New results on collectivity in small systems with ATLAS

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(on behalf of the ATLAS Collaboration)
Motivation and recent ATLAS results

- Evidence for long range correlations in $\Delta \eta$ in case of particle pairs produced at small $\Delta \phi$ (ridge) in $p+Pb$ and $pp$ systems.
- Try to understand its origin - is it initial-, final- or mixed-state effect?

The following measurements will be reviewed in this talk:

- Measurement of long-range azimuthal correlations in $Z$-boson tagged $pp$ collisions at $\sqrt{s} = 8$ TeV. ATLAS-CONF-2017-068
- $D$ meson production and long-range azimuthal correlations in 8.16 TeV $p+Pb$ collisions with ATLAS. ATLAS-CONF-2017-073
Two-particle correlations and the template fitting method

- Two-particle correlation function (and focus on LRC):
  \[
  C(\Delta \eta, \Delta \phi) = \frac{S(\Delta \eta, \Delta \phi)}{B(\Delta \eta, \Delta \phi)} \frac{\int_2^5 d|\Delta \eta|}{\int_2^5 d|\Delta \eta|} S(\Delta \phi) = C(\Delta \phi)
  \]

  - \(S\) and \(B\) are pair distributions constructed from the same and from "mixed events" (similar \(N_{\text{ch}}^\text{rec}\) and \(z_{\text{vtx}}\)).
  - Ratio \(S/B\) removes correlations due to detector effects.

- Per-trigger-particle yield - the average number of associated particles per trigger particle in a given \(\Delta \phi\) range:
  \[
  Y(\Delta \phi) = \frac{1}{2\pi N_{\text{trig}}} \left( \int_{-\pi/2}^{3\pi/2} B(\Delta \phi) d\Delta \phi \right) C(\Delta \phi)
  \]

- Template fitting procedure is applied to separate ridge from other sources of angular correlations (e.g. dijets):
  \[
  Y_{\text{templ}}(\Delta \phi) = G \left( 1 + \sum_{n=2}^{\infty} 2v_{n,n} \cos(n \Delta \phi) \right) + F Y_{\text{periph}}(\Delta \phi)
  \]

  where \(Y_{\text{periph}}(\Delta \phi)\) is obtained from low-multiplicity events.

- Importance of higher order harmonics is also visualized.
Cumulants in the standard method

- The standard cumulant method is based on the \( k \)-particle azimuthal correlations, \( \langle \{k\} \rangle \):
  \[
  \langle \{2\}_n \rangle = \langle e^{in(\phi_1 - \phi_2)} \rangle, \quad \langle \{3\}_n \rangle = \langle e^{in(\phi_1 + \phi_2 - 2\phi_3)} \rangle, \quad \langle \{4\}_{n,m} \rangle = \langle e^{in(\phi_1 - \phi_2) + im(\phi_3 - \phi_4)} \rangle
  \]

- The 2- and 4-particle cumulants are then defined as:
  \[
  c_n \{2\} = \langle \langle \{2\}_n \rangle \rangle, \quad c_n \{4\} = \langle \langle \{4\}_n \rangle \rangle - 2\langle \langle \{2\}_n \rangle \rangle^2 \quad \text{where} \quad \langle \{4\}_n \rangle \equiv \langle \{4\}_{n,n} \rangle
  \]

- The multi-particle symmetric and asymmetric cumulants are obtained from \( \langle \{k\} \rangle \) as:
  \[
  ac_n \{3\} = \langle \langle \{3\}_n \rangle \rangle, \quad sc_{n,m} \{4\} = \langle \langle \{4\}_{n,m} \rangle \rangle - \langle \langle \{2\}_n \rangle \rangle \langle \langle \{2\}_m \rangle \rangle
  \]

- In the absence of non-flow correlations, \( c_n \{2\}, \ c_n \{4\}, \ ac_n \{3\} \) and \( sc_{n,m} \{4\} \) read:
  \[
  c_n \{2\} = v_n^2, \quad c_n \{4\} = -v_n^4, \quad sc_{n,m} \{4\} = \langle v_n^2 v_m^2 \rangle - \langle v_n^2 \rangle \langle v_m^2 \rangle
  \]
  \[
  ac_n \{3\} = \langle v_n^2 v_{2n} \cos 2n(\Phi_n - \Phi_{2n}) \rangle
  \]

