Jacobi Mapping Approach for a precise Cosmological Weak Lensing Formalism

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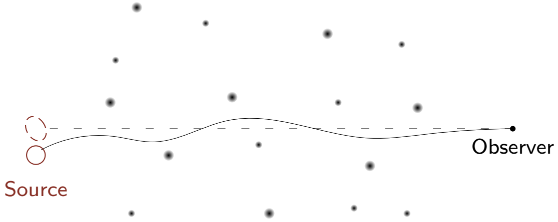
Cosmological Weak Lensing

- Ideal probe for large-scale matter distribution of the universe.
- Constrain cosmological parameters, test modified gravity, ...
- Upcoming observations: DES, LSST, Euclid, WFIRST.



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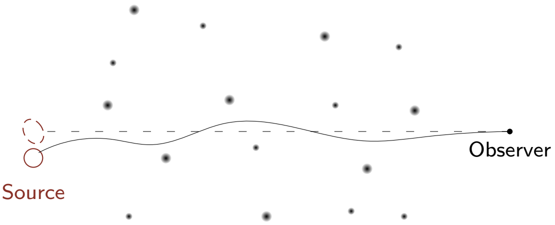

We need a precise theoretical framework to correctly interpret results!

Standard Weak Lensing Formalism

A source that we observe at an angle θ_i has been distorted by an angle $\alpha_i = \theta_i - \beta_i$, determined by the **lensing potential**,

$$\alpha_i = \partial_i \mathbf{\Phi}\,, \quad \mathbf{\Phi} = \int_0^{\bar{r}_s} \mathrm{d}\bar{r} \left(\frac{\bar{r}_s - \bar{r}}{\bar{r}_s \bar{r}} \right) 2\psi\,,$$

ψ ... gravitational potential along the light path.

Amplification (distortion) matrix:

$$\mathbb{D}_{ij} = \frac{\partial \beta_i}{\partial \theta_j} = \delta_{ij} - \partial_j \partial_i \mathbf{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa + \gamma_1 & \omega + \gamma_2 \\ -\omega + \gamma_2 & \kappa - \gamma_1 \end{pmatrix}\,,$$

κ ... convergence, (γ_1, γ_2) ... shear components, ω ... rotation.

Gauge Freedom

Perturbed flat FLRW metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\,a^2(\tau)(1+2\mathcal{A})\mathrm{d}\tau^2 - a^2(\tau)\mathcal{B}_\alpha\mathrm{d}\tau\mathrm{d}x^\alpha + a^2(\tau)\left(\delta_{\alpha\beta}+2\mathcal{C}_{\alpha\beta}\right)\mathrm{d}x^\alpha\mathrm{d}x^\beta\,.$$

Scalar, vector and tensor components:

$$\mathcal{A}=\alpha,\quad \mathcal{B}_\alpha=\beta_{,\alpha}+B_\alpha,\quad \mathcal{C}_{\alpha\beta}=\varphi\,\delta_{\alpha\beta}+\gamma_{,\alpha\beta}+C_{(\alpha,\beta)}+C_{\alpha\beta}\,.$$

10 degrees of freedom, **but only 6 of them are physical!**

Newtonian Gauge

Newtonian gauge:

$$\mathrm{d}s^2 = -a^2(\tau)(1+2\psi)\mathrm{d}\tau^2 + a^2(\tau)(1+2\phi)\delta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \,.$$

- One specific gauge choice, only scalar modes are considered.
- Observable quantities should not depend on the gauge choice.

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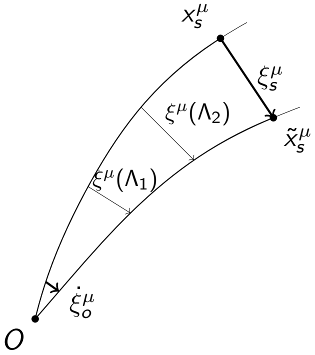
Do we get the same results in any other gauge?

Result: The Standard Formalism does not yield gauge-invariant expressions for the convergence, the shear components and the rotation!

- The expressions for κ, γ_1, γ_2 and ω obtain gauge-dependent terms evaluated at the source position.
- The standard formalism does not correctly capture all physical effects.
- For high precision cosmology, we need an alternative formalism!

