are the speed and rapidity of the c.m. frame w.r.t. the lab frame (see next problem for more).

In the massless limit, the rapidity y defines the pseudo-rapidity:

$$y \to \eta = \frac{1}{2} \ln \frac{1 + \cos\theta}{1 - \cos\theta} = \ln \cot \frac{\theta}{2}.$$

`Sol. to Exercise 4.2:`

(To understand the Jacobian keep in p_T distribution.)

Let $p_T = p \sin\theta$, then $dp_T = p d\cos\theta$. Or

$$dp_T = p \frac{\cos\theta}{\sin\theta}\ d\cos\theta = p \frac{\sqrt{1 - p_T^2/p^2}}{p_T/p}\ d\cos\theta.$$

For a two-body massless final state with an invariant mass squared s, we have $s = 4p^2$. Thus

$$dp_T = \frac{p^2 \sqrt{1 - p_T^2/p^2}}{p_T}\ d\cos\theta.$$
$$\frac{d\hat{\sigma}}{dp_T} = \frac{4p_T}{s\sqrt{1 - 4p_T^2/s}}\ \frac{d\hat{\sigma}}{d\cos\theta}.$$

The integrand is singular at $p_T = \sqrt{s}/2$, but the integration is finite. Also, the sharp singularity will be smeared by the finite width of the resonant particle around $M^2 \approx s$.