W. G. Scott (PPD/RAL)
NEXT-Meeting 9 May 2018

"MANIFEST BOSONIC DEMOCRACY"

THE SU(3) ALGEBRA IN A CYCLIC BASIS

"The SU)3) Algebra in A Cyclic Basis" P.F. Harrison, R. Krishnan and W. G. Scott, Phys. Rev. D 90 (2014) 017302. ArXiv:1407.8360[hep-ph]

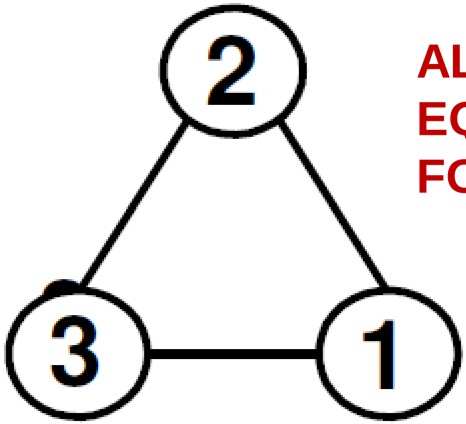
Please first consider – already familiar - :

THE SU(2) ALGEBRA IN A CYCLIC BASIS:

su(2) ⊂ A_1

Just the old
3-d vector product rule
for orthogonal
unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$

$$\hat{1} \times \hat{2} = \hat{3}$$
$$\hat{2} \times \hat{3} = \hat{1}$$
$$\hat{3} \times \hat{1} = \hat{2}$$



ALL ON EQUAL FOOTING

Defn. of Lie Algebra:

Lie Product is Anti-Symmetric, i. e. : (1 x 2) = - (2 x 1) etc.
And obeys Jacobi identity: 1 x (2 x 3) + 2 x (3 x 1) + 3 x (1 x 2) = 0

The (complex) algebra A_1 in the "Chevalley Basis"

- perhaps better aligned to the SM context ($3' = 2\, I_3$):

$$3' \times 1' = +2 \cdot 1' \qquad 3' \times 2' = -2 \cdot 2'$$

$$1' \times 2' = 3'$$

$sl(2,R)$
$\approx su(1,1)$
$\subset A_1$

Basis 1', 2', 3' related to the cyclic basis 1, 2, 3 by the (complex) basis transformation:

$$\begin{pmatrix} 1' \\ 2' \\ 3' \end{pmatrix} = \begin{pmatrix} i & -1 & 0 \\ i & +1 & 0 \\ 0 & 0 & 2i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

SM Lagrangian remains invariant with re-defined (complex) fields proportional to:

$$W^+ = (W_1 - iW_2)/\sqrt{2}$$
$$W^- = (W_1 + iW_2)/\sqrt{2}$$
$$W^0 = W_3$$

Outline:

Introduction:	su(2)	cyclic	non-cyclic
	su(3)	Gell-Mann	non-cyclic
Computer Search : →	280 Bn	→ case-1 → 972	→ case-2 → 2
case-1 3x3 rep:	an aside on:		"Tri-Maximal Mixing"
case-1 ⟷ case-2 :	circulant-transformations		assertion, conjecture
Referee's Report :	"a good turn"		proof ??
"By-The-Way" :	TSCs = Trigonometric Structure Constants		su(2,1) "case-3"
GUTS :	su(5)	"bi-cyclic"	TSC version
Challenge:	su(5)	~~£1000,000~~	Glory!

In a perhaps more acceptable notation (as in our paper):

su(2) (cyclic basis):

$$[\hat{w}_1, \hat{w}_2] = \hat{w}_3 \qquad [\hat{w}_2, \hat{w}_3] = \hat{w}_1 \qquad [\hat{w}_3, \hat{w}_1] = \hat{w}_2, \qquad (1)$$

A_1 Chevalley basis (non-cyclic):

$$[\hat{w}'_3, \hat{w}'_1] = 2\hat{w}'_1 \qquad [\hat{w}'_3, \hat{w}'_2] = -2\hat{w}'_2$$
$$[\hat{w}'_1, \hat{w}'_2] = \hat{w}'_3 \qquad (2)$$

A_2 (Gell-Mann basis - non-cyclic): **su(3) ⊂ A_2**

$$[\hat{g}_1, \hat{g}_2] = \hat{g}_3 \qquad [\hat{g}_1, \hat{g}_4] = \hat{g}_7/2 \qquad [\hat{g}_1, \hat{g}_5] = -\hat{g}_6/2$$
$$[\hat{g}_2, \hat{g}_4] = \hat{g}_6/2 \quad [\hat{g}_2, \hat{g}_5] = \hat{g}_7/2 \quad [\hat{g}_4, \hat{g}_5] = \hat{g}_3/2 + \sqrt{3}/2\hat{g}_8$$
$$[\hat{g}_4, \hat{g}_8] = -\sqrt{3}/2\hat{g}_5 \qquad [\hat{g}_6, \hat{g}_7] = -\hat{g}_3/2 + \sqrt{3}/2\hat{g}_8 \qquad (4)$$

So the question arises - can su(3) be put in cyclic form…..????

