Intercepts of π-meson correlation functions in deformed Bose gas models

Anastasiia Rebesh,
A. Gavrilik, Yu. Mishchenko

Bogolyubov Institute for Theoretical Physics
Kiev, Ukraine

WE-Heraeus Physics School
Difractive and electromagnetic properties at high energies
Bad-Honnef, August 17-21, 2015

Outline

- Background: deformed oscillators
- μ-Bose gas model
  - Intercepts of momentum correlation functions
  - Asymptotics of the intercepts
- Conclusions

- Quantum harmonic oscillator

$$aa^\dagger - a^\dagger a = 1.$$

These operators can be used to define the generators of $su(2)$ algebra.

Now if we q-deform the commutation relation we will obtain the generators of $su_q(2)$ algebra*.

- Deformed (non-linear) oscillator

$$aa^\dagger - qa^\dagger a = q^{-N},$$

where q is deformation parameter.

$$[a^\pm, a^\pm] = 0, \quad [N, a^\pm] = \pm a$$

The Fock space of the q-boson state is introduced according to the construction:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad N|n\rangle = n|n\rangle,$$

where the q-factorial $[n]_q! = [n]_q \cdot [n-1]_q \cdot ... \cdot [1]_q$, $[0]_q! = 1$.

$$a|n\rangle = \sqrt{[n]_q}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle, \qquad [N]_q = a^\dagger a.$$

Hamiltonian: $H = \frac{1}{2}(aa^\dagger + a^\dagger a)$, $\quad \hbar\omega = 1$, $\qquad H|n\rangle = E_n|n\rangle$.

* Biedenharn L.J., Phys. A: Math. Gen. 22 L873 (1989).
* Mcfarlane A., J. Phys. A: Math. Gen. 22 4581 (1989).

Now if we q-deform the commutation relation we will obtain the generators of $su_q(2)$ algebra*.

- Deformed (non-linear) oscillator

$$aa^\dagger - qa^\dagger a = q^{-N},$$

where q is deformation parameter.

$$[a^\pm, a^\pm] = 0, \quad [N, a^\pm] = \pm a$$

The Fock space of the q-boson state is introduced according to the construction:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad N|n\rangle = n|n\rangle,$$

where the q-factorial $[n]_q! = [n]_q \cdot [n-1]_q \cdot ... \cdot [1]_q$, $[0]_q! = 1$.

$$a|n\rangle = \sqrt{[n]_q}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle, \qquad [N]_q = a^\dagger a.$$

Hamiltonian: $H = \frac{1}{2}(aa^\dagger + a^\dagger a), \quad \hbar\omega = 1, \qquad H|n\rangle = E_n|n\rangle.$

*Biedenharn L.J., Phys. A: Math. Gen. 22 L873 (1989).
*Mcfarlane A., J. Phys. A: Math. Gen. 22 4581 (1989).

When studying deformed oscillators it is convenient to use the concept of structute function of deformation $\varphi(N)$:

$$a^\dagger a = \varphi(N), \qquad a a^\dagger = \varphi(N+1).$$

For the ordinary quantum oscillator: $a^\dagger a = N$, $a a^\dagger = N+1$.

Commutation relation for operators $a^\dagger$, a:

$$[a, a^\dagger] = \varphi(N+1) - \varphi(N).$$

In the q-analog of Fock space:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi(n)!}}|0\rangle, \quad N|n\rangle = n|n\rangle, \quad \varphi(N)|n\rangle = \varphi(n)|n\rangle,$$

where $\varphi(n)! = \varphi(n)\cdot\varphi(n-1)\cdot\ldots\cdot\varphi(1), \quad \varphi(0)! = 1.$