- Normalized cumulants:
  \[
  nsc_{n,m} \{4\} = \frac{sc_{n,m} \{4\}}{v_n \{2\}^2 v_m \{2\}^2} = \frac{\langle v_n^2 v_m^2 \rangle}{\langle v_n^2 \rangle \langle v_m^2 \rangle} - 1
  \]
  \[
  nac_n \{3\} = \frac{ac_n \{3\}}{v_n \{2\}^2 \sqrt{v_{2n} \{2\}^2}} = \frac{\langle v_n^2 v_{2n} \cos 2n(\Phi_n - \Phi_{2n}) \rangle}{\langle v_n^2 \rangle \sqrt{\langle v_{2n}^2 \rangle}}
  \]

where the \( v_n \{2\}^2 = \langle v_n^2 \rangle \) are flow harmonics obtained using a 2-particle correlation method based on the template fitting method.
To suppress the non-flow correlations, that usually involve few particles within a localized region in \( \eta \), the tracks are divided into several subevents, each covering a unique \( \eta \) range. Multi-particle correlations are constructed by correlating tracks from different subevents.

- **Two subevents (2SE)** - removes intra-jet correlations

\[
\langle \{ 2 \} \rangle_{a|b} = \left\langle e^{i \eta (\phi^a_1 - \phi^b_2)} \right\rangle \\
\langle \{ 3 \} \rangle_{2a|b} = \left\langle e^{i \eta (\phi^a_1 + \phi^b_2 - 2 \phi^b_3)} \right\rangle \\
\langle \{ 4 \} \rangle_{n,m}_{2a|2b} = \left\langle e^{i \eta (\phi^a_1 - \phi^b_2) + i m (\phi^a_3 - \phi^b_4)} \right\rangle
\]

- **Three subevents (3SE)** - removes inter-jet correlations

\[
\langle \{ 3 \} \rangle_{a,b|c} = \left\langle e^{i \eta (\phi^a_1 + \phi^b_2 - 2 \phi^c_3)} \right\rangle \\
\langle \{ 4 \} \rangle_{n,m}_{a,b|2c} = \left\langle e^{i \eta (\phi^a_1 - \phi^b_2) + i m (\phi^a_3 - \phi^b_4)} \right\rangle \\
\langle \{ 4 \} \rangle_{n,m}_{a,b|2c} = \left\langle e^{i \eta (\phi^a_1 - \phi^b_2) + i m (\phi^a_3 - \phi^b_4)} \right\rangle
\]

- **Four subevents (4SE)** - removes partly inter-jet correlations also in case of jets belonging to two adjacent subevents

\[
\langle \{ 4 \} \rangle_{n,m}_{a,b|c,d} = \left\langle e^{i \eta (\phi^a_1 - \phi^b_2) + i m (\phi^a_3 - \phi^b_4)} \right\rangle \\
sc^n_{a,b,c,d} \{ 4 \} = \langle \{ 4 \} \rangle_{n,m}_{a,b|c,d} - \langle \{ 2 \} \rangle_{n} \langle \{ 2 \} \rangle_{m}_{a|c} \langle \{ 2 \} \rangle_{m}_{b|d}
\]
The $c_2\{4\}$ values in $pp$ collisions and in $p+Pb$ collisions for $\langle N_{ch} \rangle < 100$ are smallest for the 3SE method and largest for the standard method.

In the $p+Pb$ collisions the $c_2\{4\}$ are consistent for all three methods for $\langle N_{ch} \rangle > 100$ suggesting that non-flow effects in $p+Pb$ collisions are much smaller than those in $pp$ collisions at comparable $\langle N_{ch} \rangle$.

The 3SE method gives negative $c_2\{4\}$ values in most of the measured $\langle N_{ch} \rangle$ range.