After >100hrs running on the RAL-PPD linux farm we found:

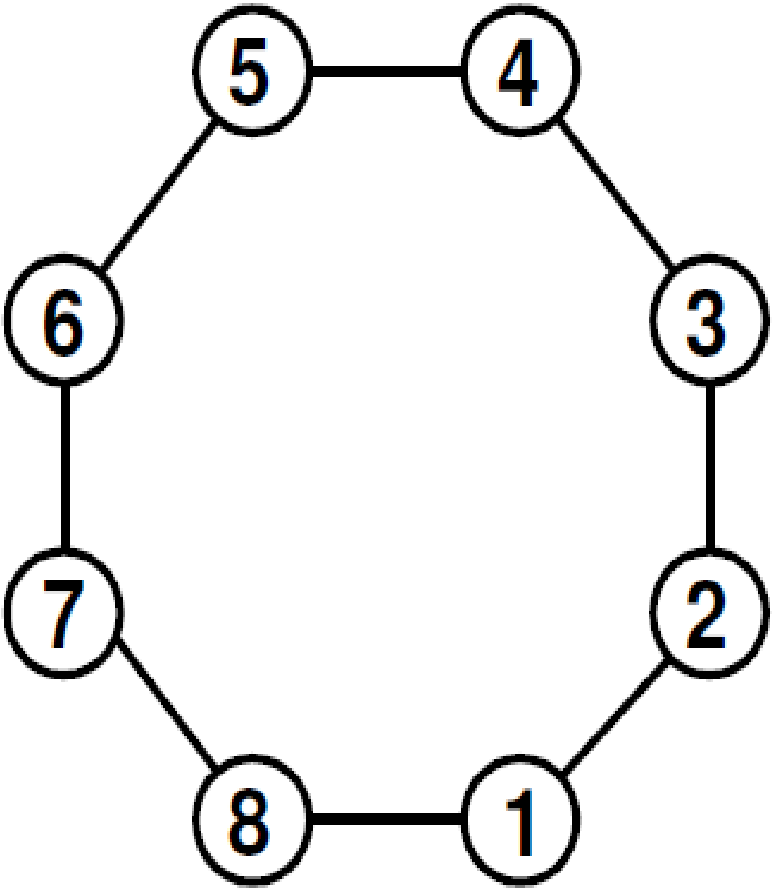
Phys. Rev. D 90 (2014) 017302;
arXiv:1407.8360[hep-ph]

$$1 \times 2 = 3 + 4 - 7 + 8$$

$$1 \times 3 = -2 - 4 - 6 + 8$$

$$1 \times 4 = -2 + 3 - 6 - 7$$

$$1 \times 5 = 0$$



Structure Constants all unit modulus and totally anti-symmetric

Killing Form is diagonal:

$$\kappa_{ab} = -24\, \delta_{ab}$$

Metric Signature : $(-24)^8$ su(3) $\subset$ A_2

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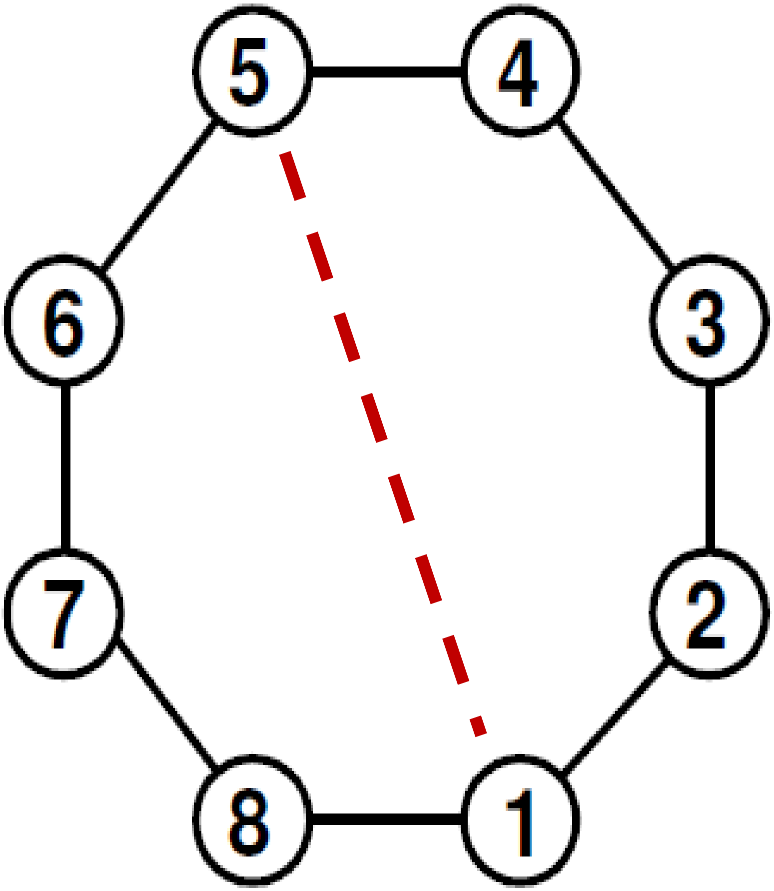
Cartan Subalgebra



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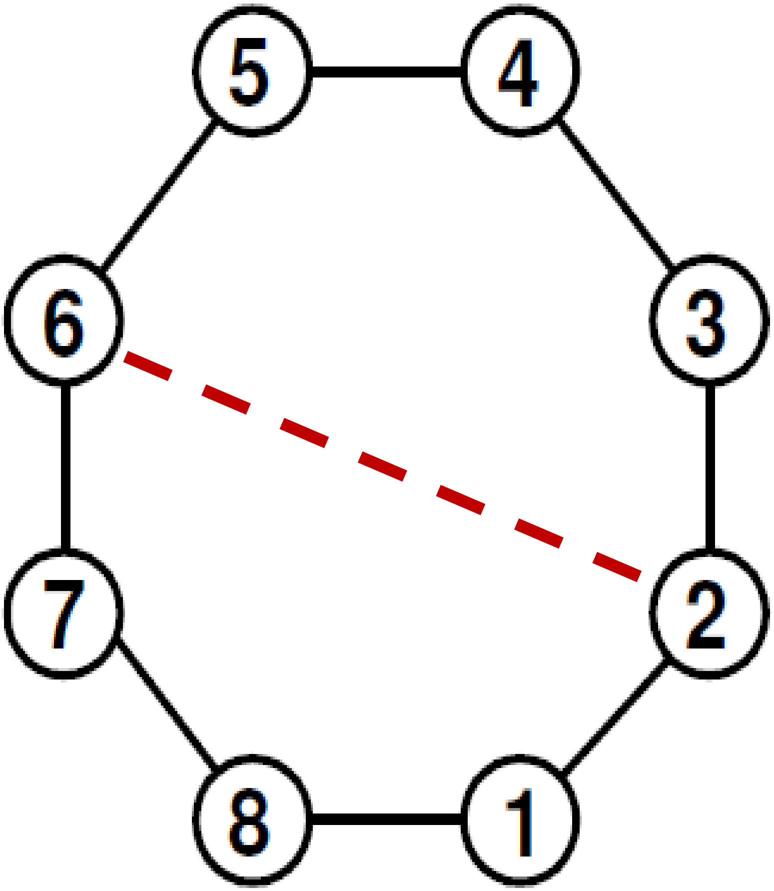

Structure Constants all unit modulus and totally anti-symmetric

Killing Form is diagonal:

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Any Opposite Pair!

