Quasi-local Energy and Compactification

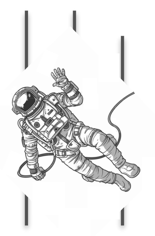
Raquel Santos-Garcia Work in collaboration with E. Alvarez, J. Anero and G. Milans del Bosch
Instituto de Física Teórica IFT UAM/CSIC, Universidad Autónoma de Madrid, Spain

Idea

The quasi-local energy definition of Brown and York is able to discriminate between the uncompactified Minkowski spacetime and the toroidal Kaluza-Klein compactification.

Energy in GR

The energy momentum cannot be a **four-vector** because it can always be made to **vanish locally** in a free falling frame.



Free falling observer ⟶ Zero gravitational energy

Covariantly conserved

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \partial_\mu T^{\mu\nu} = -\Gamma^\mu_{\mu\lambda} T^{\lambda\nu} - \Gamma^\nu_{\mu\lambda} T^{\lambda\mu}$$

The purely gravitational contribution is not a tensor

Codimension-2 spheres in M_5

Three-sphere defined by the embedding

$$y_1 = T, \qquad \sum_{i=2}^{i=5} (y_i)^2 \equiv L^2 .$$

The two **normal** vectors read

$$n_A \equiv \left(\frac{\partial}{\partial t}, \frac{y^i}{L} \frac{\partial}{\partial y^i} \right) \qquad A = 1,2$$

The three **tangent** vectors read

$$t_{\theta_1} = L\left(0, c\,\theta_1\, s\,\theta_2\, s\,\theta_3, c\,\theta_1\, s\,\theta_2\, c\theta_3, c\,\theta_1\, c\theta_2, -s\theta_1\right)$$
$$t_{\theta_2} = L\left(0, s\,\theta_1\, c\,\theta_2\, s\,\theta_3, s\,\theta_1\, c\,\theta_2\, c\theta_3, -s\theta_1\, s\theta_2, 0\right)$$
$$t_{\theta_3} = L\left(0, s\,\theta_1\, s\,\theta_2\, c\,\theta_3, -s\,\theta_1\, s\,\theta_2\, s\theta_3, 0, 0\right)$$

(Spherical coordinates)

So that the **induced metric** is

$$d\sigma^2 = h_{ab} dy^a dy^b = -L^2 \left(d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2\, d\theta_3^2 \right)$$

Hence, the **extrinsic curvature tensor** yields $K^A_{ab} = \left(0, -\frac{1}{L}\delta_{\alpha\beta} t^\alpha_a t^\beta_b\right)$

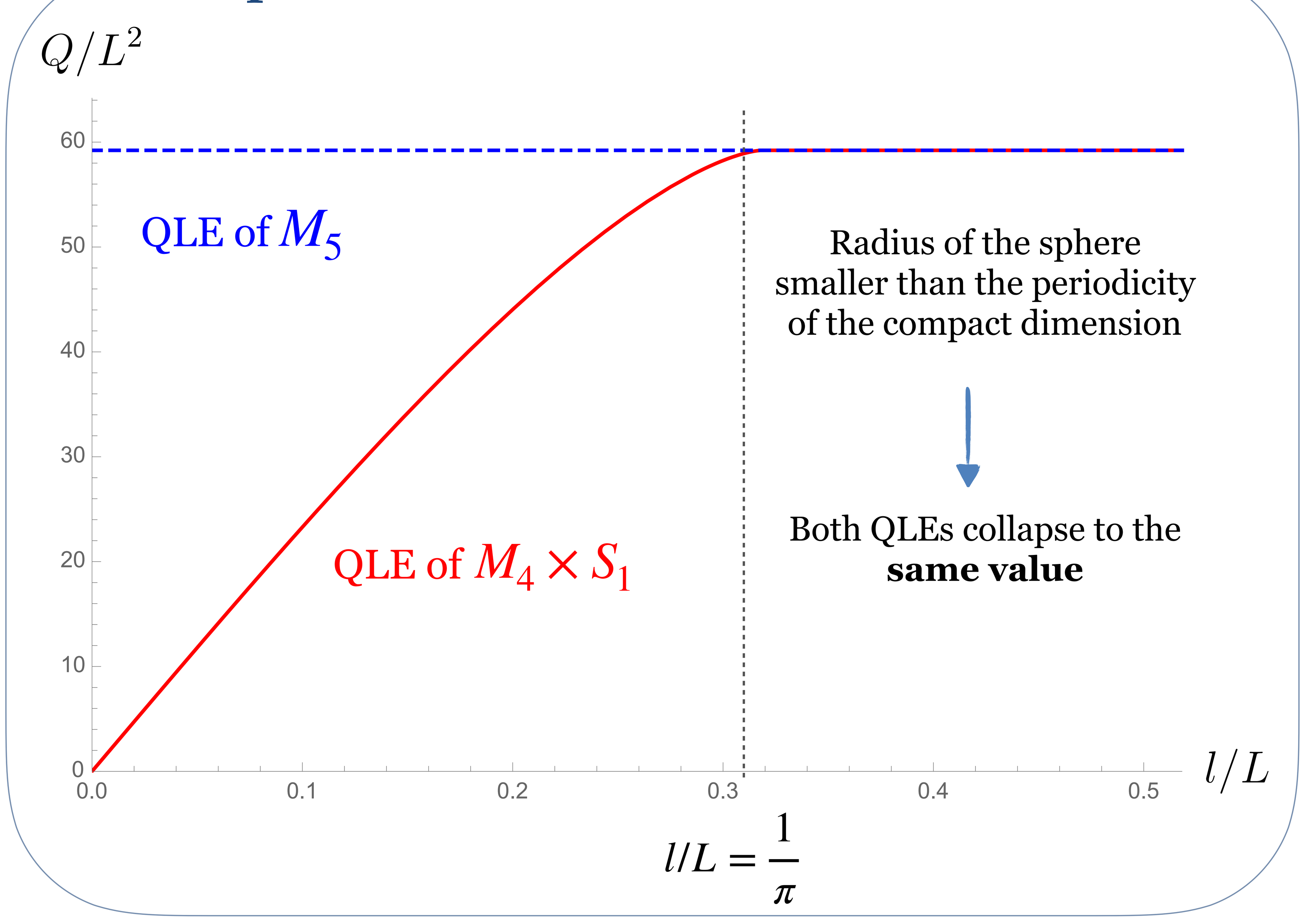
The trace of the extrinsic curvature is given by $K^\alpha = K^A n^\alpha_A = h^{ab} K^A_{ab} n^\alpha_A$