Models of deformed oscillators

Structure function	Energy spectrum
Arik-Coon model $\varphi_n^{\text{AC}} = \frac{q^n - 1}{q - 1}$	$E_n^{\text{AC}} = \frac{1}{2}\left(\frac{q^{n+1}-1}{q-1} + \frac{q^n-1}{q-1}\right)$
Biedenharn-Macfarlane model $\varphi_n^{\text{BM}} = \frac{q^n - q^{-n}}{q - q^{-1}}$	$E_n^{\text{BM}} = \frac{1}{2}\left(\frac{q^{n+1}-q^{-(n+1)}}{q-q^{-1}} + \frac{q^n-q^{-n}}{q-q^{-1}}\right)$
(p,q)-oscillator model $\varphi_n^{(\text{p,q})} = \frac{q^n - p^n}{q - p}$	$E_n^{(\text{p,q})} = \frac{1}{2}\left(\frac{q^{n+1}-p^{n+1}}{q-p} + \frac{q^n-p^n}{q-p}\right)$
Tamm-Dancoff model $\varphi_n^{\text{TD}} = nq^{n-1}$	$E_n^{\text{TD}} = \frac{1}{2}\left((n+1)q^n + nq^{n-1}\right)$
μ-oscillator * $\varphi_n^{\mu} = \frac{n}{1+\mu n}$	$E_n^{\mu} = \frac{1}{2}\left(\frac{n}{1+\mu n} + \frac{n+1}{1+\mu(n+1)}\right)$

* Jannussis A. J. Phys. A: Math. Gen. **26**, L233–L237, 1993

Towards application

Deformed oscillators have application in different fields of physics:

- molecular and nuclear spectroscopy,
- quantum optics,
- statistical mechanics,
- particle physics.

On the set of μ-oscillators we develop the respective deformed analog of Bose gas model (μ-Bose gas).

Why deformed analog of Bose gas model?

It allows us to take into account

- interaction between particles
- the internal structure of the particles
- interaction between particles + their internal structure*

* A.M. Gavrilik, Yu.A. Mishchenko, Ukr.J.Phys., 2013, 58, 1171-1177.

Towards application

Deformed oscillators have application in different fields of physics:

- molecular and nuclear spectroscopy,
- quantum optics,
- statistical mechanics,
- particle physics.

On the set of μ-oscillators we develop the respective deformed analog of Bose gas model (μ-Bose gas).

Why deformed analog of Bose gas model?

It allows us to take into account

- interaction between particles
- the internal structure of the particles
- interaction between particles + their internal structure*

* A.M. Gavrilik, Yu.A. Mishchenko, Ukr.J.Phys., 2013, **58**, 1171-1177.

In the experiments of relativistic heavy ion collisions, as the result of collisions, the secondary particles (π-mesons, K-mesons ect) are produced and then registered.

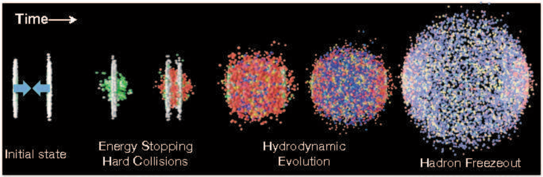
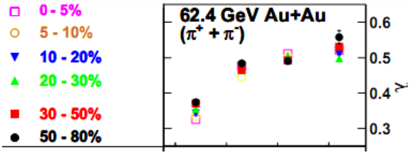


Fig.1 *The time evolution of a high energy heavy ion collision.*
(Tapan K. Naya, Pramana **79**, 719-735 (2012))



Two-particle momentum correlation function

$$C^{(2)}(k_1, k_2) = \gamma \frac{P_2(k_1, k_2)}{P_1(k_1)P_1(k_2)}$$

$$P_2(k_1, k_2) = E_1 E_2 \frac{dN}{d^3k_1 d^3k_2}, \qquad P_1(k_1) = E_1 \frac{dN}{d^3k_1}$$

Two-particle momentum correlation function $C^{(2)}(k_1, k_2)$:

$$C^{(2)}(k_1, k_2) = \gamma \frac{P_2(k_1, k_2)}{P_1(k_1)P_1(k_2)},$$

can be rewritten in variables $Q = k_1 - k_2$, $K = (k_1 + k_2)/2$:

$$C^{(2)}(Q, K) \xrightarrow{k_1=k_2} C^{(2)}(Q=0, K) = 1 + \lambda^{(2)}(m, \mathbf{K}),$$