The Fundamental (3 x 3) Representation in Cyclic Form:

$b = (x_- \omega + x_0 + x_+ \varpi) / 3$

$\bar{b} = (x_- \varpi + x_0 + x_+ \omega) / 3$

(where $x_- = 1/\sqrt{3} - 1$, $x_0 = -2/\sqrt{3}$, $x_+ = 1/\sqrt{3}+1$ with ω , ϖ complex Cube-roots of unity)

$$\lambda_1 = \begin{pmatrix} x_- & 0 & 0 \\ 0 & x_0 & 0 \\ 0 & 0 & x_+ \end{pmatrix} \quad \lambda_2 = -\begin{pmatrix} 0 & b\omega & \bar{b} \\ \bar{b}\varpi & 0 & b\varpi \\ b & \bar{b}\omega & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 0 & b\varpi & \bar{b} \\ \bar{b}\omega & 0 & b\omega \\ b & \bar{b}\varpi & 0 \end{pmatrix}$$

$$\lambda_4 = -\begin{pmatrix} 0 & \bar{b} & b \\ b & 0 & \bar{b} \\ \bar{b} & b & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} x_+ & 0 & 0 \\ 0 & x_0 & 0 \\ 0 & 0 & x_- \end{pmatrix} \quad \lambda_6 = -\begin{pmatrix} 0 & \bar{b}\omega & b \\ b\varpi & 0 & \bar{b}\varpi \\ \bar{b} & b\omega & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & \bar{b}\varpi & b \\ b\omega & 0 & \bar{b}\omega \\ \bar{b} & b\varpi & 0 \end{pmatrix} \quad \lambda_8 = -\begin{pmatrix} 0 & b & \bar{b} \\ \bar{b} & 0 & b \\ b & \bar{b} & 0 \end{pmatrix}$$

"Tri-Maximal Mixing":
Harrison & Scott
PLB 333 (1994) 471
hep-ph/9406351; see
also: hep-ph/9909431

Any choice of "non-opposite" pair λ_i, λ_j are maximally non-commuting !!!

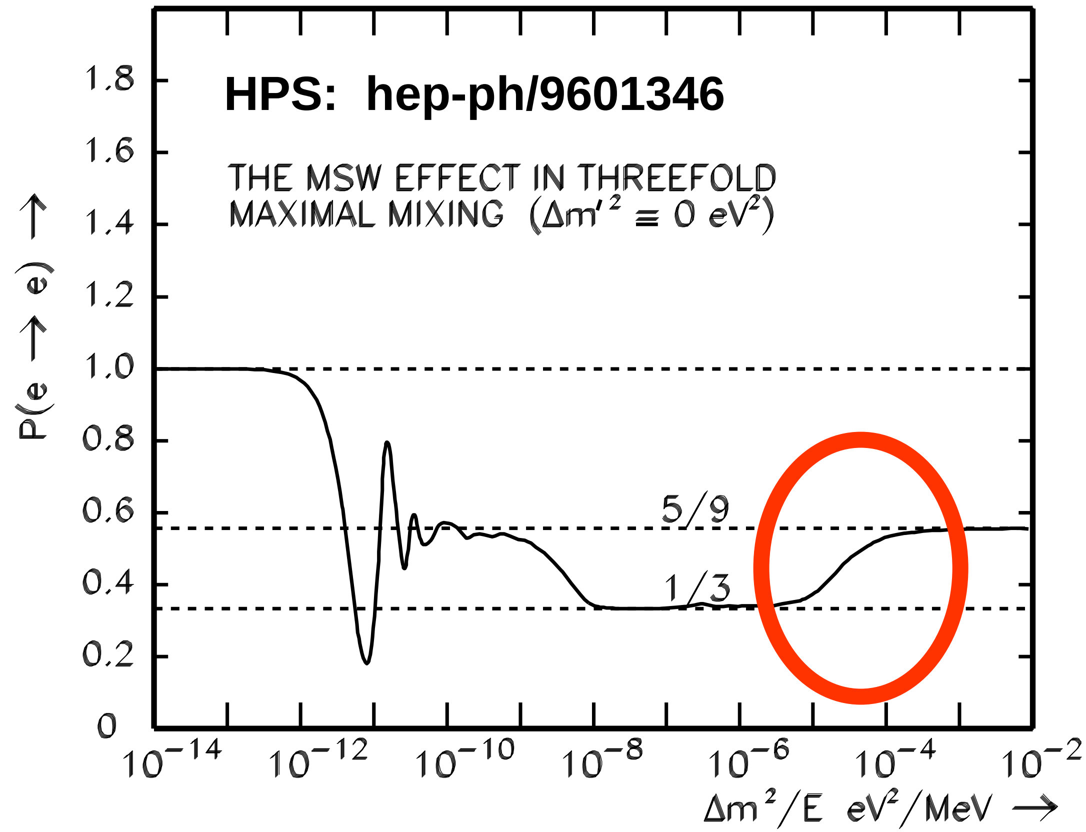
"Threefold-Maximal Mixing"
"Tri-Maximal Mixing"

P. F. Harrison, D. H. Perkins, W.G. Scott
Phys. Lett. B 349 (1995) 357 – No Arxiv.
Also: hep-ph/9702243, hep-ph/9904297
hep-ph/0402006 (BHS), hep-ph/0511201

$$\begin{array}{c} \\ e \\ \mu \\ \tau \end{array}\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