The $v_2\{4\}$ values obtained from the 3SE method are smaller than $v_2\{2\}$ extracted from 2PC, possibly due to EbyE flow fluctuations associated with the initial state (PRL 112, 082301 (2014)).
Symmetric cumulant $s c_{2,3}\{4\} = \langle v_2^2 v_3^2 \rangle - \langle v_2^2 \rangle \langle v_3^2 \rangle$

- Measured in two $p_T$ intervals: $0.3 < p_T < 3$ GeV and $0.5 < p_T < 5$ GeV.
- In $pp$ and at low values of $\langle N_{ch} \rangle$ in $p+Pb$ the $s c_{2,3}\{4\}$ obtained from the standard method are positive and significantly differ from SE.
- Non-flow is largely suppressed in SE; some residual non-flow in $pp$ at low $\langle N_{ch} \rangle$ in 2SE.
- For the $p_T$ region of $0.5 < p_T < 5$ GeV results from 2SE are systematically lower than the 3SE/4SE, suggesting that the 2SE method may be affected by negative non-flow contribution (recently observed in PYTHIA, PLB777 (2018) 201).
- Anticorrelation between $v_2$ and $v_3$ in SE.
- In case of Pb+Pb collisions the measured $s c_{2,3}\{4\}$ is consistent among all four methods across most of the $\langle N_{ch} \rangle$ range.
- 3SE method seems to be good enough for non-flow removal.
Symmetric cumulant \( s_{c2,4}\{4\} = \langle v_2^2 v_4^2 \rangle - \langle v_2^2 \rangle \langle v_4^2 \rangle \)

- Measured in two \( p_T \) intervals: 0.3 < \( p_T \) < 3 GeV and 0.5 < \( p_T \) < 5 GeV.

- In \( pp \) and in \( p+Pb \) the \( s_{c2,4}\{4\} \) obtained from the standard method are positive and significantly differ from SE methods.

- Non-flow is largely suppressed in SE; some residual non-flow in \( pp \) at low \( \langle N_{ch} \rangle \) in 2SE.

- Positive correlation between \( v_2 \) and \( v_4 \) is observed in all methods due to non-linear effect \( v_4 = v_4L + \chi^2 v_2^2 \).

- In case of \( Pb+Pb \) collisions the measured \( s_{c2,4}\{4\} \) is consistent among all SE methods while the standard method gives values systematically larger.

- 3SE method seems to be good enough for non-flow removal.
Asymmetric cumulant $ac_2\{3\} = \langle v_2^2 v_4 \cos 4(\Phi_2 - \Phi_4) \rangle$

- Measured in two $p_T$ intervals: $0.3 < p_T < 3$ GeV and $0.5 < p_T < 5$ GeV.
- The $ac_2\{3\}$ are positive for all methods.
- The standard method gives much larger results than the SE methods ⇒ standard method is dominated by non-flow effects.
- The $ac_2\{3\}$ from the 3SE method in $pp$ collisions show some increase at low $\langle N_{ch} \rangle$, but are nearly constant at large $\langle N_{ch} \rangle$. This suggests that 3SE method contains some non-flow contribution at $\langle N_{ch} \rangle < 40$ which is negligible at larger $\langle N_{ch} \rangle$.
- In SE methods in both $p+Pb$ and $Pb+Pb$ collisions the influence of non-flow effects is very small for $\langle N_{ch} \rangle > 60$. The $ac_2\{3\}$ from SE increase with $\langle N_{ch} \rangle$ reflecting the $\langle N_{ch} \rangle$ dependence of the $v_2$ and $v_4$.
- Difference between standard and SE methods at large $\langle N_{ch} \rangle$ could be due to flow decorrelation (EPJC 78 (2018) 142).
System size dependence

▶ Use 3SE method as it is sufficient to suppress most of the non-flow effects.
▶ The results for $sc_{2,3}\{4\}$, $sc_{2,4}\{4\}$ and $ac_{2}\{3\}$ support in small collision systems, an anti-correlation between $v_2$ and $v_3$ and a positive correlation between $v_2$ and $v_4$, the pattern observed previously in large collision systems.
▶ In the multiplicity range covered by the $pp$ collisions, $\langle N_{ch} \rangle < 150$, the $sc_{2,3}\{4\}$ and $sc_{2,4}\{4\}$ are similar among the three systems.
▶ For $\langle N_{ch} \rangle > 150$, $sc_{2,3}\{4\}$ and $sc_{2,4}\{4\}$ are larger in Pb+Pb than in $p+Pb$ collisions.
▶ The results for $ac_{2}\{3\}$ are similar among the three systems at $\langle N_{ch} \rangle < 100$, but they deviate from each other at higher $\langle N_{ch} \rangle$: $pp$ data are approximately constant, while $p+Pb$ and Pb+Pb data show significant increases as a function of $\langle N_{ch} \rangle$. 

