STATISTICAL MECHANICS IN RELATIVISTIC NUCLEUS-NUCLEUS COLLISIONS

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1. Basic concepts of statistical mechanics.
2. Hadron resonance gas: mean particle multiplicities in nucleus-nucleus collisions.
3. Quark-gluon plasma: lattice QCD and phenomenological models.
4. Phase diagram of strongly interacting matter: 1-st order phase transition and critical point.
5. Event-by-event fluctuations in high energy collisions: statistical models.

The units used in high energy physics:

$$\frac{h}{2\pi} = c = k = 1 .$$

All dimensional quantities are then measured in the energy units "MeV" (or GeV=10^3 MeV)

$$m_\pi \cong 140 \text{ MeV} , \quad m_p \cong 938 \text{ MeV} , \quad T = 10 - 200 \text{ MeV} ,$$
$$1 \text{ fm} \cong 10^{-13} \text{ cm} , \quad 1 \text{ fm}/c \cong 10^{-23}\text{sec} , \quad 1 \text{ fm} = \frac{1}{197 \text{ MeV}} .$$

Statistical Mechanics. Partition function = sum over all possible microstates. Microstates are found at some special conditions. These conditions are known as the statistical ensembles.

(V, E, N) – MCE

$$Z(V, E, N) = \sum_{\text{states}} \delta[E - E_s]\ \delta[N - N_s]$$

(V, T, N) – CE

$$Z(V, T, N) = \sum_{\text{states}} \exp\left[-E/T\right] \delta[N - N_s]$$

(V, T, μ) – GCE

$$Z(V, T, \mu) = \sum_{\text{states}} \exp\left[\mu N/T\right]\ \exp\left[-E/T\right]$$

I. BASIC CONCEPTS OF STATISTICAL MECHANICS

In this section we briefly remind the basic concepts of statistical mechanics and the properties of the simplest statistical systems where one can make all calculations analytically.

A. Canonical (V,T,N) Ensemble

We start from the canonical ensemble (CE) of non-relativistic classical particles (with Boltzmann statistic) that applies for the system with fixed volume V, temperature T and number of particles N. The partition function in CE reads [**?**]:

$$Z_{ce}(V,T,N) \;=\; \frac{1}{N!}\,\frac{g}{(2\pi)^3}\int d\mathbf{x}_1 d\mathbf{p}_1 \ldots \frac{g}{(2\pi)^3}\int d\mathbf{x}_N d\mathbf{p}_N\;\exp\left[-\,\frac{E_N}{T}\right]\;. \qquad (1)$$

Here g is a particle degeneracy factor; $N!$ stands because we consider classical indistinguishable particles (**Gibbs**); $d\mathbf{x} \equiv d\vec{x} \equiv d^3x \equiv dx\,dy\,dz$ and similarly $d\mathbf{p}$ are the spatial and momentum coordinates of a particle, and E_N is the microscopic N-particle energy usually presented as the sum of potential and kinetic terms:

$$E_N \;=\; U_N(\mathbf{x}_1,\ldots,\mathbf{x}_N) \;+\; \sum_{i=1}^{N}\epsilon_i\;, \qquad (2)$$

where

$$\epsilon_i \;=\; \sqrt{m^2+\mathbf{p}_i^2} \;\rightarrow\; m \;+\; \frac{\mathbf{p}_i^2}{2m} \;\rightarrow\; \frac{\mathbf{p}_i^2}{2m} \qquad (3)$$

is non-relativistic one-particle energy with with m being the particle mass (**to omit term** m **is only possible if** N**=const**). For the N-particle energy given by (3) the momentum distribution is universal (**Maxwell-distribution**). The integration over momentum in Eq. (1) can be done explicitly. The integral depends only on $\mathbf{p}_i^2 = p_i^2$ therefore we can make the integration in the spherical coordinate system, $d\mathbf{p}_i = 4\pi p^2 dp$:

$$\int_0^\infty \frac{d\mathbf{p}_i}{(2\pi)^3}\;\exp\left[-\,\frac{\mathbf{p}_i^2}{2mT}\right] \;=\; \int_0^\infty \frac{p_i^2 dp_i}{2\pi^2}\;\exp\left[-\,\frac{p_i^2}{2mT}\right] \;=\; \left(\frac{mT}{2\pi}\right)^{3/2}\;.$$

It gives for the partition function (we assume $g=1$ for simplicity):

$$Z_{ce}(V,T,N) \;=\; \frac{1}{N!}\left(\frac{mT}{2\pi}\right)^{3N/2}\int_V d\mathbf{x}_1\ldots\; d\mathbf{x}_N\exp\left[-\,\frac{U_N}{T}\right]\;.$$

The particle coordinates $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are integrated over the system volume V. The CE thermodynamical functions can be expressed in terms of the Helmholtz free energy:

$$F(V,T,N) ~=~ -T~\ln~Z_{ce}(V,T,N)~. \tag{4}$$

The CE entropy S, pressure P, average energy $\overline{E}$ and chemical potential μ are equal to:

$$P ~=~ -\left(\frac{\partial F}{\partial V}\right)_{T,N}~, \qquad\qquad S ~=~ -\left(\frac{\partial F}{\partial T}\right)_{V,N}~, \tag{5}$$

$$\overline{E} ~=~ F ~+~ TS~, \qquad\qquad \mu ~=~ \left(\frac{\partial F}{\partial N}\right)_{T,V}~. \tag{6}$$

For the non-interacting particles, the potential energy vanishes, $U_N = 0$, thus, the coordinate integration gives the system volume:

$$\int_V d\mathbf{x}_1 \ldots d\mathbf{x}_N ~=~ V^N~. \tag{7}$$

The ideal gas partition function and the free energy are:

$$Z_{ce} ~=~ \frac{V^N}{N!}\left(\frac{mT}{2\pi}\right)^{3N/2}~,$$

$$F ~\cong~ -NT ~-~ NT~\ln\left[\frac{V}{N}\left(\frac{mT}{2\pi}\right)^{3/2}\right]~,$$

where we have assumed $N \gg 1$ and used Stirling's formula: $\ln N! \cong N(\ln N - 1)$. The thermodynamical functions of the ideal gas from Eqs. (5-6) read:

$$P ~=~ \frac{NT}{V}~, \qquad\qquad S ~=~ \frac{5}{2}N ~+~ N~\ln\left[\frac{V}{N}\left(\frac{mT}{2\pi}\right)^{3/2}\right]~, \tag{8}$$

$$\overline{E} ~=~ \frac{3}{2}NT~, \qquad\qquad \mu ~=~ -T~\ln\left[\frac{V}{N}\left(\frac{mT}{2\pi}\right)^{3/2}\right]~. \tag{9}$$

Ideal gas equations $PV = NT$ and $E = 3TN/2$ are known from the **school text-books**. The entropy (8) goes to $-\infty$ at $T \to 0$. This is in contradiction with 3rd law of thermodynamics. To have $S = +0$ at $T = 0$ one needs **quantum mechanics – Bose and Fermi statistics**. Classical formulation considered above is called the **Boltzmann approximation**.

B. Grand Canonical (V,T,μ) Ensemble

The system with fixed volume, temperature and chemical potential is described by the grand canonical ensemble (GCE),

$$Z_{gce}(V,T,\mu) \;=\; \sum_{N=0}^{\infty} \exp\left(\frac{\mu\, N}{T}\right)\; Z_{ce}(V,T,N)\;, \tag{10}$$

where $Z_{ce}(V,T,N)$ is given by Eq. (1) and μ is the **chemical potential**. The thermodynamic potential in GCE is called grand potential and is defined similar to the CE (4), and MCE (21):

$$\Omega(V,T,\mu) \;=\; -T\,\ln\, Z_{gce}(V,T,\mu)\;. \tag{11}$$

The pressure, entropy, energy and the number of particles are obtained similarly:

$$P \;=\; -\left(\frac{\partial\Omega}{\partial V}\right)_{T,\,\mu} \;=\; P(T,\mu)\;, \qquad\qquad S \;=\; -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\,\mu}\;, \tag{12}$$

$$\overline{E} \;=\; \Omega \;+\; TS \;+\; N\mu\;, \qquad\qquad \overline{N} \;=\; \left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V}\;. \tag{13}$$

The substitution of Z_{ce} in (10) gives the GCE partition function of non-relativistic non-interacting Boltzmann particles:

$$Z_{gce} \;=\; \exp\left[\,V\,\left(\frac{mT}{2\pi}\right)^{3/2}\, e^{\mu/T}\,\right]\;. \tag{14}$$

It gives:

$$P \;=\; T\,\left(\frac{mT}{2\pi}\right)^{3/2}\, e^{\mu/T}\;, \qquad\qquad S \;=\; V\,\left(\frac{mT}{2\pi}\right)^{3/2}\, e^{\mu/T}\left(\frac{5}{2}\;-\;\frac{\mu}{T}\right)\;, \tag{15}$$

$$\overline{E} \;=\; \frac{3}{2}\,\overline{N}\,T\;, \qquad\qquad \overline{N} \;=\; V\,\left(\frac{mT}{2\pi}\right)^{3/2}\, e^{\mu/T}\;, \tag{16}$$

and

$$Z_{gce} \;=\; \exp\left[\;\overline{N}\;\right]\;. \tag{17}$$

Substituting $\overline{N}$ in the expressions for P and S one can see that the GCE and CE (8), (9) coincide for $\overline{N} = N$. Thus, CE (T,N,V) (8, 9) and GCE (T,μ,V) (15, 16) are equivalent. **This thermodynamical equivalence of statistical ensembles are only valid in the**

thermodynamical limit $V \to \infty$. Statistical fluctuations are different in different ensembles even in the thermodynamical limit.

The function $p(T,\mu)$ is the main function in the GCE:

$$\frac{N}{V} \equiv n \;=\; \left(\frac{\partial p}{\partial T}\right)_{\mu}~, \qquad \frac{S}{V} \;\equiv s \;=\; \left(\frac{\partial p}{\partial T}\right)_{\mu}~, \qquad \frac{E}{V} \;\equiv\; \varepsilon \;=\; Ts \;+\; \mu n \;-\; p~. \tag{18}$$

In the relativistic gas particles can be created and annihilated. The number of particles is not conserved. In relativistic gas only the **charges** (e.g., baryonic number, electric charge, and strangeness are conserved).