"Tri-Bi-Maximal Mixing"

$$\begin{array}{c} \\ e \\ \mu \\ \tau \end{array}\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}$$



Tri-Bi-Maximal Mixing (hep-ph/0202074)
dominated phenomenology for a decade

We also found ("Case-2"):

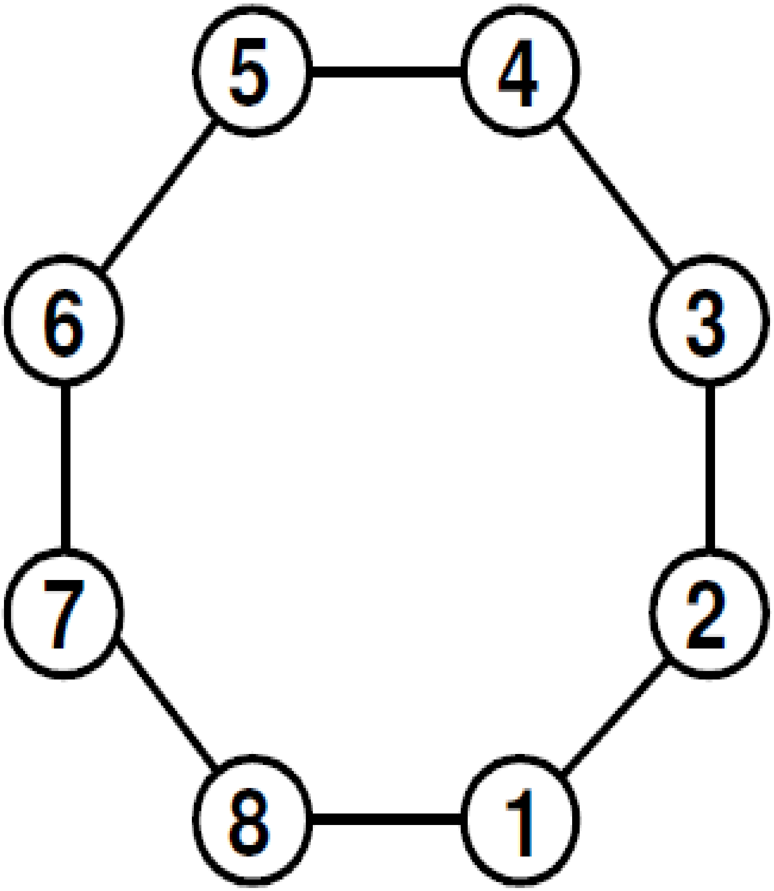
Phys. Rev. D 90 (2014) 017302;
arXiv:1407.8360[hep-ph]

1 x 2 = 4 - 7 + 8

1 x 3 = - 2 - 4 - 6

1 x 4 = - 2 + 3 - 6

1 x 5 = 0



Structure Constants all unit modulus
but are not totally anti-symmetric

Killing Form is off-diagonal
but circulant:

$$K_{ab} = \text{circ}(-12,0,0,0,6,0,0,0)$$

Metric Signature : $(-18)^4$, $(-6)^4$

su(3) $\subset$ A_2

Again - as presented in our paper: Phys. Rev. D 90 (2014) 017302; arXiv:1407.8360[hep-ph]

Cyclic su(3) – case-1:

$$\left[\hat{g}^{(1)}_{a+1}, \hat{g}^{(1)}_{a+2}\right] = \hat{g}^{(1)}_{a+3} + \hat{g}^{(1)}_{a+4} - \hat{g}^{(1)}_{a+7} + \hat{g}^{(1)}_{a+8} \qquad (15)$$

$$\left[\hat{g}^{(1)}_{a+1}, \hat{g}^{(1)}_{a+3}\right] = -\hat{g}^{(1)}_{a+2} - \hat{g}^{(1)}_{a+4} - \hat{g}^{(1)}_{a+6} + \hat{g}^{(1)}_{a+8} \qquad (16)$$

$$\left[\hat{g}^{(1)}_{a+1}, \hat{g}^{(1)}_{a+4}\right] = -\hat{g}^{(1)}_{a+2} + \hat{g}^{(1)}_{a+3} - \hat{g}^{(1)}_{a+6} - \hat{g}^{(1)}_{a+7} \qquad (17)$$

$$\left[\hat{g}^{(1)}_{a+1}, \hat{g}^{(1)}_{a+5}\right] = 0, \qquad a = 1 \ldots 8 \quad \bmod 8, 1. \qquad (18)$$

Cyclic su(3) – case-2:

$$\left[\hat{g}^{(2)}_{a+1}, \hat{g}^{(2)}_{a+2}\right] = \hat{g}^{(2)}_{a+4} - \hat{g}^{(2)}_{a+7} + \hat{g}^{(2)}_{a+8} \qquad (19)$$

$$\left[\hat{g}^{(2)}_{a+1}, \hat{g}^{(2)}_{a+3}\right] = -\hat{g}^{(2)}_{a+2} - \hat{g}^{(2)}_{a+4} - \hat{g}^{(2)}_{a+6} \qquad (20)$$

$$\left[\hat{g}^{(2)}_{a+1}, \hat{g}^{(2)}_{a+4}\right] = -\hat{g}^{(2)}_{a+2} + \hat{g}^{(2)}_{a+3} - \hat{g}^{(2)}_{a+6} \qquad (21)$$

$$\left[\hat{g}^{(2)}_{a+1}, \hat{g}^{(2)}_{a+5}\right] = 0, \qquad a = 1 - 8 \quad \bmod 8, 1. \qquad (22)$$