And the integration measure $\sqrt{h}\, n_\alpha\, dS = n_\alpha\, L^3 \sin^2\theta_1 \sin\theta_2\, d\theta_1 d\theta_2 d\theta_3$

The **QLE** corresponding to M_5

$$Q_{M_5} = \int_{\partial\Sigma} \sqrt{h}\, K^\alpha n_\alpha dS = 6\pi^2 L^2$$

Comparison of the QLE



Quasi-local energy

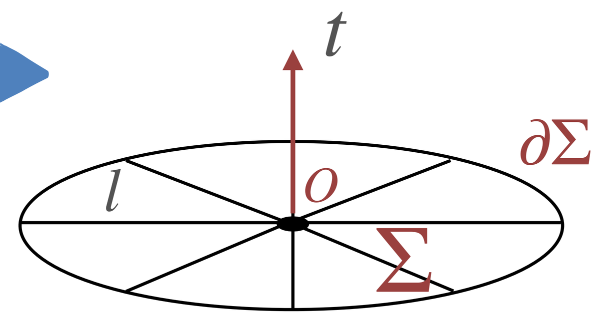
Associate to a given hypersurface of a spacetime,

m-dimensional $\Sigma \hookrightarrow M$ n-dimensional

the **integral of the trace of the extrinsic curvature**.

Solution

Geodesic ball



Take a point and send geodesics of fixed length normal to the time direction.

This defines a spacelike hypersurface Σ .

The **quasi-local energy** (QLE) is defined as

$$Q(\Sigma) \equiv \int_{\partial\Sigma} K - E_0$$

where the **extrinsic curvature tensor** is given by

$$K^A_{ab} \equiv -t^\lambda_b t^\alpha_a \nabla_\lambda n^A_\alpha$$

t^α_a Tangent vectors $a = 1,\dots,m$

n^A_α Normal vectors $A = 1,\dots,p$

The extrinsic curvature of the hypersurface is sensitive to the strength of the gravitational field

$p \equiv (n-m)$ Codimension of the hypersurface

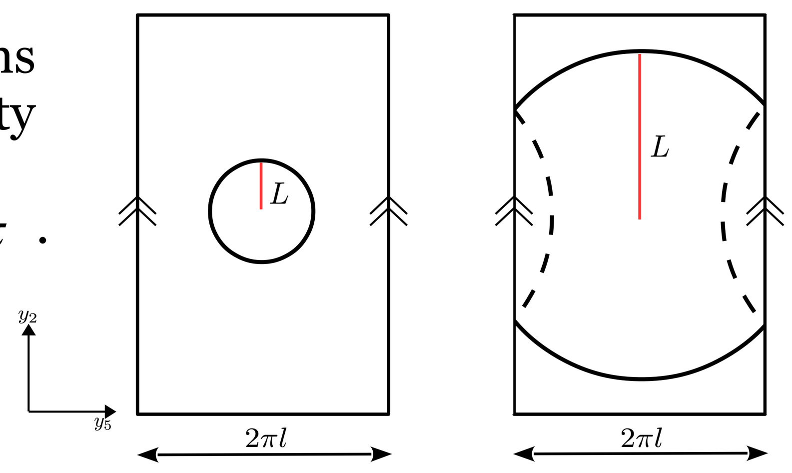
Codimension-2 spheres in $M_4 \times S_1$

Everything looks the same as in the previous case, but now **the integration is different** for the compact coordinate y_5 .

For **small 3-spheres** that completly lie within the compact dimension,

$$-\frac{L}{2} \le y_5 \le \frac{L}{2}$$

When $L < 2\pi l$, there are self intersections of the hypersurface, due to the periodicity of the compact dimension.
The **range is restricted** $-l\pi \le y_5 \le l\pi$.



The **QLE** yields

$$Q_{M_4 \times S_1} = 6\pi^2 L^2 \quad for \quad L \le l\pi$$

$$Q_{M_4 \times S_1} = 12\pi^2 l \sqrt{L^2 - \pi^2 l^2} + 12\pi L^2 \tan^{-1}\left(\frac{\pi l}{\sqrt{L^2 - \pi^2 l^2}}\right) \quad for \quad L > l\pi$$

Outlook

The QLE could provide an **energetical argument** in favour of compactified or uncompactified spacetimes.

More general setups need to be studied, such as the introduction of fluxes in order to stabilise the compact dimensions.

It could be interesting to see whether the QLE can be used to compute the **total energy** of full spacetimes in a **more covariant way**.

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