$\lambda^{(2)}$ - intercept of two-particle correlation function. In terms of $a^\dagger$, a:

$$\lambda^{(2)}(K) = \frac{\langle a^\dagger a^\dagger aa\rangle}{\langle a^\dagger a\rangle^2} - 1 = \frac{\langle [N]_\mu [N-1]_\mu \rangle}{\langle [N]_\mu\rangle^2} - 1, \qquad a^\dagger a = [N]_\mu = \frac{N}{1+\mu N}.$$

Statistical average for a system with Hamiltonian H:

$$\langle N\rangle = \frac{TrNe^{-\beta\sum_k H_k}}{Tre^{-\beta\sum_k H_k}} = \frac{\sum_n \langle n|Ne^{-\beta\sum_k H_k}|n\rangle}{\sum_n \langle n|e^{-\beta\sum_k H_k}|n\rangle} = \frac{\sum_n ne^{-\beta\varepsilon n}}{\sum_n e^{-\beta\varepsilon n}} = \frac{1}{e^{\beta\varepsilon} - 1},$$

where $\beta = \frac{1}{T}$, $k = 1$. Analogously one can obtain $\langle N^r\rangle$, $r \geq 2$.

We choose the Hamiltonian in the form

$$H = \sum_{\mathbf{k}} \hbar \varepsilon_{\mathbf{k}} N_{\mathbf{k}}, \qquad \varepsilon = (m_\pi^2 + \mathbf{K}^2)^{1/2}.$$

Intercept of two-particle correlation function*:

$$\lambda_\mu^{(2)} = \Big\{ X^{-1} - \Big(\frac{1}{\mu} + \frac{1}{\mu^2}\Big)\Phi(e^{-\beta}, 1, \mu^{-1}) - \Big(\frac{1}{\mu} - \frac{1}{\mu^2}\Big)\Phi(e^{-\beta}, 1, \mu^{-1} - 1)\Big\} \times$$
$$\times \Big(X^{-1} - \mu^{-1}\Phi(e^{-\beta}, 1, \mu^{-1})\Big)^{-2} X^{-1} - 1, \quad (1 - e^{-\beta}) = X$$

Here Φ is Lerch transcedent: $\Phi = \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$.

Intercept of three-particle correlation function: $\lambda^{(3)}(K) = \frac{\langle a^\dagger a^\dagger a^\dagger aaa \rangle}{\langle a^\dagger a \rangle^3} - 1$

$$\lambda_\mu^{(3)} = X^{-2}\Big\{ X^{-1} - \Big(\frac{1}{\mu} + \frac{3}{2\mu^2} + \frac{1}{2\mu^3}\Big)\Phi(e^{-\beta}, 1, \mu^{-1}) - \Big(\frac{1}{\mu} - \frac{1}{\mu^3}\Big)\Phi(e^{-\beta}, 1, \mu^{-1} - 1) -$$
$$- \Big(\frac{1}{\mu} - \frac{3}{2\mu^2} + \frac{1}{2\mu^3}\Big)\Phi(e^{-\beta}, 1, \mu^{-1} - 2)\Big\} \cdot \Big(X^{-1} - \mu^{-1}\Phi(e^{-\beta}, 1, \mu^{-1})\Big)^{-3} - 1.$$

* Gavrilik A.M., Mishchenko Yu.A., Phys. Lett. A **376**, 2484-2489 (2012).

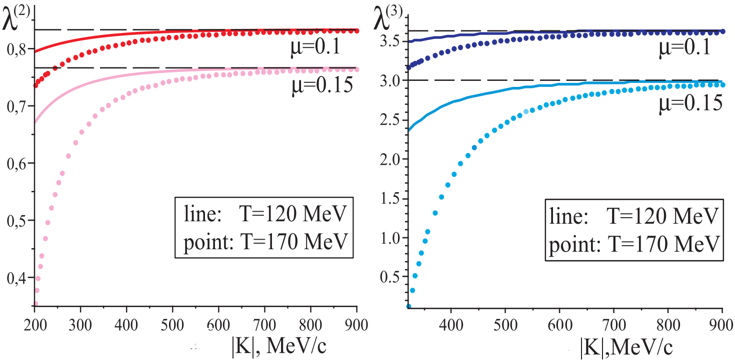


Fig.2. *Left*: Dependence of intercept $\lambda^{(2)}$ on the momentum. For curve μ=0.1 the asymptote $\lambda^{(2)}_{as.} = 0.83$, for curve μ=0.15 asymptote $\lambda^{(2)}_{as.} = 0.7664$.