C. Micro Canonical (V,E,N) Ensemble

The system of non-interacting particles with fixed volume, number of particles and energy, instead of temperature, is described by the micro canonical ensemble (MCE). The corresponding partition function reads:

$$Z_{mce}(V,E,N) \;=\; \frac{1}{N!}\,\frac{V^N}{(2\pi)^{3N}}\int d\mathbf{p}_1 \ldots \int d\mathbf{p}_N\;\delta\,[\,E - E_N\,] \tag{19}$$

The difference of the MCE partition function (19) from the CE one (1) is in the δ-function that provides energy conservation. One finds,

$$Z_{mce} \;=\; \frac{V^N}{N!}\left(\frac{mE}{2\pi}\right)^{3N/2}\frac{1}{E\;\Gamma(3N/2)}~, \tag{20}$$

The MCE entropy plays the role of the CE Helmoltz free energy (4). It is defined as,

$$S(V,E,N) \;=\; \ln\left[\,E_0\;Z_{mce}(V,E,N)\,\right]~. \tag{21}$$

where E_0 is an arbitrary constant with a dimension of energy. The MCE temperature, pressure, and chemical potential are the following:

$$P \;=\; T\left(\frac{\partial S}{\partial V}\right)_{E,N}~, \qquad\qquad \frac{1}{T} \;=\; \left(\frac{\partial S}{\partial E}\right)_{V,N}~,$$

$$E \;=\; const~, \qquad\qquad \mu \;=\; -\,T\left(\frac{\partial S}{\partial N}\right)_{V,E}~,$$

Assuming $N \gg 1$, using Stirling's formula $\Gamma(3N/2) \cong 3N/2\ [\ln(3N/2) - 1]$, and choosing $E_0 = E$, which gives $\ln(E_0 \cdot E^{3N/2-1}) = \ln(E^{3N/2})$, one obtains:

$$P = \frac{2}{3}\frac{E}{V} = \frac{NT}{V}\,, \qquad S = \frac{5}{2}N + N\ln\left[\frac{V}{N}\left(\frac{mE}{3N\pi}\right)^{3/2}\right]\,, \tag{22}$$

$$E = const\,, \qquad \mu = -\,T\ln\left[\frac{V}{N}\left(\frac{mE}{3N\pi}\right)^{3/2}\right]\,, \tag{23}$$

If $\overline{E} = E$, the ideal gas of N particles in the volume V has the same temperature, pressure, chemical potential and entropy in the MCE (22,23) and in the CE (8,9). This means the **thermodynamical equivalence** of CE and MCE at $N \gg 1$. The statistical mechanics can be therefore reduced to a single postulate: at fixed system energy all microstates have the same probability (MCE).

A). MCE $\rightarrow$ CE. Let us consider the MCE with energy $E_{\rm tot}$ which consists of the system with energy E and thermostat with N_0 number of classical noninteracting particles. It is assumed that: $E_0 = E_{\rm tot} - E \equiv 3TN_0/2 \gg E$.

$$f(E) \sim \int d\mathbf{k}_1 \ldots d\mathbf{k}_{N_0}\, \delta\left(E_{\rm tot} - \sum_{j=1}^{N_0}\frac{\mathbf{k}_j^2}{2m} - E\right) \sim (E_{\rm tot} - E)^{3N_0/2}$$
$$\sim \left(1 - \frac{E}{3TN_0/2}\right)^{3N_0/2} \cong \exp\left(-\frac{E}{T}\right) \tag{24}$$

B). MCE $\rightarrow$ GCE.

C). $\exp(-E/T)$ versus $\delta(E - E_0)$.

D. Ultra-Relativistic Gas

This is the Boltzmann approximation for the photon gas. It corresponds to the ultra-relativistic limit where particle momentum is much bigger than the mass,

$$\varepsilon_i = \sqrt{m^2 + p_i^2} \rightarrow p_i\,. \tag{25}$$

Number of photons is not fixed. Thus the GCE should be considered. Number of photons is not conserved, Thus $\mu = 0$. The GCE partition function reads:

$$Z_{gce}(V,T,\mu=0) \;=\; \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{gV}{2\pi^2}\right)^N \int_0^{\infty} p_1^2 dp_1 \ldots p_N^2 dp_N \;\exp\left[-\sum_{i=1}^{N}\frac{p_i}{T}\right]$$
$$=\; \sum_{N=0}^{\infty} \frac{1}{N!}\;\left(\frac{gVT^3}{\pi^2}\right)^N \;=\; \exp[\;\overline{N}\;]\;, \qquad (26)$$

It gives:

$$P \;=\; \frac{g}{\pi^2}T^4\;, \qquad\qquad s \;=\; \frac{4g}{\pi^2}T^3\;, \qquad (27)$$
$$\varepsilon \;=\; \frac{3g}{\pi^2}T^4 \;\equiv\; \sigma_{\rm SB}T^4\;, \qquad\qquad n \;=\; \frac{g}{\pi^2}T^3\;. \qquad (28)$$

E. Bose and Fermi gases

Let us consider non-interacting quantum particles in the box

$$V \;=\; L_x \cdot L_y \cdot L_z$$

with periodic boundary conditions. Particle momenta have the discrete values

$$k_i = \frac{2\pi}{L_i}\,l_i\;, \qquad l_i = 0, \pm 1, \pm 2, \ldots, \qquad i = x, y, z\;.$$

Particle energy is

$$\epsilon(k) \;=\; \sqrt{m^2+\mathbf{k}^2}\;, \qquad \mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2\;.$$

The microscopic state of the gas is defined by the occupation numbers $\{n_k\}$.

$$E \;=\; \sum_{\mathbf{k}} n_k\,\epsilon(k)\;, \qquad N \;=\; \sum_{\mathbf{k}} n_k\;.$$

The grand canonical partition function reads:

$$Z_{gce}(V,T,\mu) \;=\; \sum_{\{n_k\}} \exp\left[\frac{1}{T}\sum_{\mathbf{k}}(\mu \;-\; \epsilon(k))\,n_k\right]\;.$$

$$\exp\left[\frac{1}{T}\sum_{\mathbf{k}}(\mu \;-\; \epsilon(k))\,n_k\right] \;=\; \prod_{\mathbf{k}} x^{n_k}\;, \qquad x \equiv \exp\left[\frac{1}{T}\;(\mu \;-\; \epsilon(k))\,n_k\right]$$

Thus, different $\mathbf{k}$ levels are independent (uncorrelated). Let us consider one $\mathbf{k}$-level. Single level partition function reads:

$$z_k \;=\; \sum_{n_k} x^{n_k} \;.$$

There are two type of particle in nature: Bosons with $n_k = 0, 1, 2, ...$ and Fermions with $n_k = 0, 1$. One finds for Bosons,

$$z_k \;=\; \sum_{n_k=0}^{\infty} x^{n_k} \;=\; \frac{1}{1-x} \;,$$

and for Fermions

$$z_k \;=\; \sum_{n_k=0,\,1} x^{n_k} \;=\; 1 + x \;.$$

The average values of the occupation numbers can be calculated as

$$\langle n_k \rangle \;=\; \frac{1}{z_k} \sum_{n_k} n_k \, x^{n_k} \;.$$

One finds

$$\langle n_k \rangle \;=\; \frac{x}{1-x} \;=\; \frac{1}{\exp\left[\left(\epsilon(k) \;-\; \mu\right)/T\right] \;-\; 1} \quad \text{for Bosons}$$

and

$$\langle n_k \rangle \;=\; \frac{x}{1+x} \;=\; \frac{1}{\exp\left[\left(\epsilon(k) \;-\; \mu\right)/T\right] \;+\; 1} \quad \text{for Fermions.}$$

In large systems

$$\sum_{\mathbf{k}} \ldots \;=\; \frac{g\,V}{(2\pi)^3} \int d^3k \;\ldots$$

. This is not valid for $\mathbf{k} = 0$ in a case of the Bose condensation.

The system pressure equals to:

$$P(T,\mu) \;=\; \frac{gV}{6\pi^2} \int_0^{\infty} k^2 dk \, \frac{k^2}{\sqrt{m^2 + k^2}} \left[\exp\left(\frac{\sqrt{m^2+k^2} \;-\; \mu}{T} \right) \;-\; \gamma \right]^{-1} \;, \tag{29}$$

with $\gamma = 1$ for Bosons, $\gamma = -1$ for Fermions, and $\gamma = 0$ corresponds to the Boltzmann approximation. With Eq.(29) one obtains:

$$s \;=\; \left(\frac{\partial p}{\partial T}\right)_{\mu} \;, \quad n \;=\; \left(\frac{\partial p}{\partial \mu}\right)_{T} \;, \quad \varepsilon \;=\; Ts \;+\; \mu n \;-\; P \;. \tag{30}$$

F. Partition function and mean multiplicity in the CE

Let us consider the system which consists of one sort of positively and negatively charged particles (e.g. π^+ and π^- mesons) with total charge equal to zero $Q = 0$. The chemical potential μ_Q regulates the conserved charge Q. In the case of the Boltzmann ideal gas (the interactions and quantum statistics effects are neglected) in the volume V and at temperature T the GCE partition function reads:

$$Z_{gce}(V,T) \;=\; \sum_{N_+=0}^{\infty}\sum_{N_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!}\,\frac{(\lambda_- z)^{N_-}}{N_-!} \;=\; \exp\left(\lambda_+ z \;+\; \lambda_- z\right) \;=\; \exp(2z)\;, \tag{31}$$

where $\lambda_+ = \exp(\mu_Q/T)$ and $\lambda_- = \exp(-\mu_Q/T)$, and $\mu_Q = 0$ leads to $\langle Q\rangle = 0$. In Eq. (31) z is a single particle partition function

$$z \;=\; \frac{V}{(2\pi)^3}\int d^3k\;\exp[-(k^2+m^2)^{1/2}/T] \;=\,=\; \frac{V}{2\pi^2}\;T\;m^2\;K_2(m/T)\;. \tag{32}$$

The CE partition function is obtained by an explicit introduction of the charge conservation constrain, $N_+ - N_- = 0$ **for each microscopic state of the system** and it reads:

$$Z_{ce}(V,T) \;=\; \sum_{N_+=0}^{\infty}\sum_{N_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!}\,\frac{(\lambda_- z)^{N_-}}{N_-!}\;\delta(N_+ - N_-) \;= \tag{33}$$

$$=\; \frac{1}{2\pi}\int_0^{2\pi} d\phi\;\;\exp\left[z\left(\lambda_+\,e^{i\phi} \;+\; \lambda_-\,e^{-i\phi}\right)\right] \;=\; I_0(2z)\;.$$

The average number of N_+ and N_- can be calculated:

$$\langle N_\pm\rangle_{gce} \;=\; z\;, \qquad \langle N_\pm\rangle_{ce} \;=\; z\,\frac{I_1(2z)}{I_0(2z)}\;. \tag{34}$$

The exact charge conservation leads to the CE suppression $(I_1(2z)/I_0(2z) < 1)$ of the charged particle multiplicity relative to the result for the GCE (91). The ratio of $\langle N_\pm\rangle$ calculated in the CE and GCE is plotted as a function of z in Fig. 5.