Jacobi Mapping Approach



Geodesic deviation equation:

$$\frac{\mathrm{D}^2\xi^\mu(\Lambda)}{\mathrm{d}\Lambda^2} = R^\mu{}_{\nu\sigma\tau}k^\nu k^\sigma \xi^\tau \,.$$

Jacobi Map $\mathcal{J}_\mu^\nu(\Lambda)$:

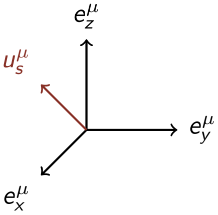
$$\xi^\nu(\Lambda) \equiv \mathcal{J}_\mu^\nu(\Lambda)\dot{\xi}_o^\mu \,, \quad \dot{\xi}^\mu(\Lambda_o) \equiv \frac{\mathrm{d}}{\mathrm{d}\Lambda}\xi^\mu(\Lambda)\bigg|_{\Lambda_o} \,.$$

Propagation equation of Jacobi Map:

$$\frac{\mathrm{D}^2\mathcal{J}_\mu^\nu(\Lambda)}{\mathrm{d}\Lambda^2} = \left(R^\nu{}_{\sigma\tau\kappa}k^\sigma k^\tau\right)\mathcal{J}_\mu^\kappa(\Lambda)\,.$$

- The vectors $\dot{\xi}_o^\mu$ and ξ_s^μ live in the global spacetime manifold.
- Gravitational lensing affects the **size** and **shape** of an object.

$\Rightarrow$ We need to transform $\dot{\xi}_o^\mu$ (or ξ_s^μ) to the local Lorentz frame of an observer moving with velocity u_o^μ (or u_s^μ).



Local orthonormal tetrads:

$$e_t^\mu = u^\mu \,, \quad \eta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu} \,.$$

Separation vector in local coordinates:

$$\xi_s^t = (\xi^\mu u_\mu)_s = 0 \,, \quad \xi_s^i = \left(\xi^\mu e_\mu^i\right)_s \,.$$

2-dimensional separation vector: $\xi_s^1 = \left(\xi^i \theta_i\right)_s$ and $\xi_s^2 = \left(\xi^i \phi_i\right)_s$.

The 2×2-dimensional Jacobi Map

Define 2 tetrads: $[e_1]^\mu_o = \left(e^\mu_i \theta^i\right)_o$ and $[e_2]^\mu_o = \left(e^\mu_i \phi^i\right)_o$.

Parallel transport:

$$\frac{\mathrm{D}[e_I]^\mu(\Lambda)}{\mathrm{D}\Lambda} = 0\,, \quad I = 1, 2.$$

The 2×2-dimensional Jacobi map $\mathfrak{D}^I{}_J$ is defined by:

$$\xi^I \equiv \xi^\mu [e^I]_\mu\,, \qquad \xi^I \equiv \mathfrak{D}^I{}_J(\Lambda)\dot{\xi}^J_o\,, \qquad \dot{\xi}^I_o \equiv \frac{\mathrm{d}}{\mathrm{d}\Lambda}\xi^I\bigg|_{\Lambda_o}\,.$$

Propagation equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\Lambda^2}\mathfrak{D}^I{}_J = -\mathfrak{R}^I{}_K \mathfrak{D}^K{}_J\,, \qquad \text{where} \qquad \mathfrak{R}^I{}_J \equiv \left(R^\mu{}_{\nu\sigma\tau} k^\nu k^\tau\right)[e^I]_\mu [e_J]^\sigma\,.$$

Simplification: Use conformally transformed metric, $a^2 \hat{g}_{ab} = g_{ab}$.

Result to Linear Order

Jacobi Map to first order:

$$\hat{\mathfrak{D}}^I{}_J(\lambda_s) = \lambda_s \delta^I_J + \lambda_s \hat{\mathfrak{D}}^{I(1)}_J = \lambda_s \delta^I_J - \lambda_s \int_0^{\lambda_s} \mathrm{d}\lambda \left(\frac{\lambda_s - \lambda}{\lambda_s \lambda} \right) \lambda^2 \hat{\mathfrak{R}}^I{}_J(\lambda) \,.$$

Amplification (distortion) matrix:

$$\xi^I_s = \tilde{\mathbb{D}}^I{}_J \bar{\xi}^J_s \,, \quad \tilde{\mathbb{D}}^I{}_J = \frac{1}{\lambda_z} (1 + \delta z) \, \hat{\mathfrak{D}}^I{}_J \,.$$

Decomposition:

$$\tilde{\mathbb{D}}^I{}_J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \tilde{\kappa} + \tilde{\gamma}_1 & \tilde{\gamma}_2 + \tilde{\omega} \\ \tilde{\gamma}_2 - \tilde{\omega} & \tilde{\kappa} - \tilde{\gamma}_1 \end{pmatrix} \,,$$

$\tilde{\kappa} \ldots$ convergence, $(\tilde{\gamma}_1, \tilde{\gamma}_2) \ldots$ shear components, $\tilde{\omega} \ldots$ rotation.