The transformations back and forth between the two forms 1 <–> 2:

$$C^{(21)} = \mathrm{circ}\{(1+\sqrt{3})/4, 0, 0, 0, (1-\sqrt{3})/4, 0, 0, 0\}$$
$$C^{(12)} = \mathrm{circ}\{(1+1/\sqrt{3}), 0, 0, 0, (1-1/\sqrt{3}), 0, 0, 0\}$$

where `circ' denotes a `circulant' matrix: (a cyclic transformation)

$$(\tilde{g}'_a)^T \;=\; C\,.\,(\tilde{g}_a)^T \qquad C = \mathrm{circ}(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$$

$$\begin{pmatrix} \tilde{g}'_1 \\ \tilde{g}'_2 \\ \tilde{g}'_3 \\ \tilde{g}'_4 \\ \tilde{g}'_5 \\ \tilde{g}'_6 \\ \tilde{g}'_7 \\ \tilde{g}'_8 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ c_8 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ c_7 & c_8 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_6 & c_7 & c_8 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_5 & c_6 & c_7 & c_8 & c_1 & c_2 & c_3 & c_4 \\ c_4 & c_5 & c_6 & c_7 & c_8 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_1 \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \\ \tilde{g}_4 \\ \tilde{g}_5 \\ \tilde{g}_6 \\ \tilde{g}_7 \\ \tilde{g}_8 \end{pmatrix}, \qquad (8)$$

We assert that all cyclic bases are accessible by such cyclic transformations and conjectured that cyclic bases exist for other lie algebras, e.g. su(n), n>3.

Our PLB referee was clearly `rattled' by our paper:

**"I urge the editors to reject this contribution……..
it is quite easy to show that such a basis exists for any finite-dimensional complex semisimple Lie algebra by combining the expression of the structure constants in a basis orthogonal with respect of the Killing form and the existence of an orthogonal automorphism of the Lie algebra of order dimension of the Lie algebra. (Such an automorphism can even be chosen to be inner.) …."**

Our response to the editor was highly non-confrontational requesting only a copy of the referee's proof::

**"We accept the referee's judgement….
However….would it be asking too much that he/she kindly passes us his/her proof in a little more detail (anonymously via the editor) for our edification?"**

While we prepared our paper for submission to PRD (where it was accepted after only minor modifications !!!) Incredibly we received the PLB referee's `proof':

His/her latex is beautiful …. but is the `proof' valid??

Let $\mathfrak{g}$ a finite-dimensional complex simple Lie algebra. Denote by κ the Killing form. It is non-degenerate. An orthonormal basis is a basis $(T_i)_{i=1,\dots \dim \mathfrak{g}}$ of $\mathfrak{g}$ such

$$\kappa(T_i, T_j) = \delta_{ij}$$

Orthonormal bases exist by an adaptation of the Gram-Schmidt procedure. Structure constants of the Lie bracket

$$[T_i, T_j] = f_{ij}^k T_k$$

in an orthonormal basis are expressed as

$$f_{ij}^k = \kappa([T_i, T_j], T_k)$$

Suppose that we have an endomorphism $S : \mathfrak{g} \to \mathfrak{g}$ orthogonal with respect to the Killing form of order $n := \dim \mathfrak{g}$. Such an automorphism can be obtained e.g. from an inner automorphism of the compact real form of $\mathfrak{g}$. If we find a basis (T_i) such that $ST_i = T_{i+1}$ with the indices taken modulo n for convenience, we have

$$[T_{i+1}, T_{j+1}] = [ST_i, ST_j] = S[T_i, T_j] = f_{ij}^k ST_k = f_{ij}^k T_{k+1}$$

and thus

$$f_{ij}^k = f_{i+1,j+1}^{k+1} \, .$$

This is the cyclic invariance the authors were looking for. Using now a basis given by the Frobenius normal form of the endomorphism T and carefully checking that orthonormality is preserved, the existence of such a basis of $\mathfrak{g}$ follows.

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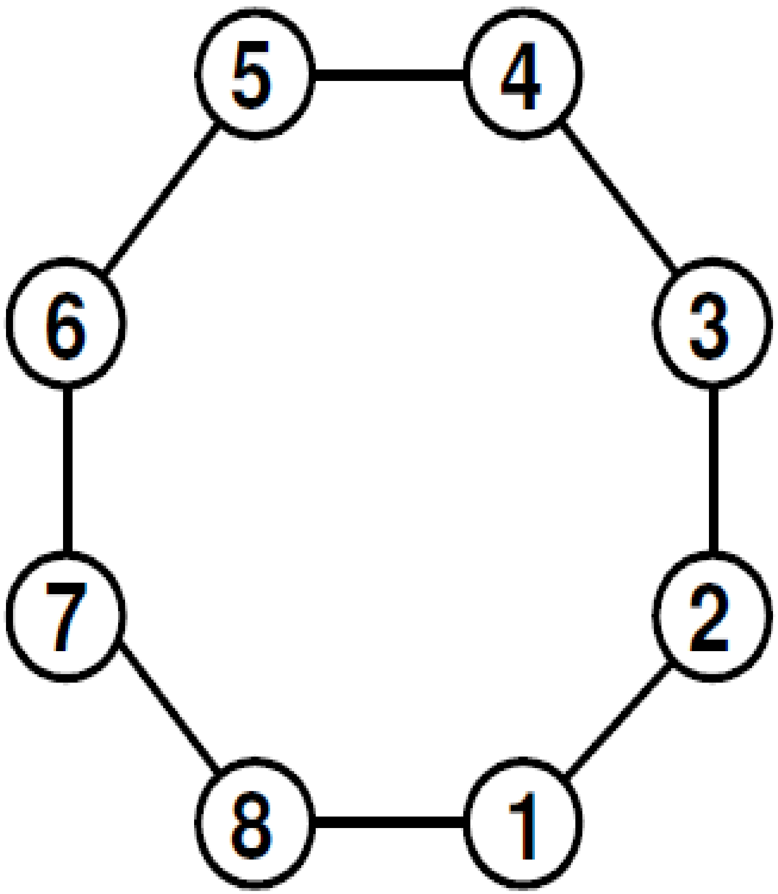
BTW:
We also found
In our search **- and rejected:**

c.f.“Trigonometric Structure Constants”
e.g. Fairlie, Zachos, Vershik….

$$1 \times 2 = 3$$
$$1 \times 3 = 8$$
$$1 \times 4 = -7$$
$$1 \times 5 = 0$$

This is A_2
but is NOT strictly su(3) !!!