In the large volume limit ($V \to \infty$ corresponds also to $z \to \infty$) the results for mean quantities in the *c.e.* and *g.c.e.* are equal. This result is referred as an equivalence of the canonical and grand canonical ensembles. It can be obtained using an asymptotic expansion of the modified Bessel function [**?**]:

$$\lim_{z\to\infty} I_n(2z) \;=\; \frac{\exp(2z)}{\sqrt{4\pi z}}\;\left[1 \;-\; \frac{4n^2-1}{16z} \;+\; O\left(\frac{1}{z^2}\right)\right]\;, \tag{35}$$

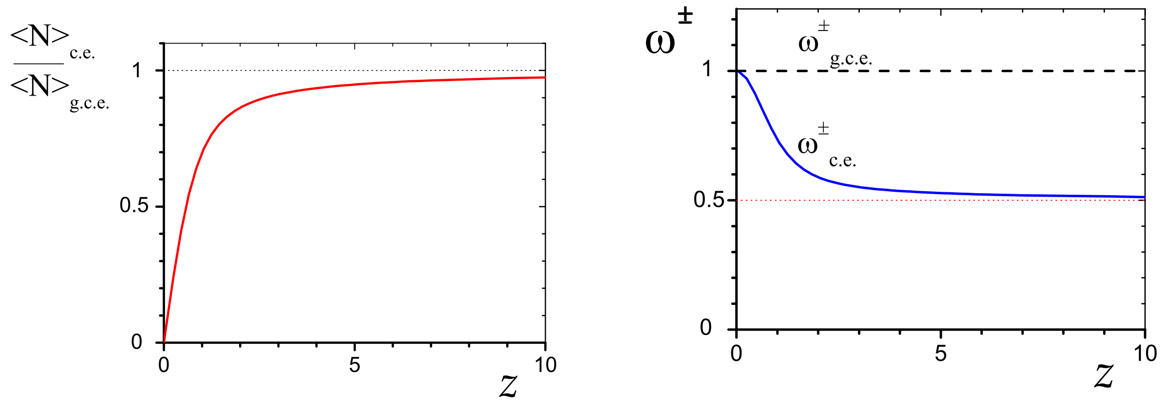

Figure 1: *Left*: The ratio of $\langle N_\pm\rangle_{c.e.}$ to $\langle N_\pm\rangle_{g.c.e.}$ as a function of z. *Right*: The scaled variances of $N_\pm$ calculated within the *g.c.e.*, $\omega^\pm_{g.c.e.}=1$, and *c.e.*, $\omega^\pm_{c.e.}$.

which gives $I_1(2z)/I_0(2z)\to 1$ and therefore

$$\langle N_\pm\rangle_{ce} \;\cong\; \langle N_\pm\rangle_{gce} = z\ . \tag{36}$$

Using the series expansion one gets [?] for small systems ($z\ll 1$):

$$I_n(2z) \;=\; \frac{z^n}{n!} \;+\; \frac{z^{n+2}}{(n+1)!} \;+\; O\left(z^{n+4}\right)\ , \tag{37}$$

and consequently $I_1(2z)/I_0(2z)\cong z$ which results in

$$\langle N_\pm\rangle_{c.e.} \;\cong\; z^2 \;\;\ll\;\; \langle N_\pm\rangle_{g.c.e.} \;=\; z\ . \tag{38}$$

The asymptotics of the mean multiplicity discussed above are clearly seen in Fig. **??**.

G. Baryon-antibaryon statistical system

One can consider the statistical production of anti-baryons within the canonical ensemble (CE) formulation. In this case the material conservation laws are imposed on each microscopic state of the system. This condition introduces a significant correlation between particles which carry conserved charges. The correlation reduces the effective number of degrees of freedom and consequently leads to the CE suppression of the charged particle multiplicity when compared with the result of the calculations done within GCE.

Let us consider the system of baryons 'b' and anti-baryons 'a' with total baryon number B as the Boltzmann ideal gas in the volume V, at temperature T. The CE partition function is

$$Z(T,V,B) \;=\; \sum_{N_b,N_a}^{\infty} \frac{(\lambda_a z_a)^{N_a}}{N_a!} \frac{(\lambda_b z_b)^{N_b}}{N_b!} \,\delta\left[B-(N_b-N_a)\right] \;, \tag{39}$$

where the single baryon (anti-baryon) partition function reads (see Eq. **??** in Appendix):

$$\begin{aligned} z_{a,b} \;=\; z_j(T,V) \;&=\; \frac{g_j V}{(2\pi)^3} \int d^3k \;\exp[-(k^2+m_j^2)^{1/2}/T] \;= \\ &=\; \frac{g_j V}{2\pi^2}\, T\, m_j^2\, K_2(m_j/T) \;\equiv\; V\; f_j(T)\;. \end{aligned} \tag{40}$$

The (anti-)baryon mass and degeneracy factor are denoted here by m_j and g_j, respectively. Auxiliary parameters λ_b and λ_a are introduced in order to calculate the mean number of baryons and anti-baryons and they are set to unity in the final formulae. By expressing the δ-function in (39) as

$$\delta(n) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{-in\phi}\;,$$

Eq. (39) becomes

$$\begin{aligned} Z(T,V,B) \;&=\; \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{-iB\phi} \sum_{N_b=0}^{\infty}\sum_{N_a=0}^{\infty} \frac{(\lambda_b z_b\; e^{i\phi})^{N_b}}{N_b!} \frac{(\lambda_a z_a\; e^{-i\phi})^{N_a}}{N_a!} \\ &=\; \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{-iB\phi}\; \exp\left[\sum_j z_j(\lambda_b\; e^{i\phi} \;+\; \lambda_a\; e^{-i\phi})\right]\;. \end{aligned} \tag{41}$$

This form of the CE partition function allows one to derive the mean numbers of baryons and anti-baryons

$$\langle N_j\rangle \;=\; \frac{1}{Z}\frac{\partial Z}{\partial \lambda_j}\Big|_{\lambda_b=\lambda_a=1}\;, \tag{42}$$

For $\lambda_b=\lambda_a=1$ the partition function (41) can be presented as the modified Bessel function

$$Z(T,V,B) \;=\; \frac{1}{2\pi}\int_0^{2\pi} d\phi\; e^{-iB\phi}\; \exp(2z\;\cos\phi) \;=\; I_B(2z)\;, \tag{43}$$

where $z \equiv z_a+z_b$. This yields final expressions for the mean number of baryons and anti-baryons

$$\langle N_b\rangle \;=\; z_b\,\frac{I_{B-1}(2z)}{I_B(2z)}\;, \qquad \langle N_a\rangle \;=\; z_a\,\frac{I_{B+1}(2z)}{I_B(2z)}\;. \tag{44}$$

As the exact baryon number conservation is imposed on each microscopic state it is evidently fulfilled also by the average values (44):

$$\langle N_b \rangle \;-\; \langle N_a \rangle \;=\; B \;, \tag{45}$$

as indeed can be easily seen from the identity $I_{n-1}(x) - I_{n+1}(x) = 2nI_n(x)/x$ [**?**]. Eq. (44) is valid for all combinations of B and z values. [1]

The CE expressions for the mean number of baryons and anti-baryons can be further simplified for the two limiting cases: $z \ll 1$ (small systems) and $z \gg 1$ (large systems). Using the representation of I_n as the infinite series [**?**]

$$I_n(2z) \;=\; \sum_{k=0}^{\infty} \frac{z^{n+2k}}{k!(n+k)!} \;,$$

one obtains for small systems

$$\langle N_b \rangle \;\cong\; B\,\frac{z_b}{z} + \frac{z_b \cdot z}{B+1} \;+\; o(z_b \cdot z^3)\;, \qquad \langle N_a \rangle \;\cong\; \frac{z_a \cdot z}{B+1} \;+\; o(z_a \cdot z^3)\;. \tag{46}$$

The dependence $\langle N_a \rangle \propto V^2/(B+1)$ is therefore observed from Eq. (46) for the anti-baryon yield in small systems.

One can show that for large systems ($z \gg 1$) the CE becomes equivalent to the GCE. In the thermodynamical limit $V \to \infty$, $B \to \infty$ with $B/V \equiv \rho_B = const(V)$. The uniform asymptotic expansion of the modified Bessel functions at $n \to \infty$ [**?**] can be used

$$I_n(nx) \cong \frac{1}{\sqrt{2\pi n}}\,\frac{\exp(n\eta)}{(1+x^2)^{1/4}}\left[1 + o\left(\frac{1}{n}\right)\right] \;;\;\; \eta \equiv \sqrt{1+x^2} + \ln\frac{x}{1+\sqrt{1+x^2}}\;.$$

It gives the same partition function and the average number of baryons and anti-baryons in the GCE and CE:

$$Z(V,T,\mu_B) = \sum_b \exp\left(\frac{\mu_B b}{T}\right) Z(T,V,b) = \exp\left[z\left(e^{\mu_B/T} + e^{-\mu_B/T}\right)\right]\;, \tag{47}$$

$$\langle N_b \rangle \;=\; z_j\;\exp(\mu_B/T)\;, \qquad \langle N_a \rangle \;=\; z_j\;\exp(-\mu_B/T)\;, \tag{48}$$

[1] The specific case of $B = 0$ in the nucleon–antinucleon gas (i.e., no resonances included) the Eq. (44) reduces to the result of Rafelski and Danos [**?**].

where μ_B is a baryon chemical potential which is defined as:

$$\exp(\mu_B/T) \;=\; \frac{B}{2z} \;+\; \sqrt{1+\left(\frac{B}{2z}\right)^2}\;. \tag{49}$$

From Eqs. (44) and (48) the anti-baryon densities for the CE and GCE are equal to

$$\frac{\langle N_a\rangle}{V}\;|_{CE} \;\equiv\; n_a\;|_{CE}=\;f_j\;\frac{I_{B+1}(2z)}{I_B(2z)}=f_j\frac{I_{B+1}(xB)}{I_B(xB)}\cong f_j\frac{x}{2}\frac{B}{B+1}\;, \tag{50}$$

$$\frac{\langle N_a\rangle}{V}\;|_{GCE} \;\equiv\; n_a\;|_{GCE}=\;f_j\;\exp(-\mu_B/T)\;=\;f_j\;\frac{x}{1+\sqrt{1+x^2}}\;, \tag{51}$$

where $f \equiv \sum_j f_j$, $x \equiv 2z/B = 2f/\rho_B$, and the last approximation in Eq. (50) is valid for small systems only [2]. Equations (50,51) give us the primary thermal density for all individual anti-baryon states j. Each non-strange resonance (anti)baryon state decays finally into (anti)nucleon plus meson(s). Therefore, the total (primary plus resonance decay) antinucleon density equals to the total thermal anti-baryon density, $n_a = \sum_j n_a$ and is given by Eqs. (50,51) with the substitution of f_j by a sum $f = \sum_j f_j$.

We define a canonical suppression factor

$$F_{cs} \;\equiv\; \frac{(n_a)_{CE}}{(n_a)_{GCE}}\;. \tag{52}$$

It quantifies the antinucleon suppression due to the exact baryon number conservation and is the same for any individual anti-baryon state.

In the $B=0$ case the baryon and anti-baryon densities are equal and Eqs. (50) and (51) yield

$$n_a\;|_{CE}=n_b\;|_{CE}=\;f_j\;\frac{I_1(2z)}{I_0(2z)}\;\cong\;f_j\;z,\quad n_a\;|_{GCE}=n_b\;|_{GCE}=\;f_j\;, \tag{53}$$

where the approximation for the CE density is valid for small system only. The canonical suppression factor (52) for $B=0$ is equal to

$$F^0_{cs}\;=\;\frac{I_1(2z)}{I_0(2z)}\;\cong\;f\;V\;. \tag{54}$$

The approximation in Eq. (54) is valid for small system only ($z \equiv fV \ll 1$).

[2] Introducing the variable x we have transformed the finite size V-dependence of the CE density (50) into its dependence on the baryon number B.

The behavior of the canonical suppression factor F_{cs}^0 (54) is shown by the solid lines in Fig. 2, Left for $T = 160$ MeV, 170 MeV and 180 MeV, assuming that f is the sum of f_j over all non-strange baryons. The lines start from $V = 5$ fm^3, which is approximately equal to the estimate of the hadronization volume for $e^+ + e^-$ interactions at $\sqrt{s} = 29$ GeV [?]. One observes (see Fig. 2, Left) a strong CE suppression of the (anti)baryon density. For $T = 160$ MeV the (anti)baryon density increases by a factor of 10 from its value at $V = 5$ fm^3 to its $V \to \infty$ *GCE* limit. For the small systems the (anti)baryon density increases approximately linearly with V, i.e., the (anti)baryon multiplicity for the small systems is proportional to V^2. The CE suppression becomes less pronounced and the volume region with linear increase of the (anti)baryon density is reduced for increasing temperature.