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- In the multiplicity range covered by the $pp$ collisions, $\langle N_{ch} \rangle < 150$, the $sc_{2,3}\{4\}$ and $sc_{2,4}\{4\}$ are similar among the three systems.
- For $\langle N_{ch} \rangle > 150$, $sc_{2,3}\{4\}$ and $sc_{2,4}\{4\}$ are larger in Pb+Pb than in $p+Pb$ collisions.
- The results for $ac_{2}\{3\}$ are similar among the three systems at $\langle N_{ch} \rangle < 100$, but they deviate from each other at higher $\langle N_{ch} \rangle$: $pp$ data are approximately constant, while $p+Pb$ and Pb+Pb data show significant increases as a function of $\langle N_{ch} \rangle$. 

- [Graphs showing correlation between $N_{ch}$ and collective flow parameters for different collision systems.]
Normalized cumulants

- Remove dependence on harmonics magnitude and focus only on correlation strength.
- Normalization removes most of $\langle N_{ch} \rangle$ dependence at $\langle N_{ch} \rangle > 100$.
- Normalized cumulants are similar among different collision systems at large $\langle N_{ch} \rangle$, although some splitting at the level of $20 - 30\%$ is observed for smaller $\langle N_{ch} \rangle$.
- Values of $nsc_{2,3}\{4\}$ in $pp$ collisions are very different from those in $p+Pb$ and $Pb+Pb$ collisions. This suggests that the $\langle v_3^2 \rangle$ values from the template fitting method may be significantly underestimated.
- Normalized cumulants are consistent between the two $p_T$ ranges. These results suggest that $p_T$ dependence of symmetric and asymmetric cumulants largely reflects the $p_T$ dependence of $v_n$ at the single-particle level.
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- Normalized cumulants are consistent between the two $p_T$ ranges. These results suggest that $p_T$ dependence of symmetric and asymmetric cumulants largely reflects the $p_T$ dependence of $v_n$ at the single-particle level.
Strong dependence of $v_2$ on $\langle N_{ch} \rangle$ in $p+$Pb and Pb+Pb collisions is attributed to the dependence on centrality.

In $pp$ collisions $v_2$ is independent of event multiplicity.

Try to control collision geometry in $pp$ by requiring presence of a $Z$ boson, produced in a hard scattering. $Z$-boson tagged events have smaller impact parameter $b$ and in consequence smaller $v_2$ then in inclusive events.

A new technique was developed to subtract pileup contribution in 2PC measurements.

A template fitting method was used to extract $v_2$.

Pileup subtraction changes $v_2$ by 20% on average.

$Z$-tagged $v_2$ is $8 \pm 6\%$ higher than the inclusive one.

No multiplicity dependence of $v_2$ in the $Z$-tagged data.
$D^*$ meson production in $p+Pb$ collisions

- Heavy quarks are primarily produced at early stages of HI collisions in gluon-gluon fusion and can carry information about early stage properties of the QGP.
- Compared with gluons and light quarks, heavy quarks lose less energy when traversing the medium due to the dead-cone effect (JPhys G17 (1991) 1481).
- Reconstruction of $D^*$ mesons in “golden channel”:
  \[ D^*+ \rightarrow D^0 \pi^+_{\text{slow}} \rightarrow (K^-\pi^+)\pi^+_{\text{slow}} + C.C. \]
- $D^0$ candidates are constructed from opposite-sign pairs of tracks with $p_T > 1$ GeV each. Both combinations of kaon and pion masses are considered for the tracks, since no particle identification is applied. The invariant mass of the pair is required to be in the range $1.75 < m(K\pi) < 1.96$ GeV.
- $D^*$ candidates are built by adding a soft pion track with $p_T > 250$ MeV to $D^0$ candidates.
\(D^*-\text{hadron correlations in } p+\text{Pb collisions}\)

- Study of azimuthal angular correlations between charged particles and inclusive \(D^*\) candidates with \(3 < p_T < 30 \text{ GeV}\) and \(-1.5 < y^* < 0.5\) in \(p+\text{Pb collisions}\).

- \(D^*-\text{hadron (}D^*-h\text{)}\) correlations are quantified using the two-particle correlation function \(C(\Delta \phi)\) obtained from all pairs of \(D^*\) candidates and charged particle tracks, separated in pseudorapidity by \(\Delta \eta > 1\).