Structure Constants all unit modulus but are not totally anti-symmetric

Killing Form is off-diagonal but circulant:

$$K_{ab} = \mathrm{circ}(0,0,0,0,-6,0,0,0)$$

Metric Signature : $(-6)^4 , (+6)^4$

- So actually SU(2,1) !!!

Trigonometric Structure Constants (TSCs)

In particular for SU(N):

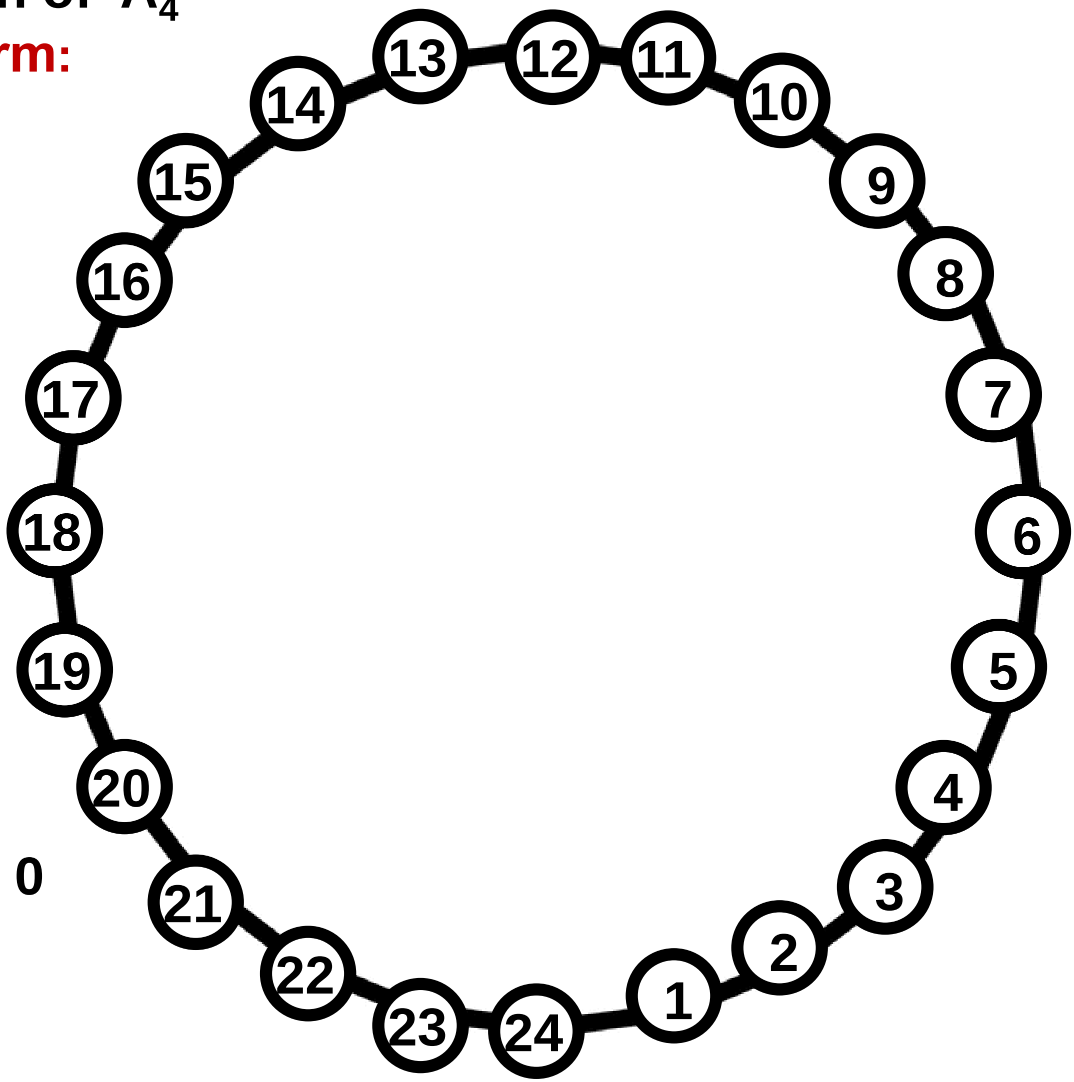
$$[\,\hat{G}_m\,,\ \hat{G}_n\,] = \sin(\,2\pi/N\,(n \times m)\,)\ \hat{G}_{m+n}$$

where: $m = (\,m_1\,,\,m_2\,)$, $n = (\,n_1\,,\,n_2\,)$ mod N

In a wider context: "sine algebras", "cross-product algebras" - mostly infinite dimensional

J. Patera and H. Zasseninaus J. Math Phys. 29 665 (1988).
D. Fairlie, P. Fletcher and C. K. Zachos, PLB 218 203 (1989).
D. Fairlie and C. K. Zachos, PLB 224 (1989) 101.
M. V. Saveliev and A. M. Vershik, PLA 143 (1990) 121.
D. Fairlie, P. Fletcher and C. K. Zachos JMP 31 (1990) 1088.
D. Fairlie and C. K. Zachos PLB 620 (2005) 195. hep-th/0505053
D. Fairlie and C. K. Zachos PLB 637 (2006) 123. hep-th/0603017

For SU(5) - from the TSC version of A_4 -
we found only this ‘bi-cyclic’ form:



$1 \times 3 \;=\; (-4 + 6 - 16 - 18)$
$2 \times 4 \;=\eta\; (-5 + 7 - 17 - 19)$

$1 \times 9 \;=\eta\; (-5 - 8 + 17 - 20)$
$2 \times 10 \;=\; (-6 + 9 + 18 + 21)$

$1 \times 5 \;=\eta\; (9 - 12 + 21 + 24)$
$2 \times 6 \;=\; (10 + 13 + 22 - 1)$

$1 \times 11 \;=\; (-2 + 4 + 14 + 16)$
$2 \times 12 \;=\eta\; (-3 + 5 + 15 + 17)$

$1 \times 1 = 1 \times 7 = 1 \times 13 = 1 \times 19 = 0$

Structure constants are modulus unity or golden ratio $\eta = (1+\sqrt{5})/2$ and are totally anti-symmetric.