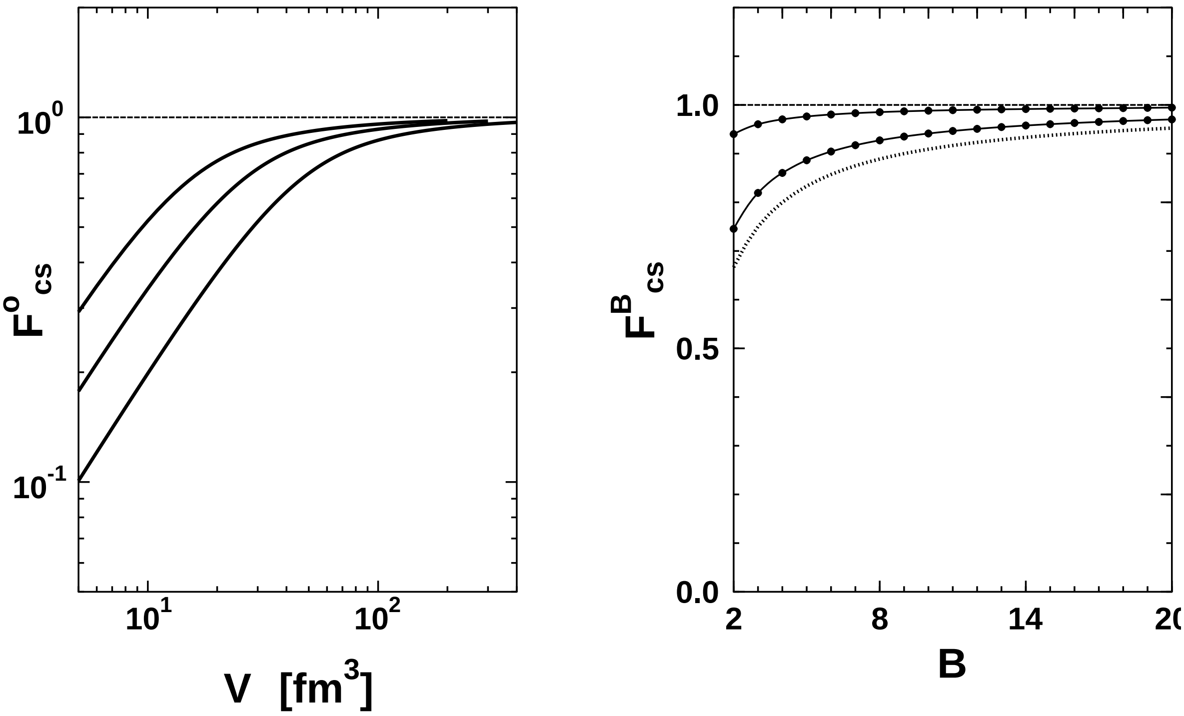

Figure 2: *Left* The solid lines show the CE suppression factor F_{cs}^0 (52) for $T = 160$ MeV, 170 MeV and 180 MeV (from bottom to top) for $B = 0$. *Right* The finite size B-dependence of the anti-baryon production in baryon rich ($B \geq 2$) systems at different values of the variable x ($x \equiv 2z/B = 2f/\rho_B$). The solid lines show the CE suppression factor F_{cs}^B (55) for x=1 and x=5 (from below to above). The lower dotted line corresponds to the limiting $B/(B+1)$ behavior (56).

Let us now turn to the anti-baryon production in baryon rich system. In the analysis of data on particle multiplicities in p+p, p+A and A+A collisions one usually assumes that all participating nucleons in the collisions (wounded nucleons) take part in the statistical hadronization of the system. It means that in the analysis of the NA49 results on antiprotons from p+p interactions to central Pb+Pb collisions at 158 A·GeV we should study statistical systems with $2 \leq B \leq 400$. The pion to baryon ratio in the statistical model is determined by two parameters: the temperature and baryon density. Thus as the temperature is found to be constant ($T = 175 \pm 15$ MeV) we conclude that the baryon density at hadronization in nuclear collisions at 158 A·GeV is also approximately constant.

Therefore, we study the evolution of the anti-baryon density with increasing net baryon number B at $T = const$ and $\rho_B = const$. The CE suppression factor (52) is found at these conditions from Eqs. (50,51)

$$F_{cs}^B \;=\; \frac{1+\sqrt{1+x^2}}{x}\;\frac{I_{B+1}(xB)}{I_B(xB)}\;; \qquad\qquad x \equiv \frac{2f}{\rho_B}\;. \tag{55}$$

Its B-dependence is plotted in Fig. 2, Right for several different values of the parameter x. Note that our assumption $T = const$ and $\rho_B = const$ for statistical hadronization at different values of B can be substituted by a weaker one, $x = const$. From Fig. 2, Right one observes that the CE suppression of anti-baryon density becomes stronger at high baryon density (i.e., small x). For $x < 1$ the CE suppression F_{cs}^B (55) becomes close to its $x \to 0$ limit:

$$F_{cs}^B \;=\; \frac{B}{B+1}\;. \tag{56}$$

Eq. (56) shows that the strongest CE suppression of the anti-baryon density is for the $B = 2$ (nucleon–nucleon interactions) case and it leads to the suppression factor of 2/3. This moderate effect of CE suppression is in strong contrast with the large CE suppression (i.e., $F_{cs}^0 \ll 1$) in the baryon–free system. A mathematical reason of this very different behavior for $B = 0$ and $B \geq 2$ (with $\rho_B = const(V)$) is due to the fact that in the latter case both the order of the modified Bessel functions and their arguments are dependent on B (i.e., on V) whereas in the $B = 0$ case only the argument increases with V.

The presence of non-zero baryon number $B > 0$ has a twofold effect on anti-baryon production. First, it suppresses the production of anti-baryons: the additional factors $\exp(-\mu_B/T) =$

$x/(1+\sqrt{1+x^2}) < 1$ and $1/(B+1) < 1$ appear respectively in the 'large' and 'small' systems for the anti-baryon density in comparison with the $B=0$ case. On the other hand, the CE suppression effect due to the exact baryon number conservation becomes smaller: at fixed T and V the following inequality is always valid, $F_{cs}^B > F_{cs}^0$. For fixed $B>0$ the CE suppression of anti-baryons becomes smaller when ρ_B decreases and it disappears completely (i.e., $F_{cs}^B \to 1$) in the limit $\rho_B \to 0$ (and respectively $V \to \infty$ in order to keep the B value fixed). This is because the total number of baryon–anti-baryon pairs becomes large due to large V. Note that in this case the last approximation in Eq. (50) is no more valid. Instead one should use the large argument asymptotic of the modified Bessel functions.

Thus for $B \geq 2$ systems at constant $x = 2f/\rho_B$ the CE suppression factor F_{cs}^B (55) ranges between 2/3 and 1 for $x \ll 1$ and between $(1-1/4x)$ and 1 for $x \gg 1$.

The statistical model with constant hadronization temperature correctly reproduces the weak dependence of the $\bar{p}/\pi$ ratio on the system size in p+p and nuclear collisions at the CERN SPS energy. A description of the ratio of J/ψ mesons to pions within the statistical hadronization model requires also a constant temperature parameter in p+p and A+A collisions at the CERN SPS. However, the same model with $T = const$ does not give a natural explanation of the approximate independence of the $\bar{p}/\pi$ ratio of collision energy in e^++e^- interactions. Therefore, a consistent description of hadron production within the statistical hadronization model has not yet been achieved.

II. HADRON RESONANCE GAS

	u	d	s	c	b	t
m (MeV)	~ 5	~ 10	~ 150	~ 1500	~ 5000	$\sim 1.7 \cdot 10^5$
B	1/3	1/3	1/3	1/3	1/3	1/3
Q	2/3	-1/3	-1/3	2/3	-1/3	2/3
S	0	0	- 1	0	0	0
charm	0	0	0	1	0	0
beauty	0	0	0	0	- 1	0
top	0	0	0	0	0	1

Table I: Quarks

Mesons=$(q_i, \overline{q_j})$:

$$\pi^0(135)\ ,\quad \pi^{\pm}(139)\ ,\quad \eta(548)\ ,\quad \rho(770)\ ,\quad \omega(782)\ ,\quad ...\ ,\ f_2(2340)\ ,$$
$$\text{strange mesons}:\quad K^{\pm}(494)\ ,\quad K^0(497)\ ,\quad ...\ ,\ K_4^*(2045)$$
$$\text{charm mesons}:\quad D^0(1865)\ ,\quad D^{\pm}(1870)\ ,\quad ...\ ,\ D_2^*(2460)$$

Baryons=q_i, q_j, q_k

$$B=1,\ S=0:\quad p(938)\ ,\quad n(940)\ ,\quad \eta(548)\ ,\quad \Delta(1232)\ ,\quad N(1440)\ ,\quad ...\ ,\ \Delta(2420)\ ,$$
$$B=1,\ S=-1\ :\quad \Lambda(1116)\ ,\quad ...\ \Lambda(2350)\ ,\quad \Sigma^+(1189)\ ,...,\Sigma(2250)\ ,$$
$$B=1,\ S=-2:\quad \Xi^0(1314)\ ,\quad ...\ \Xi(2030)\ ,$$
$$B=1,\ S=-3:\quad \Omega^-(1672)\ ,\quad ...\quad \Omega(2250)\ .$$

We introduce the GCE formulation of a gas of hadrons with arbitrary masses and quantum statistics. The conservation charges are: electric charge, baryon number, and strangeness. Considering all spectrum of hadrons and resonances with corresponding widths we also effectively take into account the interactions between particles.

$$\varepsilon_i \;=\; \sqrt{m_i^2 + p_i^2}\ , \tag{57}$$

The Bose and Fermi distributions ($\gamma = +1$ for Bose and -1 for Fermi statistics),

$$\frac{1}{\exp\left[\left(\sqrt{m_i^2 + p_i^2} \;-\; \mu_i\right)/T\;\right] \;-\; \gamma}\ , \tag{58}$$

are used for mesons and (anti)baryons, respectively. The chemical potential μ_i is the sum of the three chemical potentials that correspond to three conserved charges B, Q, and S:

$$\mu_i \;=\; q_i \mu_Q \;+\; b_i \mu_B \;+\; s_i \mu_S\ , \tag{59}$$

and q_i, b_i, s_i are the corresponding charges of a particle i.

Resonance decays are included by additional integration with Breit-Wigner mass distribution of resonances with corresponding widths Γ_i.

$$\langle N_i^{\mathrm{prim}} \rangle \;=\; \frac{g_i\, V}{2\pi^2} \int_0^\infty k^2 dk\ \int ds\ \frac{1}{\exp\left[\left(\sqrt{s^2 + k^2} \;-\; \mu_i\right)/T\;\right] \;-\; \gamma_i}\, \frac{1}{\pi}\, \frac{m_i\, \Gamma_i}{(s - m_i^2)^2 + m_i^2 \Gamma_i^2}\ , \tag{60}$$

Resonance decays with corresponding branching ratios Br:

$$\langle N_i \rangle \;=\; \overline{N}_i^{\mathrm{prim}} \;+\; \sum Br(j \to i)\, n_i \tag{61}$$

where $\overline{N}_i^{\mathrm{prim}}$ is the number of particles that corresponds to the given volume, temperature, and chemical potentials, and n_i is the number of particles created in the decay $j \to i$.

Chemical potentials.

$$h_1 + \ldots h_k \to H_1 + \ldots H_l\ , \qquad H_1 + \ldots H_l \to h_1 + \ldots + h_k\ . \tag{62}$$

Chemical equilibrium

$$\mu_{h_1} + \ldots \mu_{h_k} \;=\; \mu_{H_1} + \ldots \mu_{H_l}\ . \tag{63}$$

With Eq.(59) one finds

$$\begin{aligned} &\mu_B\ (b_{h_1} + \ldots b_{h_k})\ +\ \mu_Q\ (q_{h_1} + \ldots q_{h_k})\ +\ \mu_S\ (s_{h_1} + \ldots + s_{h_k}) \\ = &\mu_B\ (b_{H_1} + \ldots b_{H_l})\ +\ \mu_Q\ (q_{H_1} + \ldots q_{H_l})\ +\ \mu_S\ (s_{H_1} + \ldots + s_{H_k})\ . \end{aligned} \tag{64}$$

μ_B, μ_Q, μ_S – ?