Shear and Convergence

Convergence:

$$\tilde{\kappa} = \kappa + \frac{n_\alpha \mathcal{G}^\alpha}{\bar{r}_z} - \frac{\widehat{\nabla}_\alpha \mathcal{G}^\alpha}{2\bar{r}_z} - \delta z_\chi - \frac{\delta r_\chi}{\bar{r}_z} - \varphi_\chi + n^\alpha n^\beta C_{\alpha\beta} \,.$$

This quantity is gauge-invariant and describes the distortion in the luminosity distance: $\tilde{\kappa} = -\delta D_L$.

Shear components:

$$\tilde{\gamma}_1 = \gamma_1 + \frac{1}{2}(\phi_\alpha \phi^\beta - \theta_\alpha \theta^\beta)\mathcal{G}_{\alpha,\beta} + \frac{1}{2}(\phi^\alpha \phi^\beta - \theta^\alpha \theta^\beta) C_{\alpha\beta} \,,$$

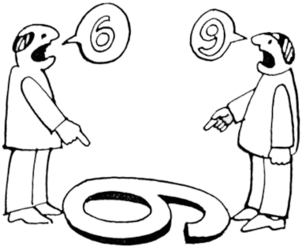
$$\tilde{\gamma}_2 = \gamma_2 - \frac{1}{2}(\theta^\alpha \phi^\beta + \theta^\beta \phi^\alpha)\mathcal{G}_{\alpha,\beta} - \theta^\alpha \phi^\beta C_{\alpha\beta} \,.$$

Comparison to standard formalism: Gauge-invariant and additional tensor contribution at source position.

The Vanishing Rotation

In the Jacobi mapping formalism, we have $\tilde{\omega} = 0$. But:

$$\omega = \Omega_o^n + \frac{1}{2}\left(\theta_\alpha\phi_\beta - \phi_\alpha\theta_\beta\right)\left[\int_0^{\bar{r}_z} \mathrm{d}\bar{r}\,\frac{\partial}{\partial x_\beta}\left(\Psi^\alpha + 2C^\alpha_\gamma n^\gamma\right) + \mathcal{G}^{\alpha,\beta}\right].$$

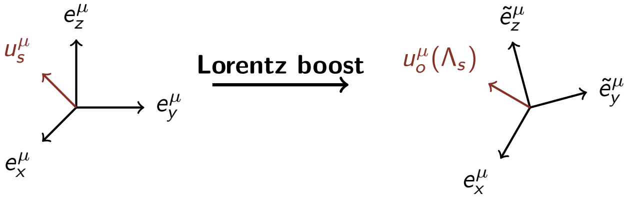


- What we measure depends on the orientation of our observer basis!
- Tetrads $[e_1]^\mu$, $[e_2]^\mu$: One rotational degree of freedom, both at the observer and the source.
- The correspondence between the source and observer bases is fixed by parallel transport!

Conclusion: The rotation is indeed vanishing (to linear order)!

Source Velocity vs. Parallel-Transported Velocity

Source velocity $u_s^a \neq$ parallel-transported velocity $u_o^a(\Lambda_s)$



- Observed photon directions n_s^i and $\tilde{n}_s^i$ differ in these frames.
- Lewis & Challinor (2006), Mitsou & Yoo (in preparation): Effect of Lorentz boost fully absorbed in transformation of n_s^i.
- Quantities orthogonal to n_s^i are unaffected:

$$\boxed{\xi_s^1 = \left(\xi_i \theta_\perp^i\right)_s = \left(\tilde{\xi}_i \tilde{\theta}_\perp^i\right)_s = \tilde{\xi}_s^1}$$

Conclusion

- Jacobi mapping approach yields gauge-invariant results for weak lensing effects.
- Standard formalism: Only accurate for scalar perturbations to linear order.
- Different results for vector and tensor perturbations.
- The rotation vanishes to linear order!

Two upcoming papers:

- "Gauge-Invariant Formalism of Cosmological Weak Lensing" (J. Yoo et al., in preparation)
- "Jacobi Mapping Approach for a precise Weak Cosmological Lensing Formalism" (N. Grimm and J. Yoo, in preparation)