- A template fitting method is used to extract the harmonic coefficients associated with the long-range ridge contribution, using the shape of peripheral contribution obtained from events with low multiplicity of \(10 < N_{\text{ch}} < 80\).

- A finite \(v_{2,2}\) is extracted from inclusive \(D^*-h\) correlations with \((1 \sim 2)\sigma\) significance.

ATLAS Preliminary
\(80 < N_{\text{ch}} < 120\)
\(p+\text{Pb } \sqrt{s_{\text{NN}}} = 8.16 \text{ TeV}\)
Simultaneous fit

\[C(\Delta \phi)\]

\(v_{2,2} \times 10^3 = 15.4 \pm 8.5(\text{stat}) \pm 0.9(\text{syst})\)

ATLAS Preliminary
\(120 < N_{\text{ch}} < 160\)
\(p+\text{Pb } \sqrt{s_{\text{NN}}} = 8.16 \text{ TeV}\)
Simultaneous fit

\[C(\Delta \phi)\]

\(v_{2,2} \times 10^3 = 8.5 \pm 3.3(\text{stat}) \pm 0.4(\text{syst})\)

ATLAS Preliminary
\(160 < N_{\text{ch}} < 240\)
\(p+\text{Pb } \sqrt{s_{\text{NN}}} = 8.16 \text{ TeV}\)
Simultaneous fit

\[C(\Delta \phi)\]

\(v_{2,2} \times 10^3 = 3.1 \pm 2.6(\text{stat}) \pm 0.3(\text{syst})\)
Summary

ATLAS has provided several new results on collectivity in small systems:

- Standard, symmetric and asymmetric cumulants have been measured with standard and subevent methods in $pp$, $p+Pb$ and $Pb+Pb$ collisions.
- Results obtained with standard method are dominated by non-flow effects. Three subevent method removes most of non-flow effects.
- Normalized cumulants show similar strength of the correlations between flow harmonics across all systems.
- Within $(1 \sim 2)\sigma$ syst. uncertainties, the $Z$-tagged $v_2$ is consistent with the min-bias $v_2$.
- The $D^*$-hadron correlation is broadly consistent with what we would expect from the observed muon-hadron correlations.
- Evidence for long-range azimuthal correlations and collectivity in small systems is confirmed and supplemented by new ATLAS measurements.

More details and results from the heavy ion physics program realized by ATLAS are available on https://twiki.cern.ch/twiki/bin/view/AtlasPublic/HeavyIonsPublicResults

Thank you for your attention!

M. Przybycień (AGH UST)
Backup slides
Heavy-ion data sets

- **A+A collisions:**
  - \( \text{Pb+Pb @ 2.76 TeV (2011), } L_{\text{int}} = 0.14 \text{ nb}^{-1} \)
  - \( \text{Pb+Pb @ 5.02 TeV (2015), } L_{\text{int}} = 0.49 \text{ nb}^{-1} \)
  - \( \text{Xe+Xe @ 5.44 TeV (2017), } L_{\text{int}} = 3 \mu\text{b}^{-1} \)

- **p + A collisions:**
  - \( \text{p+Pb @ 5.02 TeV (2013), } L_{\text{int}} = 29 \text{ nb}^{-1} \)
  - \( \text{p+Pb @ 5.02 TeV (2016), } L_{\text{int}} = 0.5 \text{ nb}^{-1} \)
  - \( \text{p+Pb @ 8.16 TeV (2016), } L_{\text{int}} = 0.16 \text{ pb}^{-1} \)

- **Reference \( pp \) samples:**
  - \( \text{pp @ 8 TeV (2012), } L_{\text{int}} = 19.4 \text{ fb}^{-1} \)
  - \( \text{pp @ 2.76 TeV (2013), } L_{\text{int}} = 4 \text{ pb}^{-1} \)
  - \( \text{pp @ 5.02 TeV (2015), } L_{\text{int}} = 28 \text{ pb}^{-1} \)
  - \( \text{pp @ 5.02 TeV (2017), } L_{\text{int}} = 270 \text{ pb}^{-1} \)
The ATLAS detector

Detector coverage:

Inner Detector (ID):
\(|\eta| < 2.5\)

Calorimeter (CAL):
\(|\eta| < 3.2\) (EM)
\(|\eta| < 4.9\) (HAD)
\(3.2 < |\eta| < 4.9\) (FCal)