1. π^+, π^-, π^0 .

$$\mu_{\pi^0} = 0\ , \qquad \mu_{pi^+} = \mu_Q\ , \qquad \mu_{\pi^-} =\ -\ \mu_Q\ .$$

$$\mu_Q > 0 \rightarrow \langle Q \rangle > 0\ , \qquad \mu_Q = 0 \rightarrow \langle Q \rangle = 0\ , \qquad \mu_Q < 0 \rightarrow \langle Q \rangle < 0\ .$$

2. p, n, $\overline{p}$, $\overline{n}$, π^+, π^-, π^0 .

$$\mu_n = \mu_B\ , \qquad \mu_{\overline{n}} = -\mu_B\ , \qquad \mu_p = \mu_B + \mu_Q\ , \qquad \mu_{\overline{p}} = -\mu_B - \mu_Q\ .$$

A+A collisions at small energies: the system of nucleons with small T (no pions) and large μ_B (no antinucleons) are created. What are μ_B and μ_Q in the system of nucleons? T and μ_B are independent variables. μ_Q is not independent variable! In nuclei $N_p \cong N_n$, i.e., $Q \cong 0.5B$ (in heavy nuclei $Q \cong 0.4B$). To have $N_p \cong N_n$ one needs $\mu_p \cong \mu_n$ and, thus, $\mu_Q \cong 0$. To have $N_p < N_n$ one needs $\mu_p < \mu_n$ and, thus, $\mu_Q < 0$. As a consequence, one finds $\mu_{\pi^+} < 0 < \mu_{pi^-}$ and therefore $N_{\pi^-} > N_{\pi^0} > N_{\pi^+}$. **It is different from p+p collisions, where $N_{\pi^-} < N_{\pi^0} < N_{\pi^+}$.** This is because $Q = B = 2$ and $\mu_Q > 0$.

3. nucleons (non strange baryons), strange baryons (Λ, Ξ, Ω), strange mesons (K, $\overline{K}$). In the system created in A+A collisions $B = 2A$, $Q \cong 0.5B$, and $S = 0$. Independent variables are T and μ_B. What are the values of μ_S? For simplicity we neglect small value of μ_Q and assume $\mu_Q \cong 0$.

$$\mu_\Lambda\ =\ \mu_B\ -\ \mu_S$$

If Λ would be the only strange particle the $S = 0$ would lead to $N_\Lambda = N_{\overline{\Lambda}}$ and $\mu_\Lambda = \mu_{\overline{\Lambda}} = 0$, or $\mu_S = \mu_B$. However, there are both strange (anti)baryons and strange mesons. Zero strangeness means

$$\langle \Lambda \rangle \; + \; \langle K^+ \rangle \; - \; \langle K^- \rangle \; \pm \ldots \cong 0$$

One finds

$$0 \; < \; \mu_S(T, \mu_B) \; < \; \mu_B$$

Fermi Gas

$$p(T, \mu) \;=\; \frac{gV}{2\pi 2} \int_0^\infty k^2 dk \; \frac{k^2}{(k^2 + m^2)^{1/2}} \; \left[\exp \left(\frac{\sqrt{k^2 + m^2} \; - \; \mu}{T} \right) \; + \; 1 \right]^{-1} \; . \tag{65}$$

$$s \;=\; \left(\frac{\partial p}{\partial T} \right)_\mu \; , \qquad n \;=\; \left(\frac{\partial p}{\partial \mu} \right)_T \; , \qquad \varepsilon \;=\; Ts \;+\; \mu n \;-\; p \; . \tag{66}$$

Particle number density:

$$n = \frac{g}{2\pi^2} \int_0^\infty k^2 dk \; \left[\exp \left(\frac{\sqrt{k^2 + m^2} \; - \; \mu}{T} \right) \; + \; 1 \right]^{-1} \; . \tag{67}$$

At $T \to 0$ the Fermi distribution should have $\mu > m$, and it goes to 1 for $k < k_F$ and to 0 at $k > k_F$, where $\sqrt{k_F^2 + m^2} = \mu$.

Ultra-relativistic approximation: $m = 0$, thus, $k_F = \mu$. It gives

$$n \;=\; \frac{g}{2\pi^2} \int_0^{k_F} k^2 dk \;=\; \frac{g}{6\pi^2} \; k_F^3 \;=\; \frac{g}{6\pi^2} \; \mu^3 \; .$$

Non-relativistic approximation

$$\sqrt{k^2 + m^2} \;-\; \mu \;\cong\; m \;+\; \frac{k^2}{2m} \;-\; \mu \;\equiv\; \frac{k^2}{2m} \;-\; \mu^{\rm nr} \; . \tag{68}$$

In this case,

$$k_F = (2m \, \mu^{\rm nr})^{1/2}$$

and

$$n \;=\; \frac{g}{2\pi^2} \int_0^{k_F} k^2 dk \;+\; \frac{g}{6\pi^2} \; k_F^3 \;=\; \frac{g}{6\pi^2} \; (2m \, \mu^{\rm nr})^{3/2} \; .$$

Bose Gas

$$p(T,\mu) \; = \; \frac{gV}{2\pi 2}\int_0^\infty k^2 dk \; \frac{k^2}{(k^2+m^2)^{1/2}} \; \left[\exp\left(\frac{\sqrt{k^2+m^2} \; - \; \mu}{T}\right) \; - \; 1\right]^{-1} \; . \tag{69}$$

$$s \; = \; \left(\frac{\partial p}{\partial T}\right)_\mu \; , \qquad n \; = \; \left(\frac{\partial p}{\partial \mu}\right)_T \; , \qquad \varepsilon \; = \; Ts \; + \; \mu n \; - \; p \; . \tag{70}$$

Particle number density:

$$n = \frac{g}{2\pi^2}\int_0^\infty k^2 dk \; \left[\exp\left(\frac{\sqrt{k^2+m^2} \; - \; \mu}{T}\right) \; - \; 1\right]^{-1} \; . \tag{71}$$

For the Bose distribution $\mu < m$ (or $\mu^{\rm nr} < 0$. If n is fixed and T decreases, the chemical potential μ increases. It reaches its maximal value $\mu = m$ at $T_{\rm BE}$ and Bose-Einstein condensation at the level $k = 0$ takes place.

III. QGP: LATTICE QCD AND PHENOMENOLOGICAL MODELS

$\mu_B = 0$: Lattice QCD, nucleus-nucleus collisions at RHIC and LHC.

Boltzmann approximation

$$\langle N \rangle \; = \; \frac{gV}{2\pi^2}\int_0^\infty k^2 dk \; \exp\left[- \; \sqrt{k^2+m^2}/T\right] \; = \; \frac{gV}{2\pi^2}\, m^2 T\, K_2(m/T) \; . \tag{72}$$

At $m \gg T$

$$\langle N \rangle \; = \; g\,V \; \left(\frac{mT}{2\pi}\right)^{3/2} \; \exp\left(- \, m/T\right) \; ,$$

and at $m \ll T$

$$\langle N \rangle \; = \; \frac{gV}{\pi^2}\, T^3$$

.

Pressure:

$$p \; = \; n\,T \; , \qquad n \; \equiv \; \frac{N}{V}$$

Energy density:

$$\varepsilon \; = \; \left(m \; + \; \frac{3}{2}T\right) \, n \; , \qquad \text{at } m \gg T \; ,$$

$$\varepsilon \;=\; 3nT \;\equiv\; \sigma\, T^4 \;\equiv\; \frac{3g}{\pi^2}\, T^4 \;, \qquad \text{at } m \ll T \;,$$

Massless Bose and Fermi gases:

$$\begin{aligned} \int_0^\infty x^2 dx\; x\; [\exp(x) \;-1]^{-1} \;&=\; 3!\;\zeta(4) \;=\; \frac{\pi^4}{15}\;, \\ \int_0^\infty x^2 dx\; x\; [\exp(x) \;+1]^{-1} \;&=\; \frac{7}{8}\int_0^\infty x^2 dx\; x\; [\exp(x) \;-1]^{-1} \end{aligned} \tag{73}$$

The Stephan-Boltzmann law for massless non-interacting particles:

$$\varepsilon \;=\; \sigma\, T^4 \;, \tag{74}$$

with Stephan-Boltzmann constant ($g = 1$),

$$\sigma_{\text{Fermi}} \;=\; \frac{7}{8}\,\frac{\pi^2}{30} \cong 0.29 \;\;<\; \sigma_{\text{Boltz}} = \frac{3}{\pi^2} \cong 0.30 \;<\; \sigma_{\text{Bose}} = \frac{\pi^2}{30} \cong 0.33\;. \tag{75}$$

Ideal QGP

$$p(T,\mu) \;=\; p_g \;+\; p_q \;+\; p_{\overline{g}}$$

$$\mu_i \;=\; b_i\mu_B \;+\; s_i\mu_S \;+\; q_i\mu_Q$$

Ideal Gluon Gas. Gluon has no conserved charges, and $\mu_g = 0$.

$$p_g \;=\; \frac{\sigma_g}{3}\, T^4 \;, \qquad \sigma_g \;=\; \frac{g_g\,\pi^2}{30}\; T^4 \;, \qquad g_g = 2\cdot 8\;, \tag{76}$$

$$\varepsilon_g \;=\; T\,\frac{dp_g}{dT} \;-\; p \;=\; \sigma_g\, T^4 \;, \qquad p = \frac{1}{3}\,\varepsilon\;. \tag{77}$$

In A+A collisions, the QGP is formed with non-zero baryonic density

$$\rho_B = \frac{B}{V}$$

, the electric charge density

$$\rho_Q = \frac{Q}{V} = \frac{B}{2V} = \frac{1}{2}\rho_B\;,$$

and zero strangeness density

$$\rho_S = 0.$$

.

One finds:

$$S = N_s + N_{\overline{s}} = 0 \; , \tag{78}$$

$$B = \frac{1}{3}\,(N_u - N_{\overline{u}} + N_d - N_{\overline{u}}) \; , \tag{79}$$

$$Q = \frac{2}{3}\,(N_u - N_{\overline{u}}) - \frac{1}{3}\,(N_d - N_{\overline{d}}) \; . \tag{80}$$

If $\mu_Q = 0$, one obtains $N_u = N_d \equiv N_q$ and $N_{\overline{u}} = N_{\overline{d}} \equiv N_{\overline{q}}$. Then

$$B = \frac{2}{3}\,(N_q - N_{\overline{q}}) \; , \tag{81}$$

$$Q = \frac{2}{3}\,(N_q - N_{\overline{q}}) - \frac{1}{3}\,(N_q - N_{\overline{q}}) = \frac{1}{3}\,(N_q - N_{\overline{q}}) = \frac{1}{2}\,B \tag{82}$$

From $N_s = N_{\overline{s}}$ it follows

$$\mu_S = \frac{1}{3}\,\mu_B$$

.

In general, there are two independent variable: T and μ_B. The strange chemical potential $\mu_S = \mu_S(T, \mu_B)$ and electric chemical potential $\mu_Q = \mu_Q(T, \mu_B)$ are obtained from $S = 0$ and $Q = (0.4 - 0.5)B$.