Right: Dependence of intercept $\lambda^{(3)}$ on the momentum. For curve μ=0.1 the asymptote $\lambda^{(3)}_{as.} = 3.6365$, for curve μ=0.15 the asymptote $\lambda^{(3)}_{as.} = 2.9964$.

* Gavrilik A.M., Rebesh A.P., Eur.Phys.J.A 47:55, 8 pp. (2011).

Intercept of r-particle correlation function

$$\lambda^{(r)}(K) = \frac{\langle (a^\dagger)^r (a)^r \rangle}{\langle a^\dagger a \rangle^r} - 1 = \frac{\langle [N]_\mu [N-1]_\mu \cdot \ldots \cdot [N-r+1]_\mu \rangle}{\langle [N]_\mu \rangle^r} - 1.$$

$$\lambda^{(r)}_\mu(k) = \Big(1 + \mu^{-1}(1-e^{-\beta\hbar\omega}) \sum_{l=0}^{r-1} A^{(r)}_l(\mu) \Phi(e^{-\beta\hbar\omega}, 1, \mu^{-1} - l)\Big) \times$$
$$\times \Big(1 + \mu^{-1}(1-e^{-\beta\hbar\omega}) A^{(1)}_0(\mu) \Phi(e^{-\beta\hbar\omega}, 1, \mu^{-1})\Big)^{-r} - 1, \; r = 2, 3, \ldots, \; \mu > 0$$

The coefficients $A^{(r)}_l$:

$$A^{(1)}_0(\mu) = -1;$$

$$A^{(2)}_0(\mu) = -1 - \frac{1}{\mu}, \quad A^{(2)}_1(\mu) = -1 + \frac{1}{\mu};$$

$$A^{(3)}_0(\mu) = -1 - \frac{3}{2\mu} - \frac{1}{2\mu^2}, \quad A^{(3)}_1(\mu) = -1 + \frac{1}{\mu^2}, \quad A^{(3)}_2(\mu) = -1 + \frac{3}{2\mu} - \frac{1}{2\mu^2};$$

$\ldots$.

Asymptotics of the intercepts: $\beta\omega \to \infty$

$$\lambda^{(r)}_{\mu,\, asympt} = \lim_{\omega\to\infty} \frac{\sum_{n=0}^{\infty} \frac{n}{1+\mu n} \cdot \ldots \cdot \frac{n-r+1}{1+\mu(n-r+1)} e^{-\beta\hbar\omega n}}{(1-e^{-\beta\hbar\omega})^{r-1} \left(\sum_{n=0}^{\infty} \frac{n}{1+\mu n} e^{-\beta\hbar\omega n} \right)^{r}} - 1 =$$

$$= \lim_{\omega\to\infty} \frac{[r]_\mu! e^{-\beta\hbar\omega r} + \ldots}{\left(\frac{1}{1+\mu}\right)^{r} e^{-\beta\hbar\omega r} + \ldots} - 1 = \left(1+\mu\right)^{r} [r]_\mu! - 1,$$

$$[r]_\mu! \equiv [r]_\mu [r-1]_\mu \ldots [1]_\mu .$$

For $r = 2$ and $r = 3$ this result is in complete agreement with the corresponding asymptotical values of the μ-Bose gas intercepts $\lambda^{(2)}$ and $\lambda^{(3)}$:

$$\lambda^{(2)}_{\mu,\, asympt} = \left(1+\mu\right)^{2} [2]_\mu! - 1 = \frac{1}{1+2\mu},$$

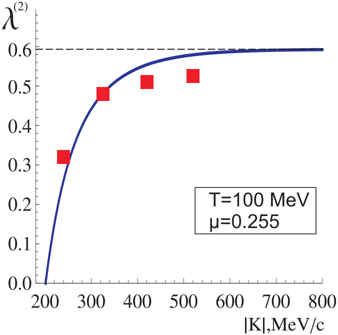
$$\lambda^{(3)}_{\mu,\, asympt} = \left(1+\mu\right)^{3} [3]_\mu! - 1 = \frac{5+7\mu}{(1+2\mu)(1+3\mu)}.$$