Killing Form is diagonal: $\kappa_{ab} = -40\,\eta\sqrt{5}\,\delta_{ab}$

For SU(5) - from the TSC version of A_4 -
we found only this 'bi-cyclic' form:

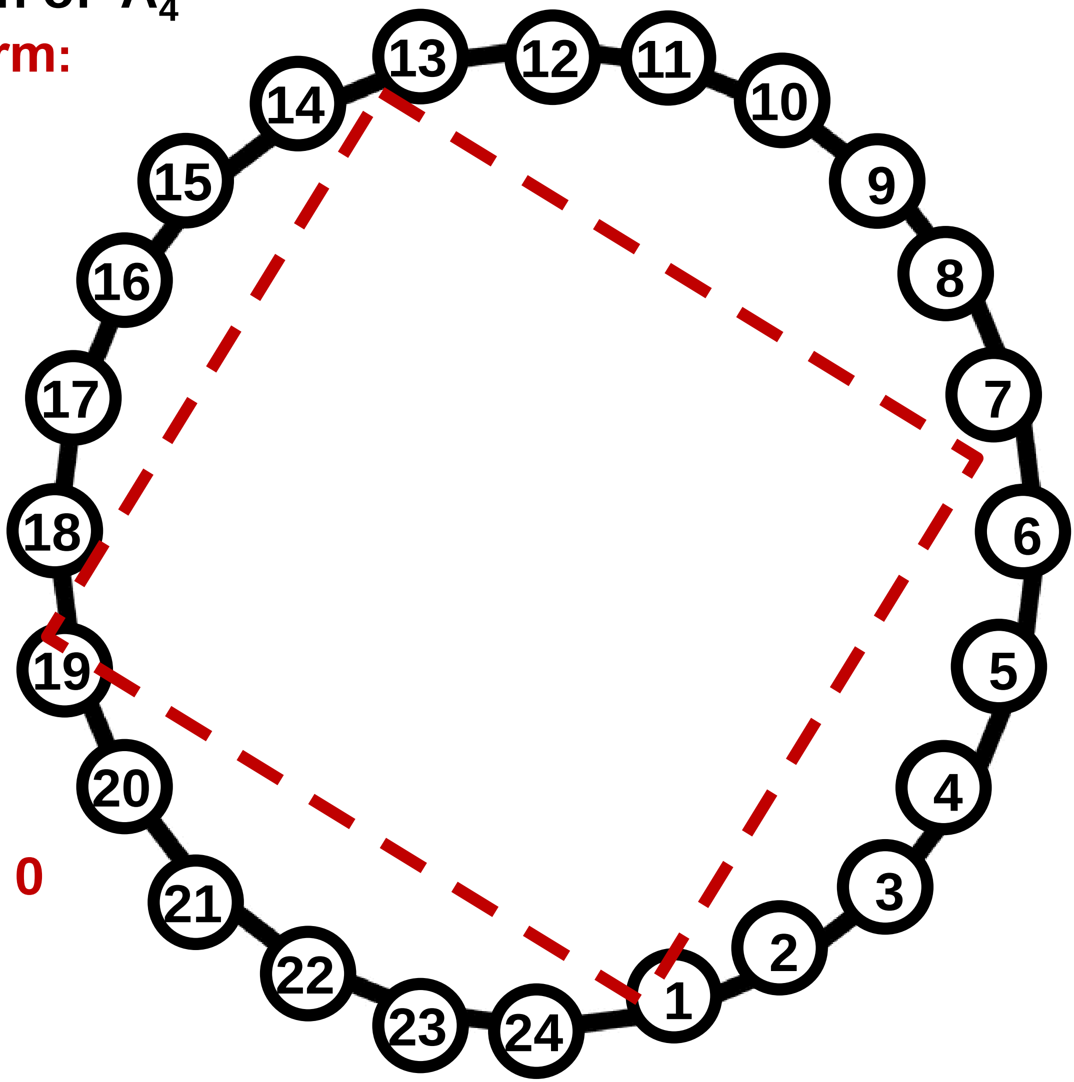
1 x 3 = (- 4 + 6 - 16 – 18)
2 x 4 = η (- 5 + 7 – 17 – 19)

1 x 9 = η (- 5 - 8 + 17 – 20)
2 x 10 = (- 6 + 9 + 18 + 21)

1 x 5 = η (9 - 12 + 21 + 24)
2 x 6 = (10 + 13 + 22 – 1)

1 x 11 = (- 2 + 4 + 14 + 16)
2 x 12 = η (- 3 + 5 + 15 + 17)