For massless on-interacting quarks the QGP can be calculated analytically

$$p_{qg}(T, \mu_B) = \frac{\sigma}{3}\,T^4 + \left(\frac{\mu_B}{3}\right)^2 T^2 + \frac{1}{2\pi^2}\left(\frac{\mu_B}{3}\right)^4 \; , \tag{83}$$

where

$$\sigma = \frac{\pi^2}{30}\left(2 \cdot 8 + \frac{7}{8} \cdot 2 \cdot 2 \cdot 3 \cdot 3\right) \; .$$

Bag model:

$$p_{QGP}(T, \mu_B) = p_{qg}(T, \mu_B) - B \; , \qquad \varepsilon(T, \mu_B) = \varepsilon_{qg}(T, \mu_B) + B \; . \tag{84}$$

Examples of the 1st order phase transition.

1). $\mu_B = 0$. Hadron phase – massless pion gas ($g_\pi = 3$):

$$p_\pi(T) \;=\; \frac{\pi^2}{10}T^4$$

QGP

$$p_{QGP}(T) \;=\; \frac{\pi^2}{30}\left(16 + \frac{21}{2}\right)T^4 \;-\; B \;. \tag{85}$$

1st order phase transition (Gibbs criterium):

$$p_\pi(T_c) \;=\; p_{QGP}(T_c) \;. \tag{86}$$

Equations (86) gives

$$T_c \;=\; \left(\frac{3}{\sigma_{QGP} \;-\; \sigma_H}\,B\right)^{1/4} \;. \tag{87}$$

2). $\mu_B \geq 0$. Hadron phase – heavy nucleons and antinucleons plus pions. The line of the 1st order phase transition in the T-μ_B plane is defined by

$$p_H(T,\mu_B) \;=\; p_{QGP}(T,\mu_B) \;. \tag{88}$$

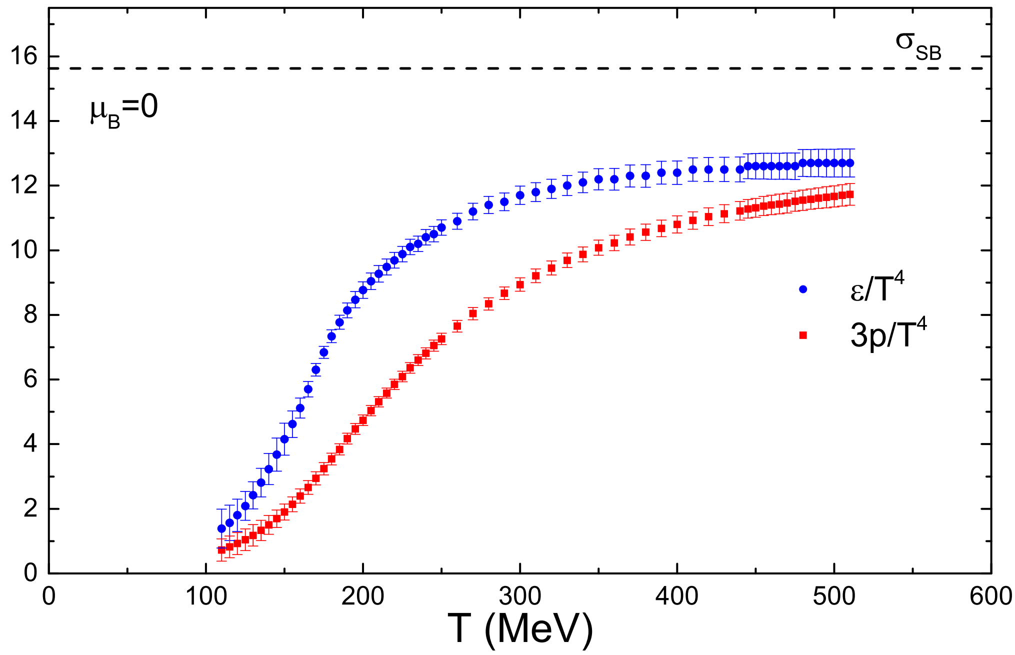

Figure 3: The lattice results from Ref. [?] for $3p/T^4$ (circles) and ε/T^4 (squares) at zero baryonic chemical potential.

IV. PARTICLE NUMBER FLUCTUATIONS

A. Partition function and mean multiplicity

The analysis of fluctuations is an important tool to study a physical system created in high energy nuclear collisions (see e.g. [?]). Recently, rich experimental data on fluctuations of particle production properties in nuclear collisions at high energies have been presented.

Let us consider the system which consists of one sort of positively and negatively charged particles (e.g. π^+ and π^- mesons) with total charge equal to zero $Q = 0$. In the case of the Boltzmann ideal gas (the interactions and quantum statistics effects are neglected) in the volume V and at temperature T the GCE partition function reads:

$$Z_{g.c.e.}(V,T) \;=\; \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!} \frac{(\lambda_- z)^{N_-}}{N_-!} \;=\; \exp\left(\lambda_+ z \;+\; \lambda_- z\right) \;=\; \exp(2z)\;. \qquad (89)$$

In Eq. (89) z is a single particle partition function

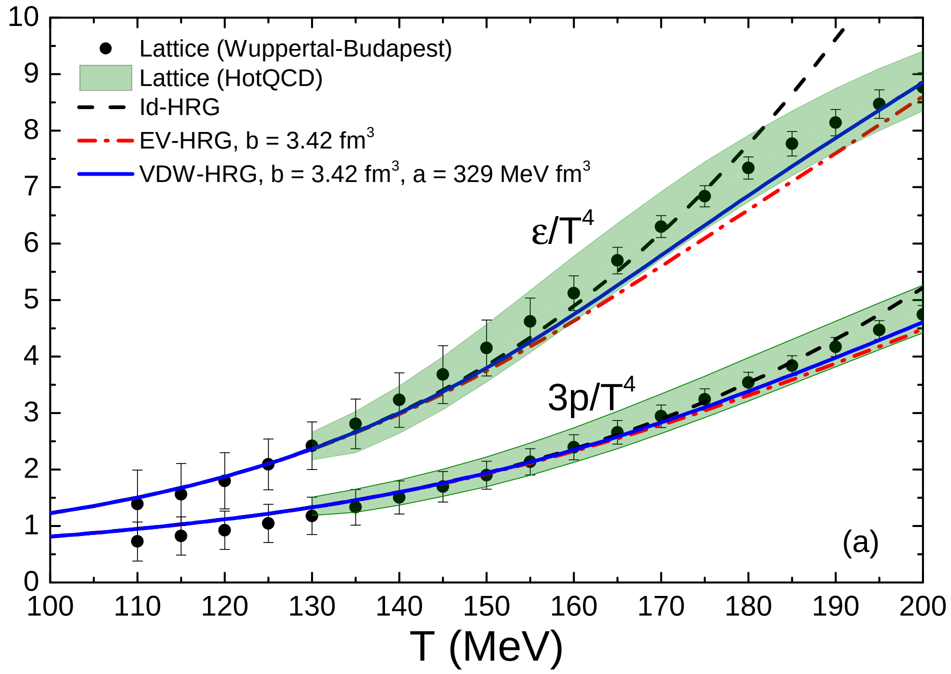

Figure 4:

The CE partition function is obtained by an explicit introduction of the charge conservation constrain, $N_+ - N_- = 0$ for each microscopic state of the system and it reads:

$$
\begin{aligned}
Z_{c.e.}(V,T) ~&=~ \sum_{N_+=0}^{\infty}\sum_{N_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!}\,\frac{(\lambda_- z)^{N_-}}{N_-!}\,\delta(N_+ - N_-) ~= \\
&=~ \frac{1}{2\pi}\int_0^{2\pi} d\phi~~\exp\left[z~(\lambda_+~e^{i\phi}~+~\lambda_-~e^{-i\phi})\right] ~=~ I_0(2z)~.
\end{aligned}
\tag{90}
$$

The average number of N_+ and N_- can be calculated:

$$
\langle N_\pm\rangle_{g.c.e.} ~=~ z~, \qquad\qquad \langle N_\pm\rangle_{c.e.} ~=~ z\,\frac{I_1(2z)}{I_0(2z)}~. \tag{91}
$$

The exact charge conservation leads to the *c.e.* suppression $(I_1(2z)/I_0(2z) < 1)$ of the charged particle multiplicity relative to the result for the *g.c.e.* (91). The ratio of $\langle N_\pm\rangle$ calculated in the *c.e.* and *g.c.e.* is plotted as a function of z in Fig. 5.

In the large volume limit ($V \to \infty$ corresponds also to $z \to \infty$) the results for mean quantities in the *c.e.* and *g.c.e.* are equal. This result is referred as an equivalence of the canonical and grand canonical ensembles. It can be obtained using an asymptotic expansion of the modified

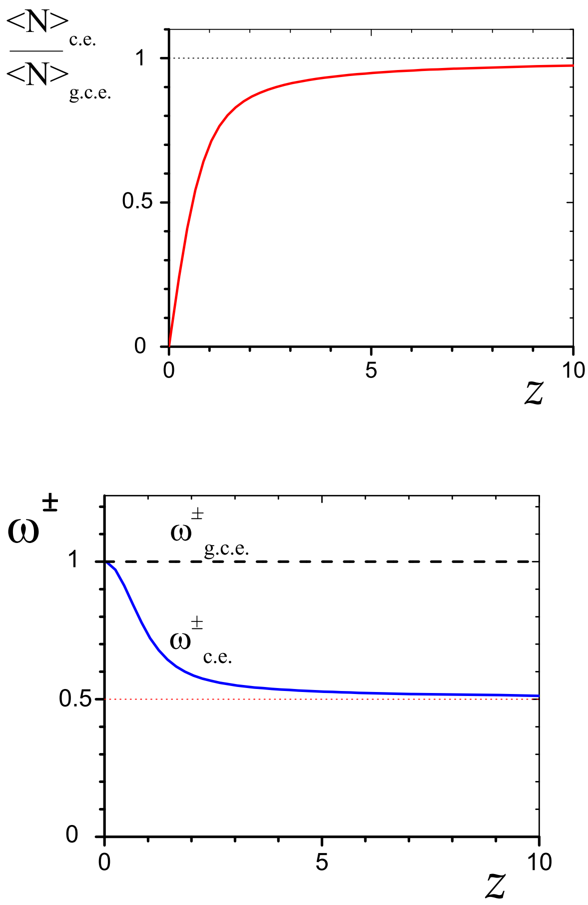

Figure 5: *Left*: The ratio of $\langle N_\pm \rangle_{c.e.}$ to $\langle N_\pm \rangle_{g.c.e.}$ as a function of z. *Right*: The scaled variances of $N_\pm$ calculated within the $g.c.e.$, $\omega^\pm_{g.c.e.} = 1$, and $c.e.$, $\omega^\pm_{c.e.}$.

Bessel function [?]:

$$\lim_{z\to\infty} I_n(2z) \;=\; \frac{\exp(2z)}{\sqrt{4\pi z}} \left[1 \;-\; \frac{4n^2-1}{16z} \;+\; O\left(\frac{1}{z^2}\right)\right] \;, \tag{92}$$

which gives $I_1(2z)/I_0(2z) \to 1$ and therefore

$$\langle N_\pm \rangle_{c.e.} \;\cong\; \langle N_\pm \rangle_{g.c.e} = z \;. \tag{93}$$

Using the series expansion one gets [?] for small systems ($z \ll 1$):

$$I_n(2z) \;=\; \frac{z^n}{n!} \;+\; \frac{z^{n+2}}{(n+1)!} \;+\; O\left(z^{n+4}\right) \;, \tag{94}$$

and consequently $I_1(2z)/I_0(2z) \cong z$ which results in

$$\langle N_\pm \rangle_{c.e.} \;\cong\; z^2 \;\;\ll\;\; \langle N_\pm \rangle_{g.c.e.} \;=\; z \;. \tag{95}$$

The asymptotics of the mean multiplicity discussed above are clearly seen in Fig. **??**.