Muon Spectrometer (MS):
\(|\eta| < 2.7\)

Zero Degree Cal. (ZDC):
\(|\eta| > 8.3\) @ \(z = \pm 140\) m

MB Trig. Scint. (MBTS):
\(2.1 < |\eta| < 3.9\)

Magnetic fields:
- 2T solenoid field in ID
- Toroidal field in MS

Identification of minimum-bias \(p+Pb\) and \(Pb+Pb\) collision measurement of spectator neutrons in ZDC and charged particle tracks (pulse height and arrival times) in MBTS.
Centrality determination in Pb+Pb and p+Pb

- Centrality is measured using forward calorimeters ($3.2 < |\eta| < 4.9$):
  - in Pb+Pb use sum of $E_T$ on both sides,
  - in p+Pb use sum of $E_T$ on Pb-going side only,
  - for Pb+Pb use Glauber MC for geometry,
  - for p+Pb use both Glauber and Glauber-Gribov color fluctuation model (PLB 633: 245 (2006)).
  - Average number of participants ($N_{\text{part}}$) for each centrality bin resulting from fits to the measured $E_T$ distribution for p+Pb.
Tracks coming from pileup are partially rejected by requiring matching to the collision vertex where $Z$-boson is produced. Then the measured distributions are corrected for pileup on a statistical basis.

Direct - tracks and track pairs that pass selection criteria and result from a single event.

Direct contributions consist from Signal and from pileup interactions (Background).

Use Mixed events constructed from tracks from different events, but with similar $\mu$ and $|z_{0}^{\text{trk}} - z_{Z-\text{boson}}^{vtx})\sin\theta| < 0.75$ mm, to obtain Background distributions as functions of both $n_{\text{trk}}^{\text{mixed}}$ and $\nu \equiv \langle n_{\text{trk}}^{\text{bkgd}} \rangle$.

One can build transition matrices, which can be used in the unfolding to restore $n_{\text{trk}}^{\text{signal}}$:

$$M(\nu, n_{\text{trk}}^{\text{signal}}, n_{\text{trk}}^{\text{direct}}) = P_{\text{Dir}}(\nu < 0.5, n_{\text{trk}}^{\text{signal}})P_{\text{Mix}}(\nu, n_{\text{trk}}^{\text{direct}} - n_{\text{trk}}^{\text{signal}})$$
Pileup correction for the pair-distribution

- \( D = \text{Direct}, \ S = \text{Signal}, \ B = \text{Bkgd} \)
  \[
  D^a \times D^b \equiv \sum_{a \in D} \sum_{b \in D, b \neq a} (\phi^a - \phi^b)
  \]
- \( D^a \times D^b = S^a \times S^b + \)
  \[
  + S^a \times B^b + B^a \times S^b + B^a \times B^b
  \]
- Averaging over many events:
  \[
  \langle S^a \times S^b \rangle = \langle D^a \times D^b \rangle - \langle B^a \times B^b \rangle -
  \]
  \[
  - \langle B^a \times S^b \rangle - \langle S^a \times B^b \rangle
  \]
- \( S \) and \( B \) are independent, and
  \[
  \langle S^a \rangle = \langle D^a \rangle - \langle B^a \rangle, \text{ hence}
  \]
  \[
  \langle S^a \times S^b \rangle = \langle D^a \times D^b \rangle - \langle B^a \times B^b \rangle -
  \]
  \[
  - \langle D^a \times \langle B^b \rangle \rangle - \langle B^a \rangle \times \langle D^b \rangle +
  \]
  \[
  + 2 \langle B^a \rangle \times \langle B^b \rangle
  \]
- Background events are statistically equivalent to \( M = \text{Mixed} \) events at the same multiplicity.

\[
(S^a \times S^b) \bigg| n_{\text{trk}}^{\text{signal}} = n_{\text{trk}}^{\text{signal,0}}
\]
\[
= \sum_{\nu}^{\nu_{\text{max}}} \sum_{n_{\text{trk}}^{\text{direct,max}}} \left( P(n_{\text{trk}}^{\text{signal}}) N_{\text{events}} (S^a \times S^b) \right) n_{\text{trk}}^{\text{direct}} \n_{\text{trk}}^{\text{signal,0}} \nu - (\nu + 0.5)
\]