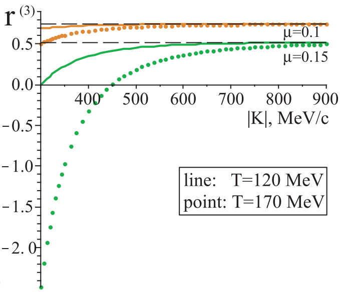
$$r^{(3)}(K) = \frac{1}{2}\frac{\lambda^{(3)}(K) - 3\lambda^{(2)}(K)}{(\lambda^{(2)}(K))^{3/2}}.$$

The importance of the function - all undesirable perverting effects are canceled (e.g. contributions from resonances).

The expression for the asymptotics of r-function

$$r^{(3)}_{as.}(\mu) = \frac{1-\mu}{1+3\mu}\sqrt{1+2\mu}.$$





B.I. Abelev *et al.* (STAR Collab.), Phys. Rev. C **80**, 024905 (2009).

Conclusions

- Deformed oscillators have unusual properties compared with ordinary quantum oscillator.
- On the set of μ-oscillators we realize the μ-Bose gas model.
- In the framework of μ-Bose gas model we obtained the intercepts of two- , three-particle momentum correlation function as well as the expression for the r-particle correlation function.
- The asymptotics of the intercepts are also obtained.
- Theoretically obtained results for π-meson two-particle correlation function are in a good agreement with experimental data on pion correlations in relativistic heavy ion collisions STAR/RHIC.

Thank you for your attention!

Connection between $a^\dagger$, a and $b^\dagger$, b

The connection between $a^\dagger$, a and the operators $b^\dagger$, b of usual quantum oscillator:

$$a^\dagger = \sqrt{\frac{[N]_q}{N}}\, b^\dagger, \quad a = b\sqrt{\frac{[N]_q}{N}}, \quad \text{де} \quad [N]_q = \frac{1-q^N}{1-q}.$$

Coordinate representation of $a^\dagger$, a:

$$a = \frac{e^{-2i\alpha x} - e^{i\alpha d/dx} e^{-i\alpha x}}{-i\sqrt{1-e^{-2\alpha^2}}}, \quad a^\dagger = \frac{e^{2i\alpha x} - e^{i\alpha} e^{i\alpha d/dx}}{i\sqrt{1-e^{-2\alpha^2}}}, \quad \text{де} \quad \alpha = \sqrt{-\ln q/2}.$$

The corresponding operators of coordinate and momentum in q-deformed quantum mechanics:

$$\hat{x} = \frac{a + a^\dagger}{\sqrt{2}} = \sqrt{\frac{2}{1-e^{-2\alpha^2}}} \left(\sin(2\alpha x) - e^{\alpha^2/2} \sin\left(\alpha x + \frac{\alpha^2}{2}\, i\right) e^{i\alpha d/dx} \right),$$

$$\hat{p} = \frac{a - a^\dagger}{i\sqrt{2}} = \sqrt{\frac{2}{1-e^{-2\alpha^2}}} \left(\cos(2\alpha x) - e^{\alpha^2/2} \cos\left(\alpha x + \frac{\alpha^2}{2}\, i\right) e^{i\alpha d/dx} \right).$$

- Usual quantum oscillator: there is no level degeneracy in the energy spectrum ($d = 1$).
- In more general and complicated cases different types of energy level can exist (V.N. Zakhariev).

Spectrum of p, q-oscillator, $0 < p \leq 1, \quad 0 < q \leq 1$.

The degeneracy of energy levels exist:

$\mathbf{E_n} = \mathbf{E_0}$,
$\mathbf{E_n} = \mathbf{E_{n+1}}$,
$\mathbf{E_n} = \mathbf{E_{n+2}}$, $n \geq 2$.

$$E.g.: \quad E_n^{(p,q)} - E_0^{(p,q)} = F_{n,0}(q,p) =$$

$$= \sum_{r=0}^{n} p^{n-r} q^r + \sum_{s=0}^{n-1} p^{n-1-s} q^s - 1 = 0,$$