B. Scaled variance

An useful measure of fluctuations of any variable X is the ratio of its variance $V(X) = \langle X^2 \rangle - \langle X \rangle^2$ to its mean value $\langle X \rangle$, referred here as the scaled variance:

$$\omega^X \;\equiv\; \frac{\langle X^2 \rangle - \langle X \rangle^2}{\langle X \rangle} \;. \tag{96}$$

Note, that $\omega^X \;=\; 1$ for the Poisson distribution. Thus, to study the fluctuations of charged particles the second moment of the multiplicity distribution $\langle N_\pm^2 \rangle$ has to be calculated. In the $g.c.e.$ (89) and CE (90) one finds:

$$\langle N_\pm^2 \rangle_{g.c.e.} \;=\; \frac{1}{Z_{g.c.e.}} \left[\frac{\partial}{\partial \lambda_\pm} \left(\lambda_\pm \frac{\partial \; Z_{g.c.e.}}{\partial \lambda_\pm} \right) \right]_{\lambda_\pm = 1} \;=\; z \;+\; z^2 \;, \tag{97}$$

$$\langle N_\pm^2 \rangle_{c.e.} \;=\; \frac{1}{Z_{c.e.}} \left[\frac{\partial}{\partial \lambda_\pm} \left(\lambda_\pm \frac{\partial \; Z_{c.e.}}{\partial \lambda_\pm} \right) \right]_{\lambda_\pm = 1} \;=\; z \, \frac{I_1(2z)}{I_0(2z)} \;+\; z^2 \, \frac{I_2(2z)}{I_0(2z)} \;=\; z^2 \;. \tag{98}$$

The corresponding scaled variances are:

$$\omega^\pm_{g.c.e.} \;=\; \frac{\langle N_\pm^2 \rangle_{g.c.e.} \;-\; \langle N_\pm \rangle_{g.c.e.}^2}{\langle N_\pm \rangle_{g.c.e.}} \;=\; 1 \;, \tag{99}$$

$$\omega^\pm_{c.e.} \;=\; \frac{\langle N_\pm^2 \rangle_{c.e.} \;-\; \langle N_\pm \rangle_{c.e.}^2}{\langle N_\pm \rangle_{c.e.}} \;=\; 1 \;-\; z \left[\frac{I_1(2z)}{I_0(2z)} \;-\; \frac{I_2(2z)}{I_1(2z)} \right] \;. \tag{100}$$

Using Eqs. (92) and (94) the asymptotic behavior of $\omega^\pm_{c.e}$ for both $z \;\to\; 0$ and $z \;\to\; \infty$ can be found. The CE fluctuations measured in terms of ω are equal to those in the $g.c.e.$ for the small system ($z \ll 1$) (another variable to treat the fluctuations in the small systems is discussed in Appendix):

$$\omega^\pm_{c.e} \;\cong\; 1 \;-\; \frac{z^2}{2} \;\cong\; 1 \;= \omega^\pm_{g.c.e} \;. \tag{101}$$

For large systems ($z \gg 1$) the scaled variance for the $c.e.$ is two times smaller than the scaled variance for the $g.c.e.$:

$$\omega^\pm_{c.e.} \;\cong\; \frac{1}{2} \;+\; \frac{1}{8z} \;\cong\; \frac{1}{2} \;=\; \frac{1}{2} \, \omega^\pm_{g.c.e.} \;. \tag{102}$$

The dependence of the scaled variance calculated within the $c.e$ and GCE on z is shown in Fig. 2.

The scaled variance shows a very different behavior than the mean multiplicity. In the limit of small z the ratio of the results for CE and GCE approaches zero for the mean multiplicity (Fig. 1) and one for the scaled variance (Fig. 2). On the other hand in the large z limit the mean multiplicity ratio approaches one and the scaled variance ratio 0.5. Thus in the case of fluctuations the canonical and grand canonical ensembles are not equivalent.

C. Multiplicity distribution.

In the GCE the multiplicity distribution of N_+ (and N_-) is equal to the Poisson one:

$$P_{g.c.e.}(N_+) \equiv \sum_{N_-=0}^{\infty} P_{g.c.e.}(N_+, N_-) = \frac{1}{Z_{g.c.e.}} \sum_{N_-=0}^{\infty} \frac{z^{N_+}}{N_+!} \frac{z^{N_-}}{N_-!} \tag{103}$$
$$= \exp(-z) \cdot \frac{z^{N_+}}{N_+!} ,$$

whereas the corresponding distribution in the $c.e.$ (90) is:

$$P_{c.e.}(N_+) \equiv \sum_{N_-=0}^{\infty} P_{c.e.}(N_+, N_-) = \frac{1}{Z_{c.e.}} \sum_{N_-=0}^{\infty} \frac{z^{N_+}}{N_+!} \frac{z^{N_-}}{N_-!} \cdot \delta(N_+ - N_-) \tag{104}$$
$$= \frac{1}{I_0(2z)} \cdot \left(\frac{z^{N_+}}{N_+!}\right)^2 .$$

As an example, the distributions in GCE and CE are plotted in Figs. 3 and 4 for $z = 0.5$ (the small system) and $z = 10$ (the large system), respectively.

As expected from the previous discussion, the $c.e.$ distribution (104) is narrower (the variance is smaller) than the $g.c.e.$ one (103). This result is valid for both the large ($z \gg 1$) and the small ($z \ll 1$) system. On the other hand, the average value of $N_\pm$ is smaller in the $c.e.$ than in the $g.c.e.$ for small z. It results in $\omega^\pm_{c.e.} \to \omega^\pm_{g.c.e} = 1$ at $z \to 0$. Moreover, for $\langle N_\pm \rangle \ll 1$ one can easily demonstrate that $\omega^\pm \cong 1$ for any $P(N_\pm)$ distribution if the conditions $P(0) \gg P(1) \gg P(k)$ (with $k \geq 2$) are satisfied. Indeed, in this limit one can neglect all $P(N_\pm)$ for $N_\pm \geq 2$ which results in:

$$\omega^\pm \equiv \frac{\langle N_\pm^2 \rangle - \langle N_\pm \rangle^2}{\langle N_\pm \rangle} \cong \frac{P(1) \cdot 1^2 - [P(1) \cdot 1]^2}{P(1) \cdot 1} \cong 1 , \tag{105}$$

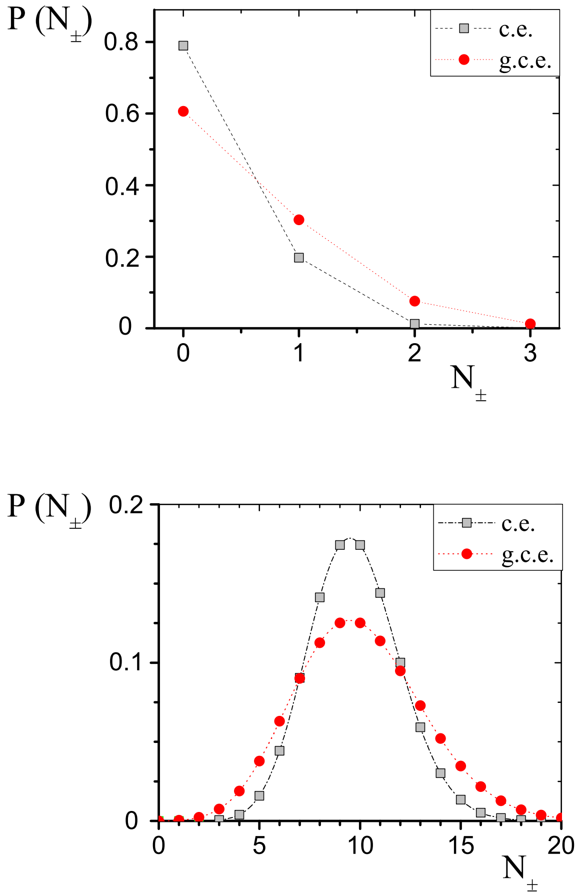

Figure 6: *Left*: Multiplicity distributions $P_{c.e.}(N_{\pm})$ (104) and $P_{g.c.e.}(N_{\pm})$ (103) for $z = 0.5$. *Right*: The same for $z = 10$.

as $P(1) \cong \langle N_{\pm} \rangle \ll 1$. In the large volume limit, see Fig. 4, the mean values of the CE and GCE distributions become equal, but the CE distribution is narrower than the GCE one.

D. Total multiplicity of charged particles

The total multiplicity of charged particles is defined as $N_{ch} = N_+ + N_-$. Its average in the $g.c.e.$ and $c.e.$ reads:

$$\langle N_{ch}\rangle_{g.c.e.} = \langle\, N_+ + N_-\,\rangle_{g.c.e.} = \langle N_+\rangle_{g.c.e.} + \langle N_-\rangle_{g.c.e.} = 2z\;, \tag{106}$$

$$\langle N_{ch}\rangle_{c.e.} = \langle\, N_+ + N_-\,\rangle_{c.e.} = \langle N_+\rangle_{c.e.} + \langle N_-\rangle_{c.e.} = 2z\,\frac{I_1(2z)}{I_0(2z)}\;. \tag{107}$$

In the $g.c.e.$ one finds:

$$\langle N_{ch}^2\rangle_{g.c.e.} = \langle N_+^2 + 2N_+N_- + N_-^2\rangle_{g.c.e.} = \langle N_+^2\rangle_{g.c.e.} + 2\langle N_+\rangle_{g.c.e.}\langle N_-\rangle_{g.c.e.} + \tag{108}$$
$$+\ \langle N_-^2\rangle_{g.c.e.} = z^2 + z\ +\ 2z^2\ +\ z^2 + z\ =\ 4z^2\ +\ 2z\;,$$

and consequently the scaled variance of N_{ch} in the $g.c.e.$ is:

$$\omega^{ch}_{g.c.e.} \equiv \frac{\langle N_{ch}^2\rangle_{g.c.e.}\ -\ \langle N_{ch}\rangle^2_{g.c.e.}}{\langle N_{ch}\rangle_{g.c.e.}} = \frac{4z^2 + 2z\ -\ (2z)^2}{2z} = 1\;. \tag{109}$$

The result (109) also follows from explicit expression on the probability distribution of N_{ch} in the $g.c.e.$:

$$P_{g.c.e.}(N_{ch}) \equiv \sum_{N_+}^{\infty}\sum_{N_-=0}^{\infty} P_{g.c.e.}\,(N_+,N_-)\cdot\delta\left[N_{ch} - (N_+ + N_-)\right] \tag{110}$$
$$= \frac{1}{Z_{g.c.e.}}\sum_{N_+}^{\infty}\sum_{N_-=0}^{\infty}\frac{z^{N_+}}{N_+!}\,\frac{z^{N_-}}{N_-!}\cdot\delta\left[N_{ch} - (N_+ + N_-)\right] = \exp(-2z)\,\frac{(2z)^{N_{ch}}}{N_{ch}!}\;.$$

Thus distributions of N_{ch} and $N_\pm$ are Poissonian in the $g.c.e.$. In the $c.e.$ the negatively and positively charged particles are correlated, $\langle N_+\cdot N_-\rangle_{c.e.} \neq \langle N_+\rangle_{c.e.}\cdot\langle N_-\rangle_{c.e.}$. The correlation term reads:

$$\langle N_+\cdot N_-\rangle_{c.e.} = \frac{1}{Z_{c.e.}}\left(\frac{\partial^2\, Z_{c.e.}}{\partial\lambda_+\partial\lambda_-}\right)_{\lambda_\pm=1} = z^2\;. \tag{111}$$

Using Eqs. (98) and (111) one obtains the scaled variance of N_{ch} in the $c.e.$:

$$\omega^{ch}_{c.e.} \equiv \frac{\langle N_{ch}^2\rangle_{c.e.}\ -\ \langle N_{ch}\rangle^2_{c.e.}}{\langle N_{ch}\rangle_{c.e.}} = 1\ +\ z\left[\frac{I_2(2z) + I_0(2z)}{I_1(2z)}\ -\ 2\,\frac{I_1(2z)}{I_0(2z)}\right]\;. \tag{112}$$

The scaled variances $\omega^{ch}_{g.c.e}$ and $\omega^{ch}_{c.e.}$ as functions of z are shown in Fig. 7 together with $\omega^{\pm}_{g.c.e}$ and $\omega^{\pm}_{c.e.}$.