1 x 1 = 1 x 7 = 1 x 13 = 1 x 19 = 0

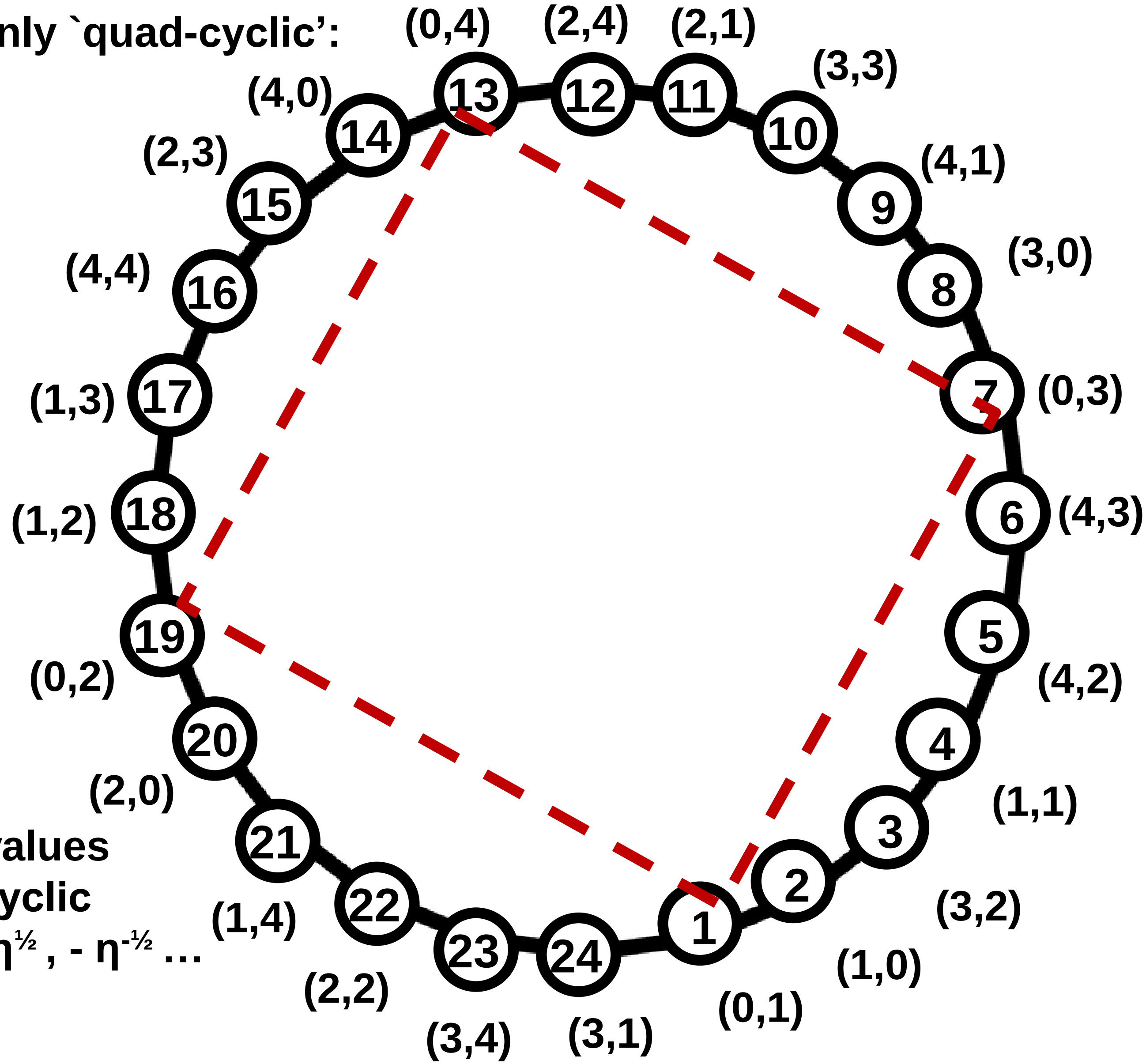


Structure constants are modulus unity or golden ratio η = (1+ √5)/2 and are totally anti-symmetric.

Killing Form is diagonal: $\kappa_{ab} = -40\,\eta\sqrt{5}\,\delta_{ab}$

TSC version of A_4 is only `quad-cyclic':

$1 \times 2 \;=\; -\,\eta^{1/2}\;\; 4$
$1 \times 3 \;=\; \eta^{-1/2}\;\; 10$
$1 \times 4 \;=\; -\,\eta^{1/2}\;\; 18$
$1 \times 5 \;=\; \eta^{1/2}\;\; 6$
$1 \times 6 \;=\; \eta^{1/2}\;\; 16$
$1 \times 7 \;=\; 0$
$1 \times 8 \;=\; \eta^{-1/2}\;\; 24$
$1 \times 9 \;=\; \eta^{1/2}\;\; 5$
$1 \times 10 \;=\; \eta^{-1/2}\;\; 23$
$1 \times 11 \;=\; -\,\eta^{-1/2}\;\; 22$
$1 \times 12 \;=\; -\,\eta^{-1/2}\;\; 20$
$1 \times 13 \;=\; 0$



All structure constants values are `quad-cyclic' in the cyclic sequence …. $-\,\eta^{1/2}$, $\eta^{-1/2}$, $\eta^{1/2}$, $-\,\eta^{-1/2}$ …

Killing Form is circulant:
κ = circ(0,0,0,0,0,0,0,0,0,0,0,-50,0,0,0,0,0,0,0,0,0,0,0,0)

The "Harrison-Krishnan-Scott" Challenge:

P. Harrison, R. Krishnan, W. Scott
arXiv:1407.8360[hep-ph]

c,f. Millenium Prize Problems
Hilbert's 23 Problems
Riemann Hypothesis ....
etc. etc.

Find a (mono-) cyclic basis for SU(5) !!

If you believe the PLB referee's proof you know it is possible/trivial !!!

~~Win 1 Million Pounds GBP !!!~~

~~£1,000,000~~

→ ~~£1000~~ → **Just Glory!**

From the same Authors:

Fully Constrained Majorana Neutrino Mass Matrices Using $\sum (72 \times 3)$
R. Krishnan, P. F. Harrison, W. G. Scott
Eur. Phys. J. C (2018) 78: 74 **arXiv:1801.10197 [hep-ph]**

Deviations from Tribimaximal Neutrino Mixing using a Model with $\Delta(27)$ Symmetry **P. F. Harrison, R. Krishnan, W. G. Scott**
IJMPA 29 (2014) 1450095 **arXiv:1406.2025 [hep0ph]**

Simplest Neutrino Mixing from S4 Symmetry
R. Krishnan, P. F. Harrison, W. G. Scott
JHEP, 087, 2013 **arXiv:1211.2000 [hep-ph]**

Exact One-Loop Evolution Invariants in the Standard Model
P. F. Harrison, R. Krishnan, W. G. Scott
Phys. Rev. D82:096004, 2010 **arXiv:1007.3810 [hep-ph]**

See also: R. Krishnan **arXiv:1211.3364, arXiv:1402.0857 [hep-ph]**