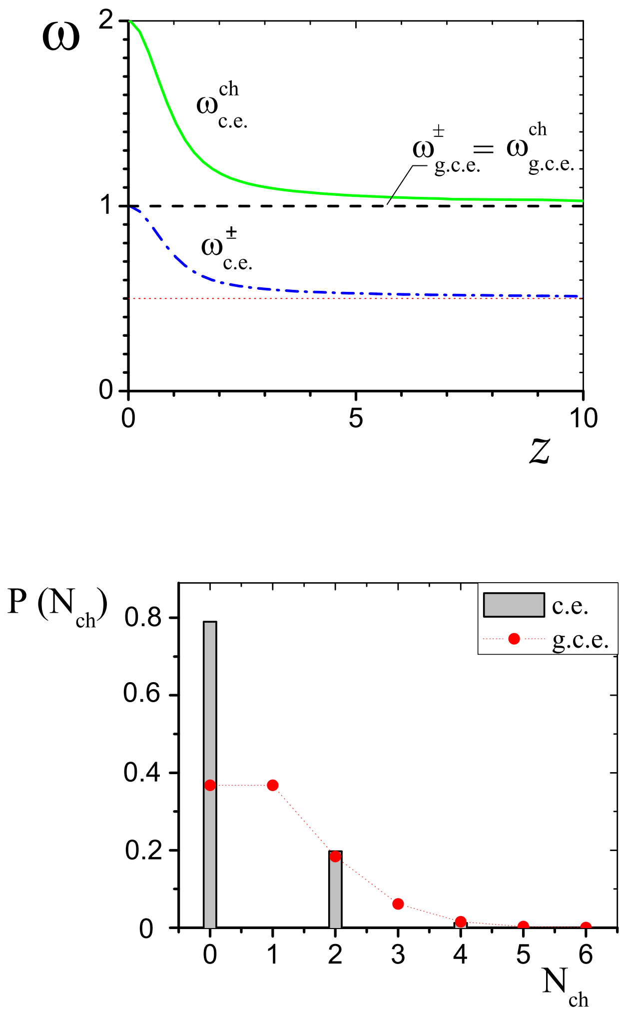

Figure 7: *Left*: The scaled variances $\omega^{ch}_{c.e.}$ (112), $\omega^{\pm}_{c.e.}$ (100) and $\omega^{\pm}_{g.c.e.} = \omega^{ch}_{g.c.e.} = 1$ (99,109) as functions of z. *Right*: Multiplicity distributions of N_{ch} for $z = 0.5$ in the *g.c.e.* and *c.e.* .

From Eqs. (100) and (112) and the recurrence relation $I_0(2z) = I_2(2z) + I_1(2z)/z$ [**?**] it follows that $\omega^{ch}_{c.e.} = 2\omega^{\pm}_{c.e.}$, i.e. the relative variance of total charge multiplicity N_{ch} is two times larger than the one of $N_{\pm}$. This is because $N_{ch} = 2N_+ = 2N_-$ in each microscopic state allowed by an exact charge conservation. One obtains a similar result for the case of particle production via decay of neutral resonances, e.g., $\rho^0 \to \pi^+ + \pi^-$. The distributions of π^+ and π^- coincide with the ρ^0 distribution, and consequently $\omega^{\pm} = \omega$, where ω is the scaled variance of the distribution of ρ^0. But because $N_{ch} = 2N_\rho$ one gets $\omega^{ch} = 2\omega$.

Probability distribution of N_{ch} in the $c.e.$ reads:

$$
\begin{aligned}
P_{c.e.}(N_{ch}) &\equiv \sum_{N_+}^{\infty}\sum_{N_-=0}^{\infty} P_{c.e.}\left(N_+, N_-\right)\cdot\delta\left[N_{ch}-(N_+ + N_-)\right] \qquad (113)\\
&= \frac{1}{I_0(2z)}\sum_{N_+=0}^{\infty}\sum_{N_-=0}^{\infty}\frac{z^{N_+}}{N_+!}\frac{z^{N_-}}{N_-!}\cdot\delta\left(N_+ - N_-\right)\cdot\delta\left[N_{ch}-(N_+ + N_-)\right]\\
&= \frac{1}{I_0(2z)}\left[\frac{z^{N_{ch}/2}}{(N_{ch}/2)!}\right]^2 .
\end{aligned}
$$

It coincides, of course, with $P_{c.e.}(N_+)$ (104) at $N_+ = N_{ch}/2$. As an example , the probability distributions $P_{g.c.e.}(N_{ch})$ (110) and $P_{c.e.}(N_{ch})$ (113) are shown for $z = 0.5$ (the small system) and for $z = 10$ (the large system) in Figs. 6 and 7, respectively. Only even multiplicities $N_{ch} = 0,\ 2,\ 4\ldots$ are allowed in the $c.e.$ because of an exact charge conservation. For the small system ($z \ll 1$) the ω^{ch} reads (both $P_{g.c.e.}(N_{ch} = 1) \ll 1$ and $P_{c.e.}(N_{ch} = 2) \ll 1$ at $z \ll 1$):

$$
\omega^{ch}_{g.c.e.} \cong \frac{P_{g.c.e.}(1)\cdot 1^2 - \left[P_{g.c.e.}(1)\cdot 1\right]^2}{P_{g.c.e.}(1)\cdot 1} \cong 1 , \qquad (114)
$$

$$
\omega^{ch}_{c.e.} \cong \frac{P_{c.e.}(2)\cdot 2^2 - \left[P_{c.e.}(2)\cdot 2\right]^2 - \left[P_{c.e.}(2)\cdot 2\right]^2}{P_{c.e.}(2)\cdot 2} \cong 2 . \qquad (115)
$$

In the large z limit the average number of charge particles $\langle N_{ch}\rangle$ and its scaled variance ω^{ch} in the $g.c.e.$, Eqs. (106) and (109), are equal to those in the $c.e.$, Eqs. (107) and (112). Nevertheless the corresponding probability distributions are different, see Fig. 7. This is because all odd multiplicities are excluded in CE as a consequence of the charge conservation. The relation between $P_{g.c.e.}(N_{ch})$ (110) and $P_{c.e.}(N_{ch})$ (113) for the large system ($z \gg 1$) can be established as follows. Let us introduce the probability distribution $P^*(N_{ch})$ defined as

$$
P^*(N_{ch}) \equiv C\cdot P_{g.c.e.}(N_{ch}) , \qquad N_{ch} = 0,\ 2,\ 4,\ldots , \qquad (116)
$$

$$
P^*(N_{ch}) \equiv 0 , \qquad N_{ch} = 1,\ 3,\ 5,\ldots , \qquad (117)
$$

where the constant C is given by a normalization condition

$$
\begin{aligned}
1 &= \sum_{N_{ch}=0,2,4,\ldots} P^*(N_{ch}) \equiv C\cdot\sum_{N_{ch}=0,2,4,\ldots} P_{g.c.e.}(N_{ch}) \qquad (118)\\
&= C\cdot\exp(-2z)\sum_{n=0}^{\infty}\frac{(2z)^{2n}}{(2n)!} = C\cdot\exp(-2z)\,\cosh(2z) .
\end{aligned}
$$

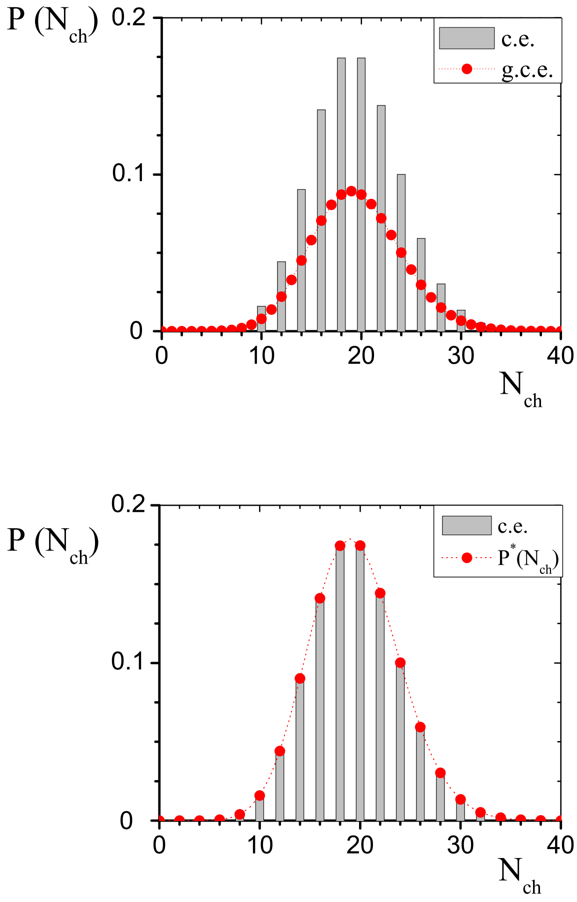

Figure 8: *Left*: Multiplicity distributions of N_{ch} for $z = 10$ in the *g.c.e.* and *c.e.* . *Right*: Multiplicity distributions $P_{c.e.}(N_{ch})$ (113) and $P^*(N_{ch})$ (116) for $z = 10$.

Using Eq. (118) one gets $C = 2 \cdot [1 + \exp(-4z)]^{-1} \cong 2$ for $z \gg 1$. The origin of the result $C \cong 2$ is the fact that

$$P_{g.c.e.}(N_{ch} + 1) \;\equiv\; P_{g.c.e.}(N_{ch}) \cdot \frac{2z}{N_{ch} + 1} \cong P_{g.c.e.}(N_{ch}) \;, \tag{119}$$

for N_{ch} close to its average value $\langle N_{ch} \rangle_{g.c.e.} = 2z \gg 1$, i.e. if the odd numbers $N_{ch} = 1, 3, 5, \ldots$ are forbidden the probabilities $P_{g.c.e.}(N_{ch})$ for the even numbers $N_{ch} = 0, 2, 4, \ldots$ should be approximately doubled to have a correct normalization for $P^*(N_{ch})$ (116).

Using the Stirling formula, $n! \cong n^n e^{-n} \sqrt{2\pi n}$, valid for $n \gg 1$, one finds that $P_{c.e.}(N_{ch}) \cong P^*(N_{ch})$ for N_{ch} close to its average value equal to $2z \gg 1$. Both distributions are plotted in

Fig. 8 for a comparison.